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Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group

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Preface

The theory of mean periodic functions is a subject which goes back to works of Littlewood, Delsarte, John and that has undergone a vigorous development in recent years. There has been much progress in a number of problems concerning local aspects of spectral analysis and spectral synthesis on homogeneous spaces. The study of these problems turns out to be closely related to a variety of questions in harmonic analysis, complex analysis, partial differential equations, integral geometry, approximation theory, and other branches of contemporary mathematics. The present book describes recent advances in this direction of research.

Symmetric spaces and the Heisenberg group are an active field of investigation at the moment. The simplest examples of symmetric spaces, the classical 2-sphere $\mathbb{S}^2$ and the hyperbolic plane $\mathbb{H}^2$, play familiar roles in many areas in mathematics. The Heisenberg group H^n is a principal model for nilpotent groups, and results obtained for H^n may suggest results that hold more generally for this important class of Lie groups. The purpose of this book is to develop harmonic analysis of mean periodic functions on the above spaces.

The book consists of four parts. Part I is devoted to symmetric spaces and related questions. After some general considerations in Chap. 1, rank one symmetric spaces play here a privileged role. There is a number of books that are characteristic of this subject from the abstract point of view. Our text differs at this point and is based on realizations of rank one spaces as domains in Euclidean space. The aim of such an approach is twofold: on the one hand, in this way we hope to contribute towards a better visualization and a better handling of these spaces; on the other hand, in addition to their intrinsic interest, these realizations will play an important role in our study of transmutation operators on rank one compact symmetric spaces in Part II. The exposition in Chaps. 2–5 of Part I has on occasion been used as a textbook for first-year graduate students without background in Lie group theory.

Part II develops the transmutation operator theory. We define appropriate analogues of the Abel–Radon transform and give a treatment of their basic properties. The generalized homomorphism property is the crucial one; it relates the mean periodicity on the spaces in question to that on $\mathbb{R}^1$ and allows many proofs in Parts III and IV to be carried out by reduction to the one-dimensional case.

Parts III and IV deal with the theory of mean periodic functions on domains of Euclidean spaces, Riemannian symmetric spaces, and the Heisenberg group. Attention was focused on Fourier-type decompositions and on the “hard analysis" problems that could be attacked with them: structure of zero sets of mean periodic functions and modern versions of John’s support theorem, local analogues of the Schwartz fundamental principle, the problem of mean-periodic continuation, Hörmander-type approximation theorems on domains without the convexity assumption, explicit reconstruction formulae in the deconvolution problem, Zalcman-type two-radii problems on domains of symmetric spaces of arbitrary rank, local versions of the Brown–Schreiber–Taylor theorem on spectral analysis and their symmetric space analogues, and so on. The difficulty in studying the above varies with spaces. Nevertheless, in all cases almost all results are the best possible, i.e., give answers to all questions which naturally arise in the topic and present a complete picture of the corresponding phenomenon. The proofs given are "minimal" in the sense that they involve only such concepts and facts which are indispensable for the essence of the subject. We shall have nothing to say in this book on the mean-periodicity on domains of compact symmetric spaces of higher ranks (except the case of the whole space) but hope that the methods that we develop will prove useful in this connection.

Each part begins with a summary and ends with comments. The reader will find here not only historical notes and further results but also many challenging conjectures and open problems and the invitation to work in this exiting field. The authors hope that the exposition in the book will be comprehensible to anyone who knows the elements of functional analysis and possesses sufficient perseverance in overcoming purely logical difficulties. All the necessary information is given in the text with references to the sources.

Some of the material in this book has been the subject of lectures delivered by the authors over a number of years. We have received helpful comments and suggestions from many colleagues; of these we mention R. Trigub, V. Zastavnyĭ, D. Zaraisky, A. Grishin, V. Ryazanov, V. Belyĭ, L. Ronkin, and B. Kotlyar. We thank them all.

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We are very indebted to the National Fund for Scientific Research for supporting our work. It is a pleasure to thank P. Masharov and O. Riznychenko for their expert and conscientious TEX setting of the manuscript. We are very grateful to our home institution, the Donetsk National University, for working conditions we enjoy.

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Donetsk, *Valery Volchkov*
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Contents

Part I
Symmetric Spaces. Harmonic Analysis on Spheres

The notion of symmetric space is among the most important notions in differential geometry. They are defined as Riemannian manifolds M with the following property: each $p \in M$ is an isolated fixed point of an involutive isometry s_p of M. There is only one such s_p. It is called the symmetry at the point p.

The theory of symmetric spaces was initiated by É. Cartan in 1926 and was vigorously developed by him in the late 1920s. Symmetric spaces have a transitive group of isometries and can be represented as coset spaces G/K, where G is a connected Lie group with an involutive automorphism σ whose fixed-point set is (essentially) K. This property was used by É. Cartan to classify them.

In Chap. 1 we review general notions and facts related to the theory of symmetric spaces which will be used throughout the book. We restrict ourselves to only a minimum of auxiliary information. Standard definitions in Riemannian geometry and Lie group theory are assumed; references for them are given, when necessary.

In Chaps. 2 and 3 symmetric spaces of rank one play a privileged role. Their geometrical structure is so rich that one can characterize them in several other ways. In particular, they, together with the Euclidean spaces $\mathbb{R}^n$ ($n = 1, 2, \dots$), comprise the two-point homogeneous spaces. These are the Riemannian manifolds M with the property that for any two pairs points (p_1, p_2) and (q_1, q_2) satisfying $d(p_1, p_2) = d(q_1, q_2)$, where d is the distance on M, there exists an isometry mapping p_1 to q_1 and p_2 to q_2.

The rank one symmetric spaces of noncompact type are the real, complex, and quaternionic hyperbolic spaces $\mathrm{SO}_0(n, 1)/\mathrm{SO}(n)$, $\mathrm{SU}(n, 1)/\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$, and $\mathrm{Sp}(n, 1)/\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ and the Cayley hyperbolic plane $F_4^*/\mathrm{Spin}(9)$. In a dual manner, the compact symmetric spaces of rank one are the various projective spaces corresponding to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{C}a$ and the Euclidean spheres. Hyperbolic spaces can be identified with the unit ball in Euclidean space. We give this identification in Chap. 2. Euclidean spaces can be regarded as parts of projective spaces. Chapter 3 contains a detailed discussion of this imbedding. We have attempted to render calculations in the text as explicit as possible. This circumstance is the source of very extensive information about symmetric spaces of rank one.

Roughly speaking, the group K acts transitively on the unit sphere S in the tangent space $\mathfrak{p}$ to a symmetric space G/K of rank one. The purpose of Chap. 4 is an explicit description of the decomposition of $L^2(S)$ into irreducible components under the action of K. Since K in its action on S is contained in the orthogonal group, we see that $\mathcal{H}^k$, the space of homogeneous kth-degree harmonic polynomials on $\mathfrak{p}$, is invariant under the action of K. Identifying elements of $\mathcal{H}^k$ with their restrictions to S, we have $L^2(S) = \sum_{k=0}^{\infty} \mathcal{H}^k$. In this way the question reduces to the problem of decomposing each $\mathcal{H}^k$ according to the action of K. To solve it we use the realizations of G/K obtained in Chaps. 2 and 3.

In Chap. 5 we consider K-finite eigenfunctions of the Laplace–Beltrami operator on rank one symmetric spaces G/K of noncompact type. Integral representations for these functions give us non-Euclidean analogues of the plane waves $e^{i\lambda\langle x,\eta\rangle_{\mathbb{R}}}$, $x \in \mathbb{R}^n$. They play an important role in harmonic analysis on G/K.

The results of Part I are used in the sequel to study mean periodic functions.

Chapter 1
General Considerations

We have put in Chap. 1 a good deal of generalities which is needed for the rest of the work. The five sections (see Contents) are rather disconnected.

Section 1.1 contains preliminaries to alternative algebras and exceptional Lie groups. Special attention is paid to the Cayley algebra $\mathbb{C}a$, the Albert algebra $\mathbb{A}1$, the exceptional compact Lie groups G_2 and F_4, and the group $\mathrm{O}_{\mathbb{C}a}(2)$ acting transitively on the sphere $\mathbb{S}^{15}$. In particular, we mention the very important algebraic characterization of F_4 due to Chevalley and Schafer.

In Sect. 1.2 we put together some background material from elementary differential geometry. Besides several standard notions such as the curvature tensor, the Riemannian connection, the Beltrami parameters, Hermitian and Kaehlerian structures, etc., the definition of ζ domain is introduced which will play an important role in Part III later.

Section 1.3 provides a brief introduction to the theory of symmetric spaces. We begin with the definition of homogeneous spaces and end with the complete classification of two-point homogeneous spaces.

In Sect. 1.4 the reader is acquainted with some basic tools of analysis on symmetric spaces G/K. The unifying object here is the convolution structure on G/K. In particular, we discuss the commutativity of convolution for K-invariant distributions on G/K, the action of invariant differential operators on convolution, and the mean value property for joint eigenfunctions on G/K.

Section 1.5 deals with harmonic analysis on compact homogeneous spaces K/M. Results coming into the picture are: the orthogonality relations of Schur, the Peter–Weyl theorem, and the Weyl decomposition of $L^2(K/M)$.

1.1 Numbers, Algebras and Groups. Some Illustrative Examples

The numbers dealt with in this book are representatives of one of the following four structures:

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$\mathbb{R}$—the field of real numbers;

$\mathbb{C}$—the field of complex numbers;

$\mathbb{Q}$—the body of quaternions;

$\mathbb{C}a$—the algebra of octaves (or the Cayley algebra).

Also, we shall use the set of natural numbers $\mathbb{N}$, the ring of integers $\mathbb{Z}$, the set of nonnegative integers $\mathbb{Z}_+$, and the set of rational numbers $\mathbf{Q}$.

The construction of $\mathbb{C}$, $\mathbb{Q}$, and $\mathbb{C}a$ is a special case of the so-called *doubling procedure*. We recall its definition (see, for instance, Postnikov [167], Lecture 14).

Let $\mathcal{A}$ be an arbitrary finite-dimensional algebra over $\mathbb{R}$ in which a conjugation, i.e., some involutory antiautomorphism $a \to \overline{a}$ is given. Consider a vector space $\mathcal{A}^2$ which is a direct sum of two copies of a vector space $\mathcal{A}$, i.e., which consists of pairs of the form (α, β), where $\alpha, \beta \in \mathcal{A}$. We introduce into $\mathcal{A}^2$ multiplication as follows:

$$(\alpha, \beta)(\gamma, \delta) = \big(\alpha\gamma - \overline{\delta}\beta, \beta\overline{\gamma} + \delta\alpha\big).$$

A simple check shows that relative to that multiplication the vector space $\mathcal{A}^2$ is an algebra. The algebra $\mathcal{A}^2$ is called a *doubling* of $\mathcal{A}$. We shall identify elements α and $(\alpha, 0)$ and thus assume that the algebra $\mathcal{A}$ is a subalgebra of $\mathcal{A}^2$. If $\mathcal{A}$ is a unit algebra, then the element $1 = (1, 0)$ will obviously be an identity element in $\mathcal{A}^2$ too. In addition, every element $(\alpha, \beta) \in \mathcal{A}^2$ is uniquely written as $\alpha + \beta e$, where $e = (0, 1)$.

For the procedure of a doubling to be iterated, it is necessary to define a conjugation in $\mathcal{A}^2$. We shall do this by the formula

$$\overline{(\alpha, \beta)} = (\overline{\alpha}, -\beta).$$

Then

$$\mathbb{C} = \mathbb{R}^2, \quad \mathbb{Q} = \mathbb{C}^2, \quad \mathbb{C}a = \mathbb{Q}^2.$$

The basis of $\mathbb{C}$ consists of $\mathbf{i}_0 = 1$ and the imaginary unit $\mathbf{i}_1 = (0, 1)$. In the sequel, $\mathbf{i}_1$ will be also denoted by i. The basis of $\mathbb{Q}$ consists of $\mathbf{i}_0 = 1$ and three elements

$$\mathbf{i}_1 = (\mathbf{i}_1, 0), \quad \mathbf{i}_2 = (0, 1), \quad \mathbf{i}_3 = \mathbf{i}_1\mathbf{i}_2.$$

Analogously, the basis of $\mathbb{C}a$ consists of $\mathbf{i}_0 = 1$ and seven elements

$$\mathbf{i}_1 = (\mathbf{i}_1, 0), \quad \mathbf{i}_2 = (\mathbf{i}_2, 0), \quad \mathbf{i}_3 = (\mathbf{i}_3, 0), \quad \mathbf{i}_4 = (0, 1),$$
$$\mathbf{i}_5 = \mathbf{i}_1\mathbf{i}_4, \quad \mathbf{i}_6 = \mathbf{i}_2\mathbf{i}_4, \quad \mathbf{i}_7 = \mathbf{i}_3\mathbf{i}_4.$$

As the construction of a doubling is iterated, the algebraic properties of the multiplication gradually deteriorate. In particular, the octave algebra $\mathbb{C}a$ is noncommutative and nonassociative. Nevertheless, in $\mathbb{C}a$ there are the relations

$$|uv| = |vu| = |u||v|, \tag{1.1}$$

$$uv = \overline{\overline{v}\,\overline{u}}, \tag{1.2}$$

$$(uv)v = u(vv), \qquad u(uv) = (uu)v, \tag{1.3}$$

$$(uv)\overline{v} = \overline{v}(vu) = u|v|^2, \tag{1.4}$$

$$(uv)v^{-1} = v^{-1}(vu) = u, \tag{1.5}$$

$$(uv)u = u(vu), \tag{1.6}$$

$$t(uv)t = (tu)(vt), \tag{1.7}$$

$$\mathrm{Re}\,(uv) = \mathrm{Re}\,(vu), \qquad \mathrm{Re}\,\big((tu)v\big) = \mathrm{Re}\,\big(t(uv)\big), \tag{1.8}$$

where

$$|u| = \sqrt{u\overline{u}} = \sqrt{\overline{u}u}, \qquad u^{-1} = \frac{\overline{u}}{|u|^2}, \qquad \mathrm{Re}\,u = \frac{u+\overline{u}}{2}.$$

Equality (1.1) means that the algebra $\mathbb{C}a$ is *normed*. Relations (1.3) and (1.6) are called the identities of *alternativity* and *elasticity*, respectively. Identity (1.7) is known as the *central Moufang identity*. We pay attention to the following well-known results (see Cantor and Solodovnikov [45], for example).

Theorem 1.1 (Hurwitz). *Every normed unit finite-dimensional algebra over the field* $\mathbb{R}$ *is isomorphic to one of the algebras* $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, *or* $\mathbb{C}a$.

Theorem 1.2 (Frobenius). *Every alternative division finite-dimensional algebra over* $\mathbb{R}$ *is isomorphic to one of the algebras* $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, *or* $\mathbb{C}a$.

Let Aut $\mathcal{A}$ be the group of automorphisms of an algebra $\mathcal{A}$. Clearly, Aut $\mathbb{R} = \{\mathrm{Id}\}$, where Id is the identity mapping. In addition, it is easy to make sure that Aut $\mathbb{C}$ consists of Id and the automorphism of complex conjugation $z \to \overline{z}$. Next, any automorphism $\mathbb{Q} \to \mathbb{Q}$ is an internal automorphism of the form $q \to pqp^{-1}$, where p is a quaternion of norm 1 (see Postnikov [167], Lecture 14). In this case, the mapping

$$(q_1, q_2) \to \big(pq_1p^{-1},\ pq_2p^{-1}\big), \quad q_1, q_2 \in \mathbb{Q}, \tag{1.9}$$

is an automorphism of the Cayley algebra. The group Aut $\mathbb{C}a$ of all automorphisms of the algebra $\mathbb{C}a$ is an exceptional compact Lie group of dimension 14 and is denoted by G_2 (see Postnikov [167], Lecture 14).

Besides the octaves themselves, one can consider matrices whose entries are octaves. Since there is a conjugation in the algebra $\mathbb{C}a$, for any octave matrix A, the *Hermitian conjugate matrix* A^* is defined as that which results from the transposed matrix A^T by replacing all its entries by conjugate octaves. By analogy with the complex case, the octave matrix A for which $A^* = A$ is called *Hermitian*.

On defining the product AB of octave matrices A and B by the usual formula, we see that the collection $\mathrm{A}(n, \mathbb{C}a)$ of all Hermitian octave matrices of a given order n is an algebra under the Jordan multiplication

$$A \circ B = \frac{AB + BA}{2}.$$

Moreover, one can prove that $(\mathrm{A}(n, \mathbb{C}a), \circ)$ is a *Jordan algebra* for $n \leqslant 3$, i.e.,

$$A \circ B = B \circ A \quad \text{and} \quad \big(A^2 \circ B\big) \circ A = A^2 \circ (B \circ A),$$

where $A^2 = A \circ A = AA$. The Jordan algebra

$$\mathbb{A}1 = \big(\mathrm{A}(3, \mathbb{C}a), \circ\big)$$

is called *Albert's algebra* (see Postnikov [167], Lecture 15).

For $A \in \mathbb{A}1$, we denote by l_A the linear map on $\mathbb{A}1$ acting according to the rule

$$l_A B = A \circ B, \quad B \in \mathbb{A}1.$$

Put

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If K_m is the kernel of l_{E_m} and

$$\mathcal{K}_m = \big\{A = \{a_{ij}\}_{i,j=1}^3 \in K_m \colon a_{jj} = 0,\ 1 \leqslant j \leqslant 3\big\}, \quad m = 1, 2, 3,$$

we have

$$\mathbb{A}1 = \mathbb{R}E_1 \oplus \mathbb{R}E_2 \oplus \mathbb{R}E_3 \oplus \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_3$$

(see Johnson [130]).

Let $\mathfrak{D}$ be the set of derivations of $\mathbb{A}1$. We set

$$\begin{aligned} \mathfrak{D}_0 &= \{D \in \mathfrak{D} \colon DE_m = 0, m = 1, 2, 3\}, \\ \mathfrak{D}_1 &= \big\{[l_{E_2 - E_3}, l_A] \colon A \in \mathcal{K}_1\big\}, \\ \mathfrak{D}_2 &= \big\{[l_{E_1 - E_3}, l_A] \colon A \in \mathcal{K}_2\big\}, \\ \mathfrak{D}_3 &= \big\{[l_{E_1 - E_2}, l_A] \colon A \in \mathcal{K}_3\big\}, \end{aligned}$$

where $[\cdot, \cdot]$ stands for the bracket operation. Then

$$\mathfrak{D} = \mathfrak{D}_0 + \mathfrak{D}_1 + \mathfrak{D}_2 + \mathfrak{D}_3, \tag{1.10}$$

and $\mathfrak{D}$ is the Lie algebra of the compact group

$$F_4 = \mathrm{Aut}\ \mathbb{A}1. \tag{1.11}$$

In addition, $\mathfrak{D}_0 + \mathfrak{D}_1 + \mathrm{i}\big(\mathfrak{D}_2 + \mathfrak{D}_3\big)$ is the Lie algebra of the noncompact connected group F_4^* (see Jonson [130]). The maximal compact subgroup of F_4^* with Lie algebra $\mathfrak{D}_0 + \mathfrak{D}_1$ is isomorphic to the *spinor group* $\mathrm{Spin}(9)$.

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{Q}$. Define

$$\mathbb{K}^n = \big\{a = (a_1, \ldots, a_n) \colon a_k \in \mathbb{K},\ 1 \leqslant k \leqslant n\big\}, \quad n \in \mathbb{N}.$$

We shall regard $\mathbb{K}^n$ as a left $\mathbb{K}$-module. This refers to the fact that the algebra of quaternions is noncommutative. The product space $\mathbb{K}^n$ is endowed with its Hermitian product

$$\langle a, b\rangle_{\mathbb{K}} = \sum_{k=1}^{n} a_k \overline{b}_k, \quad b = (b_1, \dots, b_n) \in \mathbb{K}^n,$$

and its Euclidean norm

$$|a| = \sqrt{\sum_{k=1}^{n} |a_k|^2}.$$

For the Cayley algebra, we consider the vector space

$$\mathbb{C}a^2 = \{a = (a_1, a_2) \colon a_k \in \mathbb{C}a,\ k = 1, 2\}.$$

If $b = (b_1, b_2) \in \mathbb{C}a^2$, put

$$\Phi_{\mathbb{C}a}(a, b) = |a_1|^2|b_1|^2 + |a_2|^2|b_2|^2 + 2\,\mathrm{Re}\,\big((a_1 a_2)\big(\overline{b_1 b_2}\,\big)\big).$$

Using (1.2)–(1.8), it is easy to verify that

$$\Phi_{\mathbb{C}a}(a, b) = \big|(\overline{a}_1 b_2)\big(b_2^{-1} b_1\big) + a_2 \overline{b}_2\big|^2 \tag{1.12}$$

for $b_2 \neq 0$. The form $\Phi_{\mathbb{C}a}(a, b)$ is an analogue of the form $\big|\langle a, b\rangle_{\mathbb{K}}\big|^2$.

Identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ according to the rule

$$a = (a_1, \dots, a_n) \to x = (x_1, \dots, x_{2n}), \tag{1.13}$$

where $a_k = x_k + \mathrm{i}x_{n+k}$. Putting $b_k = y_k + \mathrm{i}y_{n+k}$, $y = (y_1, \dots, y_{2n})$, where $y_k, y_{n+k} \in \mathbb{R}$, we obtain

$$\langle a, b\rangle_{\mathbb{C}} = \langle x, y\rangle_{\mathbb{C}} = \langle x, y\rangle_{\mathbb{R}} - \mathrm{i}[x, y]_{\mathbb{R}}$$

with

$$[x, y]_{\mathbb{R}} = \sum_{k=1}^{n} (x_k y_{n+k} - x_{n+k} y_k).$$

By analogy with (1.13) we may interpret elements of $\mathbb{Q}^n$ as points in $\mathbb{C}^{2n}$ using the correspondence

$$a = (a_1, \dots, a_n) \to z = (z_1, \dots, z_{2n}), \tag{1.14}$$

where $a_k = z_k + z_{n+k}\mathbf{i}_2$. Set $b_k = w_k + w_{n+k}\mathbf{i}_2$, $w = (w_1, \dots, w_{2n})$, where $w_k, w_{n+k} \in \mathbb{C}$. Then

$$\langle a, b\rangle_{\mathbb{Q}} = \langle z, w\rangle_{\mathbb{Q}} = \langle z, w\rangle_{\mathbb{C}} - [z, w]_{\mathbb{C}}\mathbf{i}_2$$

with

$$[z, w]_{\mathbb{C}} = \sum_{k=1}^{n} (z_k w_{n+k} - z_{n+k} w_k).$$

Finally, we identify $\mathbb{C}a^2$ with $\mathbb{R}^{16}$ by the map

$$a = (a_1, a_2) \to x = (x_1, \ldots, x_{16}), \tag{1.15}$$

where

$$a_1 = x_1 + x_9\mathbf{i}_1 + x_5\mathbf{i}_2 + x_{13}\mathbf{i}_3 + x_3\mathbf{i}_4 + x_{11}\mathbf{i}_5 + x_7\mathbf{i}_6 + x_{15}\mathbf{i}_7,$$
$$a_2 = x_2 + x_{10}\mathbf{i}_1 + x_6\mathbf{i}_2 + x_{14}\mathbf{i}_3 + x_4\mathbf{i}_4 + x_{12}\mathbf{i}_5 + x_8\mathbf{i}_6 + x_{16}\mathbf{i}_7.$$

Setting $y = (y_1, \ldots, y_{16})$, $y_k \in \mathbb{R}$,

$$b_1 = y_1 + y_9\mathbf{i}_1 + y_5\mathbf{i}_2 + y_{13}\mathbf{i}_3 + y_3\mathbf{i}_4 + y_{11}\mathbf{i}_5 + y_7\mathbf{i}_6 + y_{15}\mathbf{i}_7,$$
$$b_2 = y_2 + y_{10}\mathbf{i}_1 + y_6\mathbf{i}_2 + y_{14}\mathbf{i}_3 + y_4\mathbf{i}_4 + y_{12}\mathbf{i}_5 + y_8\mathbf{i}_6 + y_{16}\mathbf{i}_7,$$

we find that

$$\Phi_{\mathbb{C}a}(a, b) = \Phi_{\mathbb{C}a}(x, y) = 2\sum_{k=1}^{8} p_k(x)p_k(y) + p_9(x)p_9(y) + p_{10}(x)p_{10}(y) \tag{1.16}$$

with

$$p_1(x) = x_1x_2 - x_3x_4 - x_5x_6 - x_7x_8 - x_9x_{10} - x_{11}x_{12} - x_{13}x_{14} - x_{15}x_{16},$$
$$p_2(x) = x_1x_4 - x_9x_{12} - x_5x_8 - x_{13}x_{16} + x_3x_2 + x_{11}x_{10} + x_7x_6 + x_{15}x_{14},$$
$$p_3(x) = x_1x_6 - x_9x_{14} + x_5x_2 + x_{13}x_{10} + x_3x_8 + x_{11}x_{16} - x_7x_4 - x_{15}x_{12},$$
$$p_4(x) = x_1x_8 + x_9x_{16} + x_5x_4 - x_{13}x_{12} - x_3x_6 + x_{11}x_{14} + x_7x_2 - x_{15}x_{10},$$
$$p_5(x) = x_1x_{10} + x_9x_2 + x_5x_{14} - x_{13}x_6 + x_3x_{12} - x_{11}x_4 - x_7x_{16} + x_{15}x_8,$$
$$p_6(x) = x_1x_{12} + x_9x_4 - x_5x_{16} + x_{13}x_8 - x_3x_{10} + x_{11}x_2 - x_7x_{14} + x_{15}x_6,$$
$$p_7(x) = x_1x_{14} + x_9x_6 - x_5x_{10} + x_{13}x_2 + x_3x_{16} - x_{11}x_8 + x_7x_{12} - x_{15}x_4,$$
$$p_8(x) = x_1x_{16} - x_9x_8 + x_5x_{12} + x_{13}x_4 - x_3x_{14} - x_{11}x_6 + x_7x_{10} + x_{15}x_2,$$

$$p_9(x) = \sum_{k=1}^{8} x_{2k-1}^2, \qquad p_{10}(x) = \sum_{k=1}^{8} x_{2k}^2.$$

The polynomials $p_1, \ldots, p_{10}$ satisfy the equalities

$$\sum_{k=1}^{8} x_{2k}p_k(x) = x_1p_{10}(x), \qquad \sum_{k=1}^{8} x_{2k-1}p_k(x) = x_2p_9(x). \tag{1.17}$$

In the future, unless otherwise stated, we shall use identifications (1.13)–(1.15).

In the constructions below, an important role is played by the following groups:

$\mathrm{GL}(n;\mathbb{K})$—the group of nondegenerate $n \times n$ matrices with entries in $\mathbb{K}$;

$\mathrm{SL}(n;\mathbb{C})\big(\mathrm{SL}(n;\mathbb{R})\big)$—the group of complex (real) $n \times n$ matrices of determinant 1;

$\mathrm{U}(n,1;\mathbb{K})$—the group of matrices in $\mathrm{GL}(n+1;\mathbb{K})$ which leave invariant the Hermitian form

$$a_1\overline{b}_1 + \cdots + a_n\overline{b}_n - a_{n+1}\overline{b}_{n+1},\ a_k, b_k \in \mathbb{K};$$

$$\mathrm{O}(n,1) = \mathrm{U}(n,1;\mathbb{R}), \quad \mathrm{U}(n,1) = \mathrm{U}(n,1;\mathbb{C}), \quad \mathrm{Sp}(n,1) = \mathrm{U}(n,1;\mathbb{Q});$$

$$\mathrm{SO}(n,1) = \mathrm{O}(n,1) \cap \mathrm{SL}(n+1;\mathbb{R}), \quad \mathrm{SU}(n,1) = \mathrm{U}(n,1) \cap \mathrm{SL}(n+1;\mathbb{C});$$

$\mathrm{SO}_0(n,1)$—the identity component of $\mathrm{SO}(n,1)$;

$\mathrm{U}(n;\mathbb{K})$—the group of matrices in $\mathrm{GL}(n;\mathbb{K})$ which stabilize the form $\langle a, b\rangle_{\mathbb{K}}$;

$\mathrm{O}(n) = \mathrm{U}(n;\mathbb{R})$—the orthogonal group;

$\mathrm{U}(n) = \mathrm{U}(n;\mathbb{C})$—the unitary group;

$\mathrm{Sp}(n) = \mathrm{U}(n;\mathbb{Q})$—the symplectic group;

$\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}(n;\mathbb{R})$—the rotation group;

$$\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n;\mathbb{C});$$

$$\mathrm{U}(n) \times \mathrm{U}(1) = \left\{ \begin{pmatrix} A & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\theta} \end{pmatrix} : A \in \mathrm{U}(n), \theta \in \mathbb{R} \right\};$$

$$\mathrm{S}\big(\mathrm{U}(n) \times \mathrm{U}(1)\big) = (\mathrm{U}(n) \times \mathrm{U}(1)) \cap \mathrm{SL}(n+1;\mathbb{C});$$

$$\mathrm{Sp}(n) \times \mathrm{Sp}(1) = \left\{ \begin{pmatrix} A & 0 \\ 0 & q \end{pmatrix} : A \in \mathrm{Sp}(n), q \in \mathbb{Q}, |q| = 1 \right\};$$

$\mathrm{O}_{\mathbb{K}}(n)\big(\mathrm{O}_{\mathbb{C}a}(2)\big)$—the group of $\mathbb{R}$-linear transformations of $\mathbb{K}^n$ ($\mathbb{R}^{16}$, respectively) which preserve the form $|\langle a, b\rangle_{\mathbb{K}}|^2$ ($\Phi_{\mathbb{C}a}(x, y)$, respectively).

The mapping

$$A + \mathrm{i}B = \begin{pmatrix} A & -B \\ B & A \end{pmatrix},$$

where $A + \mathrm{i}B \in \mathrm{GL}(n;\mathbb{C})$, A and B real, is the imbedding of $\mathrm{GL}(n;\mathbb{C})$ into $\mathrm{GL}(2n;\mathbb{R})$. Likewise, the imbedding of $\mathrm{GL}(n;\mathbb{Q})$ into $\mathrm{GL}(2n;\mathbb{C})$ is given by

$$A + B\mathbf{i}_2 \to \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \quad \text{for } A + B\mathbf{i}_2 \in \mathrm{GL}(n;\mathbb{Q}), \tag{1.18}$$

where both A and B are complex $n \times n$ matrices.

Next, obviously,

$$\mathrm{O}_{\mathbb{R}}(n) = \mathrm{O}(n), \quad \mathrm{U}(n) \subset \mathrm{O}_{\mathbb{C}}(n), \quad \mathrm{Sp}(n) \subset \mathrm{O}_{\mathbb{Q}}(n). \tag{1.19}$$

To illustrate the definition of $\mathrm{O}_{\mathbb{C}a}(2)$, we consider some examples.

Example 1.1. For $a = (a_1, a_2) \in \mathbb{C}a^2$, we set

$$\tau(a) = (\overline{a}_2, \overline{a}_1),$$
$$\tau_s(a) = (\mathbf{i}_s a_1, a_2 \mathbf{i}_s), \quad 1 \leqslant s \leqslant 7.$$

By means of (1.2), (1.3), (1.7), and (1.8) it is not difficult to see that $\tau, \tau_s \in \mathrm{O}_{\mathbb{C}a}(2)$.

Example 1.2. Let $\varphi \in G_2$. Put

$$A_\varphi(a) = \big(\varphi(a_1), \varphi(a_2)\big), \quad a = (a_1, a_2) \in \mathbb{C}a^2.$$

Then $A_\varphi \in \mathrm{O}_{\mathbb{C}a}(2)$.

Proof. For $\alpha \in \mathbb{C}a$, we have

$$|\varphi(\alpha)| = |\alpha|, \qquad \mathrm{Re}\,\varphi(\alpha) = \mathrm{Re}\,\alpha, \qquad \overline{\varphi(\alpha)} = \varphi(\overline{\alpha}) \tag{1.20}$$

(see Postnikov [167], Lecture 14). By use of (1.20) one obtains

$$\begin{aligned}\varPhi_{\mathbb{C}a}\big(A_\varphi(a), A_\varphi(b)\big) &= |\varphi(a_1)|^2|\varphi(b_1)|^2 + |\varphi(a_2)|^2|\varphi(b_2)|^2 \\ &\quad + 2\,\mathrm{Re}\,\big((\varphi(a_1)\varphi(a_2))\big(\overline{\varphi(b_1)\varphi(b_2)}\,\big)\big) \\ &= |a_1|^2|b_1|^2 + |a_2|^2|b_2|^2 + 2\,\mathrm{Re}\,\big(\varphi(a_1a_2)\varphi\big(\,\overline{b_1b_2}\,\big)\big) \\ &= |a_1|^2|b_1|^2 + |a_2|^2|b_2|^2 + 2\,\mathrm{Re}\,\varphi\big((a_1a_2)\big(\,\overline{b_1b_2}\,\big)\big) \\ &= \varPhi_{\mathbb{C}a}(a, b),\end{aligned}$$

whence $A_\varphi \in \mathrm{O}_{\mathbb{C}a}(2)$. □

Example 1.3. Let $u = (t, \alpha)$, where $t \in \mathbb{R}$, $\alpha \in \mathbb{C}a$, and $t^2 + |\alpha|^2 = 1$. Define the mapping

$$R_u(a) = (-ta_1 + \overline{\alpha}\,\overline{a}_2, ta_2 + \overline{a}_1\overline{\alpha}), \quad a = (a_1, a_2) \in \mathbb{C}a^2 \tag{1.21}$$

(see Postnikov [167], Lecture 15). Then $R_u \in \mathrm{O}_{\mathbb{C}a}(2)$ and $R_u^{-1} = R_u$.

Proof. Let $b = (b_1, b_2) \in \mathbb{C}a^2$. In view of (1.2), (1.6), and (1.8),

$$\begin{aligned}&\mathrm{Re}\,\big((a_1a_2)\big(\alpha(b_1b_2)\alpha\big)\big) + |\alpha|^2\mathrm{Re}\,\big(b_2(\overline{a}_2\overline{a}_1)b_1\big) \\ &\quad = \mathrm{Re}\,\big((a_1a_2)\big(\alpha(b_1b_2)\alpha + |\alpha|^2(\overline{b}_2\overline{b}_1)\big)\big) \\ &\quad = \mathrm{Re}\,\big(\big((a_1a_2)\alpha\big)\big((b_1b_2)\alpha + \overline{\alpha}(\overline{b}_2\overline{b}_1)\big)\big) \\ &\quad = 2\mathrm{Re}\,(a_2\alpha a_1)\,\mathrm{Re}\,(b_2\alpha b_1).\end{aligned} \tag{1.22}$$

Taking (1.22) into account and using (1.2)–(1.8), we obtain that $R_u \in \mathrm{O}_{\mathbb{C}a}(2)$ by a direct calculation. Relation $R_u = R_u^{-1}$ is an easy consequence of (1.21). □

The group $\mathrm{O}_{\mathbb{K}}(n)$ acts on the unit sphere

$$\mathbb{S}^{dn-1} = \big\{a \in \mathbb{K}^n : |a| = 1\big\},$$

where $d = \dim_{\mathbb{R}} \mathbb{K}$. The action is *transitive*, i.e., for any $a, b \in \mathbb{S}^{dn-1}$, there exists $\tau \in \mathrm{O}_{\mathbb{K}}(n)$ such that $\tau(a) = b$ (see (1.19)). The following is the analogue of this result for the group $\mathrm{O}_{\mathbb{C}a}(2)$.

Proposition 1.1. *The group* $\mathrm{O}_{\mathbb{C}a}(2)$ *acts transitively on the sphere* $\mathbb{S}^{15}$.

Proof. Let $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^{16}$. Take $x \in \mathbb{S}^{15}$ arbitrarily. We write x in the form $x = (\alpha, \beta)$, where $\alpha, \beta \in \mathbb{C}a$. Put

$$\tau_x = R_{u_x} \circ R_{v_x}, \tag{1.23}$$

where

$$u_x = \big(0, -|\beta|\beta^{-1}\big), \quad v_x = \big(|\beta|, -|\beta|\beta^{-1}\alpha\big) \quad \text{if } \beta \neq 0,$$

and

$$u_x = (0, \overline{\alpha}), \quad v_x = (0, 1) \quad \text{if } \beta = 0.$$

It is easy to see that $\tau_x e_1 = x$. Since $\tau_x \in \mathrm{O}_{\mathbb{C}a}(2)$ (see Example 1.3), we obtain the desired assertion. □

Corollary 1.1. *For all* $x, y \in \mathbb{R}^{16}$, *one has*

$$\Phi_{\mathbb{C}a}(x, y) \leqslant |x|^2|y|^2. \tag{1.24}$$

Proof. Pick $\tau \in \mathrm{O}_{\mathbb{C}a}(2)$ such that $\tau(|x|e_1) = x$. Then

$$\Phi_{\mathbb{C}a}(x, y) = \Phi_{\mathbb{C}a}\big(|x|e_1, \tau^{-1}y\big).$$

Using the definition of $\Phi_{\mathbb{C}a}$, we get

$$\Phi_{\mathbb{C}a}(x, y) \leqslant |x|^2|\tau^{-1}y|^2 = |x|^2|y|^2,$$

as required. □

Let $\mathbb{C}[x]$, $x = (x_1, \dots, x_m) \in \mathbb{R}^m$, be the polynomial ring in variables $x_1, \dots, x_m$ over the field $\mathbb{C}$. Our further purpose is to describe elements $P \in \mathbb{C}[x]$ satisfying the invariance condition

$$P \circ \tau = P, \quad \tau \in K, \tag{1.25}$$

for some groups $K \subset \mathrm{O}(m)$, $m \geqslant 2$.

Proposition 1.2. *Assume that for a polynomial* $P \in \mathbb{C}[x]$, *condition* (1.25) *is fulfilled, where* K *is a subgroup of* $\mathrm{O}(m)$ *acting transitively on the sphere* $\mathbb{S}^{m-1}$. *Then* P *has the form*

$$P(x) = \sum_{k=0}^{N} c_k |x|^{2k}, \quad N \in \mathbb{Z}_+, \ c_k \in \mathbb{C}.$$

Proof. We can always write

$$P(x) = \sum_{k=0}^{j} P_k(x), \quad P_k \in \mathcal{P}_k[x],$$

where $\mathcal{P}_k[x]$ is the set of all homogeneous polynomials of degree k in $\mathbb{C}[x]$. In this case,

$$P_k \circ \tau = P_k, \quad \tau \in K, \; k = 0, \dots, j.$$

Then $|x|^{-k} P_k(x)$ is a homogeneous function of degree 0 that is invariant under the group K. In view of the transitivity of K on $\mathbb{S}^{m-1}$, this gives $P_k(x) = d_k|x|^k$, $d_k \in \mathbb{C}$, $x \in \mathbb{R}^m$. Since P_k is a polynomial, k must be even when $d_k \neq 0$. Thus, the statement is proved. □

We now consider the case $m = 2n$, $n \geqslant 2$. A variable point in $\mathbb{R}^{2n}$ will be denoted by (x, y), where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$.

Theorem 1.3. *Let $\mathcal{K}$ be a subgroup of* $\mathrm{O}(n)$*, $n \geqslant 2$. Suppose that $\mathcal{K}$ is pairwise transitive on $\mathbb{S}^{n-1}$, i.e., for any $x, y, x', y' \in \mathbb{S}^{n-1}$ with $\langle x, y\rangle_{\mathbb{R}} = \langle x', y'\rangle_{\mathbb{R}}$, there exists $\tau \in \mathcal{K}$ such that $\tau(x) = x'$, $\tau(y) = y'$. Then every polynomial $P \in \mathcal{P}_k[x, y]$ satisfying the relation*

$$P(\tau x, \tau y) = P(x, y), \quad \tau \in \mathcal{K}, \; x, y \in \mathbb{R}^n, \tag{1.26}$$

has the form

$$P(x, y) = \begin{cases} \sum_{\mu=0}^{k/2} \sum_{\nu=0}^{k-2\mu} c_{\mu,\nu} |x|^{k-2\mu-\nu} |y|^{\nu} \langle x, y\rangle_{\mathbb{R}}^{\mu} & \text{if } k \text{ is even}, \\ 0 & \text{if } k \text{ is odd}, \end{cases} \tag{1.27}$$

where $c_{\mu,\nu}$ are complex constants.

To prove Theorem 1.3 we need the following result.

Lemma 1.1. *Let $P \in \mathbb{C}[x, y]$ and $P = 0$ on the set*

$$E = \left\{(x, y) \in \mathbb{R}^{2n} \colon \langle x, y\rangle_{\mathbb{R}} = 0\right\}. \tag{1.28}$$

Then

$$P(x, y) = \langle x, y\rangle_{\mathbb{R}} Q(x, y)$$

for some $Q \in \mathbb{C}[x, y]$.

Proof. Fix $y \in \mathbb{R}^n \backslash \{0\}$. Put $\mathrm{R}_y(x) = P(\tau^{-1}x, y)$, where τ is a rotation of $\mathbb{R}^n$ for which $\tau y = (|y|, 0, \dots, 0)$. Because $P = 0$ on E, we have $\mathrm{R}_y(0, x_2, \dots, x_n) = 0$, $(x_2, \dots, x_n) \in \mathbb{R}^{n-1}$. Hence, $\mathrm{R}_y(x) = x_1 r_y(x)$ with $r_y \in \mathbb{C}[x]$. Consequently,

$$P(x, y) = \langle x, y\rangle_{\mathbb{R}} \, p_y(x), \tag{1.29}$$

where $p_y \in \mathbb{C}[x]$ for each $y \in \mathbb{R}^n$. We rewrite (1.29) as

$$\frac{P(x,y)}{\langle x,y\rangle_{\mathbb{R}}} = \sum_{|\alpha|\leqslant N} c_\alpha(y)x^\alpha, \tag{1.30}$$

where α denotes an n-tuple $(\alpha_1,\dots,\alpha_n)$ of nonnegative integers, $|\alpha| = \alpha_1+\dots+\alpha_n$, and $x^\alpha = x_1^{\alpha_1}\dots x_n^{\alpha_n}$. Applying to (1.30) the operator

$$\partial^\beta = \left(\frac{\partial}{\partial x}\right)^\beta = \left(\frac{\partial}{\partial x_1}\right)^{\beta_1}\dots\left(\frac{\partial}{\partial x_n}\right)^{\beta_n}, \quad \beta = (\beta_1,\dots,\beta_n),$$

with $|\beta| = N, N-1,\dots,0$, we obtain

$$c_\alpha(y) = |y|^{-2^{N+1}} d_\alpha(y),$$

where $d_\alpha \in \mathbb{C}[y]$. Therefore, $|y|^{2^{N+1}}P(x,y)$ belongs to the ideal $\langle x,y\rangle_{\mathbb{R}}\mathbb{C}[x,y]$. By virtue of the irreducibility of $\langle x,y\rangle_{\mathbb{R}}$, the ideal $\langle x,y\rangle_{\mathbb{R}}\mathbb{C}[x,y]$ is prime (see Zariski and Samuel [277], Chap. 3, Sect. 8). Since

$$|y|^{2^{N+1}} \notin \langle x,y\rangle_{\mathbb{R}}\mathbb{C}[x,y],$$

this gives $P(x,y) \in \langle x,y\rangle_{\mathbb{R}}\mathbb{C}[x,y]$. Thereby the lemma is established. □

Proof of Theorem 1.3. The polynomial P can be represented in the form

$$P(x,y) = \sum_{m=0}^{k} P_m(x,y)$$

with

$$P_m(x,y) = \sum_{|\alpha|=m} c_\alpha(x)y^\alpha, \quad c_\alpha \in \mathcal{P}_{k-m}[x].$$

Then, for each $t>0$ and each $\tau\in\mathcal{K}$,

$$\sum_{m=0}^{k} t^m P_m(x,y) = P(x,ty) = P(\tau x, t\tau y) = \sum_{m=0}^{k} t^m P_m(\tau x,\tau y).$$

Consequently, we must have

$$P_m(\tau x,\tau y) = P_m(x,y), \quad \tau\in\mathcal{K},\ x,y\in\mathbb{R}^n. \tag{1.31}$$

Let $(x,y)\in E$, $x,y\in\mathbb{R}^n\backslash\{0\}$ (see (1.28)). Since $\mathcal{K}$ is pairwise transitive on $\mathbb{S}^{n-1}$, (1.31) implies that

$$\frac{P_m(x,y)}{|x|^{k-m}|y|^m} = \sum_{|\alpha|=m} c_\alpha\left(\frac{x}{|x|}\right)\left(\frac{y}{|y|}\right)^\alpha = P_m\left(\frac{x}{|x|},\frac{y}{|y|}\right) = P_m(e_1,e_2),$$

where $e_1 = (1, 0, \dots, 0) \in \mathbb{S}^{n-1}$, $e_2 = (0, 1, 0, \dots, 0) \in \mathbb{S}^{n-1}$. Hence,

$$P(x, y) = \sum_{m=0}^{k} P_m(e_1, e_2)|x|^{k-m}|y|^m \quad \text{on } E. \tag{1.32}$$

Assume that k is even. Using (1.32), we obtain $P_m(e_1, e_2) = 0$ for odd $m \in \{0, \dots, k\}$. Analogously, $P_m(e_1, e_2) = 0$ for all $m \in \{0, \dots, k\}$ when k is odd. Now, by Lemma 1.1,

$$P(x, y) - \sum_{m=0}^{k} P_m(e_1, e_2)|x|^{k-m}|y|^m = \langle x, y\rangle_{\mathbb{R}} Q(x, y), \quad (x, y) \in \mathbb{R}^{2n}, \tag{1.33}$$

where $Q = 0$ if $k \leqslant 1$ and $Q \in \mathcal{P}_{k-2}[x, y]$ if $k \geqslant 2$. Moreover, owing to (1.26) and (1.33),

$$Q(\tau x, \tau y) = Q(x, y), \quad \tau \in \mathcal{K}, \ x, y \in \mathbb{R}^n.$$

Invoking induction on k, we arrive at the desired assertion. □

To close this section we present a consequence of Theorem 1.3.

In view of (1.20), we may assume that $G_2 \subset \mathrm{O}(7)$. In this case, G_2 is pairwise transitive on the sphere $\mathbb{S}^6$ (see Busemann [44], Chap. 6, Sect. 55). Applying Theorem 1.3 with $n = 7$ and $\mathcal{K} = G_2$, we obtain

Corollary 1.2. *Let P be a homogeneous polynomial of degree k in variables $x_1, \dots, x_7, y_1, \dots, y_7$. Suppose that*

$$P(\tau x, \tau y) = P(x, y) \quad \text{for all } \tau \in G_2, x, y \in \mathbb{R}^7.$$

Then P has the form (1.27).

1.2 Elements of Differential Geometry

We collect here some notation and facts from the theory of manifolds we need in the sequel. For proofs, we refer to Helgason [115, 122] and also Kobayashi and Nomizu [137].

Let M be a manifold. In the future, M will always be assumed to be real analytic. We shall use the following spaces on M:

$C^k(M)$ ($k \in \mathbb{Z}_+$ or $k = \infty$)—the space of all complex-valued C^k-functions on M;

$C(M) = C^0(M)$, $\mathcal{E}(M) = C^\infty(M)$;

RA(M)—the space of real analytic functions;

$C_c^k(M)$—the subspace of $C^k(M)$ consisting of compactly supported functions;

$\mathcal{D}(M) = C_c^\infty(M)$;

$\mathcal{D}'(M)$—the space of distributions;

$\mathcal{E}'(M)$—the space of compactly supported distributions.

Let supp T be the support of a distribution $T \in \mathcal{D}'(M)$. A linear mapping D of the space $\mathcal{D}(M)$ into itself is said to be a *differential operator* on M if

$$\operatorname{supp}(Df) \subset \operatorname{supp} f, \quad f \in \mathcal{D}(M). \tag{1.34}$$

For the definition of the order of a differential operator, see Hörmander [124], Sect. 1.8. The set of all differential operators on M is denoted by $\mathbf{E}(M)$.

Relation (1.34) implies that D can be extended to a linear operator (also denoted D) from $\mathcal{E}(M)$ to $\mathcal{E}(M)$ by means of the formula

$$(Dg)(x) = (Df)(x).$$

Here $x \in M$, $g \in \mathcal{E}(M)$ are arbitrary, and f is any function in $\mathcal{D}(M)$ which coincides with g in a neighborhood of x. The choice of f is clearly immaterial.

Suppose that ψ is a diffeomorphism of M onto itself. For $g \in \mathcal{E}(M)$, we put

$$D^{\psi} g = \big(D(g \circ \psi)\big) \circ \psi^{-1}.$$

In view of (1.34), $D^{\psi} \in \mathbf{E}(M)$. We shall say that D is *invariant under* ψ (or D *commutes* with ψ) if $D^{\psi} = D$, that is, if

$$D(g \circ \psi) = (Dg) \circ \psi \quad \text{for all } g \in \mathcal{E}(M).$$

Let μ be a measure on M which on each coordinate neighborhood is the multiple of the Lebesgue measure by a nowhere vanishing C^{∞}-function. Denote by $L^p(M)$ and $L^{p,\mathrm{loc}}(M)$, $1 \leqslant p < \infty$, the classes of complex-valued functions on M that are p-integrable and p-locally integrable with respect to μ, respectively. The space $L^{1,\mathrm{loc}}(M)$ is embedded in $\mathcal{D}'(M)$ by means of μ if to $g \in L^{1,\mathrm{loc}}(M)$ we associate the distribution

$$f \to \int_M fg \, \mathrm{d}\mu, \quad f \in \mathcal{D}(M),$$

on M.

For every $D \in \mathbf{E}(\mathrm{M})$, there exists a unique differential operator $D^* \in \mathbf{E}(\mathrm{M})$ satisfying the relation

$$\int_M \big(D^* f\big)(x) g(x) \, \mathrm{d}\mu(x) = \int_M f(x)(Dg)(x) \, \mathrm{d}\mu(x), \quad f, g \in \mathcal{D}(M).$$

The operator D^* is called the *adjoint* of D with respect to μ. The action of an operator $D \in \mathbf{E}(M)$ on distributions is given by

$$\langle DT, f \rangle = \big\langle T, D^* f \big\rangle, \quad f \in \mathcal{D}(M), T \in \mathcal{D}'(M).$$

The inequality $\operatorname{ord}(DT) \leqslant \operatorname{ord} T + n$ holds, where n is the order of D, and the symbol ord is used for the order of distributions.

For $m \in \mathbb{Z}_+$, we denote by $L^p_m(M)$ the set of all $f \in L^p(M)$ such that $Df \in L^p(M)$ for every $D \in \mathbf{E}(M)$ of order at most m. The class $L^{p,\mathrm{loc}}_m(M)$ is defined analogously.

A connected manifold M becomes a Riemannian space (or a Riemannian manifold) when provided with a metric tensor $\{g_{ij}\}$. Let g be the determinant of the matrix $\{g_{ij}\}$ and $\{g^{ij}\}$ the inverse matrix of $\{g_{ij}\}$. We need the following objects on M concerning $\{g_{ij}\}$:

$$\mathrm{d}s = \Big(\sum_{i,j} g_{ij}\,\mathrm{d}x_i\,\mathrm{d}x_j\Big)^{1/2}\text{—the length element;} \tag{1.35}$$

$$\mathrm{d}\mu = \sqrt{g}\,\mathrm{d}x_1\ldots\mathrm{d}x_n\text{—the Riemannian measure;} \tag{1.36}$$

$$\Gamma^k_{ij} = \frac{1}{2}\sum_l g^{kl}\left(\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l}\right)\text{—the Riemannian connection;} \tag{1.37}$$

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ki}}{\partial x_j} - \frac{\partial \Gamma^l_{ji}}{\partial x_k} + \sum_m \big(\Gamma^m_{ki}\Gamma^l_{mj} - \Gamma^m_{ji}\Gamma^l_{mk}\big)\text{—the curvature tensor of the Riemannian connection;} \tag{1.38}$$

$$R_{ijkl} = \sum_m g_{im}R^m_{jkl}\text{—the Riemannian curvature tensor;} \tag{1.39}$$

$d(\cdot,\cdot)$—the inner metric; $I(M)$—the isometry group;

$$\nabla_1 = \sum_{i,j} g^{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\text{—the first Beltrami parameter;}$$

$$\Delta_2 = \frac{1}{\sqrt{g}}\sum_{i,j}\frac{\partial}{\partial x_i}\left(\sqrt{g}g^{ij}\frac{\partial}{\partial x_j}\right)\text{—the second Beltrami parameter.} \tag{1.40}$$

The analog of straight lines on M are geodesics. Equations of geodesics have the form

$$\frac{\mathrm{d}^2x_i}{\mathrm{d}t^2} + \sum_{j,k}\Gamma^i_{jk}\frac{\mathrm{d}x_j}{\mathrm{d}t}\frac{\mathrm{d}x_k}{\mathrm{d}t} = 0. \tag{1.41}$$

A geodesic arc is, at least locally, the shortest arc between its end points. This is proved in calculus of variations.

A Riemannian manifold M is a space of *constant sectional curvature* c if and only if

$$R_{ijkl} = c(g_{ik}g_{jl} - g_{jk}g_{li}) \tag{1.42}$$

(see [137], Vol. 1, Chap. 5, Sect. 2).

The distance $d(p,q)$ between two points $p,q\in M$ is given by

$$d(p,q) = \inf_\gamma\int_\gamma \mathrm{d}s,$$

where γ runs over all curve segments joining p and q. The pair (M,d) is a metric space. A mapping f of M onto itself is in $I(M)$ if and only if f preserves distances, i.e.,

$$d\big(f(p),f(q)\big) = d(p,q)\quad\text{for } p,q\in M.$$

We shall always consider $I(M)$ with the *compact open topology*. This is defined as the smallest topology on $I(M)$ for which all the sets $\{\psi \in I(M)\colon \psi C \subset U\}$ are open. Here C and U, respectively, are a compact and an open subset of M. The identity component of $I(M)$ will be denoted $I_0(M)$.

Let $a \in \mathbb{R}$, $R, b \in [0, +\infty]$. We shall often use the following sets in M:

$$S_R(p) = \{q \in M\colon d(p,q) = R\}\text{—the sphere;}$$
$$B_R(p) = \{q \in M\colon d(p,q) < R\}\text{—the open ball;}$$
$$\mathring{B}_R(p) = B_R(p) \cup S_R(p)\text{—the closed ball;}$$
$$B_{a,b}(p) = \{q \in M\colon a < d(p,q) < b\}\text{—the open spherical annulus;}$$
$$\mathring{B}_{a,b}(p) = \{q \in M\colon a \leqslant d(p,q) \leqslant b\}\text{—the closed spherical annulus.}$$

Definition 1.1. A nonempty open set $\mathcal{O} \subset M$ is said to be a ζ *domain* ($\zeta \geqslant 0$) if the following conditions are satisfied:

(a) each point in $\mathcal{O}$ can be covered by a closed ball of radius ζ contained in $\mathcal{O}$;
(b) centers of two arbitrary closed balls with radius ζ contained in $\mathcal{O}$ can be joined by a curve such that any closed ball with the radius ζ and the center on this curve is contained in $\mathcal{O}$.

Let $\zeta \geqslant 0$ and assume that $\mathcal{U}$ is a subset of M. Denote by $\mathfrak{S}(\mathcal{U}, \zeta)$ the collection of all open subsets $\mathcal{O}$ of M with the property that $\mathcal{O} \in \mathfrak{S}(\mathcal{U}, \zeta)$ if and only if $\mathcal{O}$ is a ζ domain and $\mathcal{U} \subset \mathcal{O}$. The motivation for the definition of the class $\mathfrak{S}(\mathcal{U}, \zeta)$ is the theory of convolution equations developed in Volchkov [225], Part III.

A Riemannian manifold is said to be *complete* if each closed ball in M is compact.

We shall need the following results.

Theorem 1.4 ([137], Chap. 4.4). *Suppose that $I(M)$ acts transitively on M. Then M is a complete Riemannian manifold.*

Theorem 1.5 ([137], Chap. 4.4). *In a complete Riemannian manifold M with metric d, each pair $p, q \in M$ can be joined by a geodesic of length $d(p,q)$.*

Theorem 1.6 ([137], Chap. 6.7). *Let M and N be two simply connected complete Riemannian manifolds of constant sectional curvature c. Then M is isometric to the space N.*

The operator

$$L = \Delta_2 \tag{1.43}$$

is a direct generalization of the Laplacian Δ on $\mathbb{R}^n$ and is frequently referred to as the Laplace–Beltrami operator. For $u \in \mathcal{D}(M)$ and $v \in \mathcal{E}(M)$, one has

$$\int_M u(x)(Lv)(x)\,\mathrm{d}\mu(x) = \int_M (Lu)(x)v(x)\,\mathrm{d}\mu(x), \tag{1.44}$$

i.e., L is symmetric. A diffeomorphism f of a Riemannian manifold M is an isometry if and only if L commutes with f (see [122], Chap. 2, Sect. 2.4).

Let $\mathcal{M}^0(M)$ be the set of all complex-valued compactly supported measures on M, and let $\mathcal{M}^1(M)$ be the set of all distributions $f \in \mathcal{E}'(M)$ such that $Df \in \mathcal{M}^0(M)$ for each differential operator D on M of order at most one. We define the class $\mathcal{M}^\nu(M)$ for each $\nu \in \mathbb{Z}$ as follows.

If $\nu \in \mathbb{N}$ and ν is even (respectively ν is odd), we denote by $\mathcal{M}^\nu(M)$ the set of all distributions $f \in \mathcal{E}'(M)$ such that $L^{[\nu/2]} f \in \mathcal{M}^0(M)$ (respectively $L^{[\nu/2]} f \in \mathcal{M}^1(M)$), where $[\nu/2]$ is the integer part of $\nu/2$.

If $\nu \in \mathbb{Z}$, $\nu < 0$ and ν is even (respectively ν is odd) we write $\mathcal{M}^\nu(M)$ for the set of all distributions $f \in \mathcal{E}'(M)$ such that $f = p(L)u$ for some $u \in \mathcal{M}^0(M)$ (respectively $u \in \mathcal{M}^1(M)$) and some polynomial p of degree at most $[(1-\nu)/2]$.

The class $\mathcal{M}^\nu(M)$ plays an important role in Part III below.

Assume now that M is a Riemannian manifold of dimension $\geqslant 2$. Let $\mathcal{O}$ be a nonempty subset of M, and let $\alpha > 0$. Denote by $\mathrm{QA}(\mathcal{O})$ (respectively $G^\alpha(\mathcal{O})$) the set of all functions $f \in C^\infty(\mathcal{O})$ such that for each compact set $\mathcal{K} \subset \mathcal{O}$,

$$\sum_{\nu=1}^{\infty}\left(\inf_{q\geqslant\nu}\left(\int_{\mathcal{K}} |L^q f(x)| \mathrm{d}\mu(x)\right)^{1/2q}\right)^{-1} = +\infty \tag{1.45}$$

(respectively

$$\int_{\mathcal{K}} \left|L^q f(x)\right| \mathrm{d}\mu(x) \leqslant c^q (1+q)^{2\alpha q}, \quad q = 1, 2, \ldots, \tag{1.46}$$

where the constant $c > 0$ is independent of q). Next, suppose that $\mathcal{O}$ is bounded and let $\mathrm{Cl}\,\mathcal{O}$ be the closure of $\mathcal{O}$. We shall write $f \in \mathrm{QA}(\mathrm{Cl}\,\mathcal{O})$ (respectively $f \in G^\alpha(\mathrm{Cl}\,\mathcal{O})$) if $f \in C^\infty(\mathrm{Cl}\,\mathcal{O})$ and property (1.45) (respectively (1.46)) is satisfied for $\mathcal{K} = \mathrm{Cl}\,\mathcal{O}$.

Let M be a connected complex manifold with a *Hermitian metric* $H = \{h_{ij}\}_{i,j=1}^n$, and let $\{z_1, \ldots, z_n\}$ be local coordinates on M. We write

$$H = A + \mathrm{i}B, \quad z_j = x_j + \mathrm{i}x_{n+j},$$

where $A = \mathrm{Re}\, H$, $B = \mathrm{Im}\, H$, $x_j, x_{n+j} \in \mathbb{R}$. The matrix

$$G = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

gives rise to a Riemannian structure on M. The Riemannian measure on M has the form

$$\mathrm{d}\mu = h\, \mathrm{d}x_1 \ldots \mathrm{d}x_{2n}, \tag{1.47}$$

where h is the determinant of H. The Laplace–Beltrami operator L on M is calculated by

$$L = \frac{2}{h}\sum_{j=1}^{n}\left(\frac{\partial}{\partial z_j}\left(\sum_{i=1}^{n} h^{ij} h \frac{\partial}{\partial \overline{z}_i}\right) + \frac{\partial}{\partial \overline{z}_j}\left(\sum_{i=1}^{n} \overline{h^{ij}} h \frac{\partial}{\partial z_i}\right)\right), \tag{1.48}$$

where $\{h^{ij}\}$ is the inverse matrix of $\{h_{ij}\}$,

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \mathrm{i}\frac{\partial}{\partial x_{n+j}}\right), \qquad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \mathrm{i}\frac{\partial}{\partial x_{n+j}}\right).$$

Let F be a real-valued function in a coordinate neighborhood of a connected complex manifold M. We set

$$g_{i\overline{j}} = \frac{\partial^2 F}{\partial z_i \partial \overline{z}_j}. \tag{1.49}$$

Since F is real, $\{g_{i\overline{j}}\}$ is always Hermitian. If it is positive definite, then

$$\mathrm{d}s^2 = 2\sum_{i,j} g_{i\overline{j}}\,\mathrm{d}z_i\,\mathrm{d}\overline{z}_j \tag{1.50}$$

is a *Kaehler metric* on M. Every Kaehler metric can be locally written in form (1.50) with $\{g_{i\overline{j}}\}$ given by (1.49) (see [137], Vol. 2, Chap. 9, Sect. 5).

For a Kaehler metric $\{g_{i\overline{j}}\}$, we put

$$R_{i\overline{j}k\overline{l}} = \frac{\partial^2 g_{i\overline{j}}}{\partial z_k \partial \overline{z}_l} - \sum_{\alpha,\beta} g^{\alpha\overline{\beta}} \frac{\partial g_{i\overline{\beta}}}{\partial z_k}\frac{\partial g_{\alpha\overline{j}}}{\partial \overline{z}_l}, \tag{1.51}$$

where $\{g^{i\overline{j}}\}$ is defined by the relation

$$\sum_j g^{i\overline{j}} g_{k\overline{j}} = \delta_{i,k}$$

with

$$\delta_{i,k} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k. \end{cases}$$

The metric $\{g_{i\overline{j}}\}$ has *constant holomorphic sectional curvature* c if and only if

$$R_{i\overline{j}k\overline{l}} = -\frac{c}{2}(g_{i\overline{j}}g_{k\overline{l}} + g_{i\overline{l}}g_{k\overline{j}}). \tag{1.52}$$

The proof of this statement can be found in [137], Vol. 2, Chap. 9, Sect. 7.

The following is a Kaehlerian analogue of Theorem 1.6.

Theorem 1.7 ([137], Chap. 9.7). *Let M and N be two simply connected complete Kaehlerian manifolds of constant holomorphic sectional curvature c. Then M is holomorphically isometric to N.*

1.3 Homogeneous and Symmetric Spaces

Let M be a Hausdorff space and G a topological group such that to each $g \in G$ there is associated a homeomorphism $p \to gp$ of M onto itself with the following properties:

(a) $(g_1 g_2)p = g_1(g_2 p)$ for $p \in M$, $g_1, g_2 \in G$;
(b) the mapping $(g, p) \to gp$ is a continuous mapping of the product space $G \times M$ onto M.

The group G is then called a *topological transformation group* of M. The space M is said to be *homogeneous* with respect to G if the group G acts transitively on M.

To illustrate the definition we consider an example.

Suppose that G is a topological group and H a closed subgroup of G. Denote by G/H the system of left cosets gH, $g \in G$. Let π be the *natural mapping* of G onto G/H, i.e.,

$$\pi g = gH, \quad g \in G.$$

The set G/H can be given a topology, the *natural topology*, which is uniquely determined by the condition that π is a continuous and open mapping. This makes G/H a Hausdorff space, and it is not difficult to see that if to each $g \in G$ we assign the mapping $\mathbf{g}H \to g\mathbf{g}H$, $\mathbf{g} \in G$, then G is a topological transformation group of G/H. The coset space G/H is a homogeneous space with respect to G.

The example is universal as the following result shows.

Theorem 1.8 ([115], Chap. 2.3). *Let G be a locally compact group with a countable base. Suppose that G is a transitive topological transformation group of a locally compact Hausdorff space M. Let p be any point in M and H the subgroup of G which leaves p fixed. Then H is closed, and the mapping $gH \to gp$ is a homeomorphism of G/H onto M.*

The group H is called the *isotropy group* at p (or the *isotropy subgroup* of G at p).

Let G be a Lie group and M a manifold. Suppose that G is a topological transformation group of M. The group G is said to be a *Lie transformation group* of M if the mapping $(g, p) \to gp$ is an infinitely differentiable mapping of $G \times M$ onto M. It follows that for each $g \in G$, the mapping $p \to gp$ is a diffeomorphism of M onto itself.

Theorem 1.9 ([115], Chap. 2.4). *Let G be a Lie group, H a closed subgroup of G, and G/H the space of left cosets gH with the natural topology. Then G/H has a unique analytic structure with the property that G is a Lie transformation group of G/H.*

In the sequel the coset space G/H (G a Lie group, H a closed subgroup) will always be taken with the analytic structure described in Theorem 1.9.

Among homogeneous manifolds of particular interest, there are the so-called *symmetric spaces*.

Recall that a mapping is called *involutive* if its square, but not the mapping itself, is the identity. A Riemannian manifold M is said to be *Riemannian globally symmetric* if each $p \in M$ is an isolated fixed point of an involutive isometry s_p of M.

It can be proved that there is only one such s_p (see [115], Chap. 4, Sect. 3). It is called the *symmetry* at the point p.

Theorem 1.10 ([115], Chap. 4.3). *Let M be a Riemannian globally symmetric space. Then $I(M)$ has an analytic structure compatible with the compact open topology in which it is a transitive Lie transformation group of M.*

We can look at the definition of symmetric spaces from a different angle.

Let G be a connected Lie group and H a closed subgroup. The pair (G, H) is called a *symmetric pair* if there exists an involutive analytic automorphism σ of G such that $(H_\sigma)_0 \subset H \subset H_\sigma$, where H_σ is the set of fixed points of σ, and $(H_\sigma)_0$ is the identity component of H_σ. If, in addition, the image of H under the adjoint representation Ad_G of the group G is compact, (G, H) is said to be a *Riemannian symmetric pair* (see [115], Chap. 2, Sect. 5, and Chap. 4, Sect. 3).

Theorem 1.11 ([115], Chap. 4.3). *Let (G, K) be a Riemannian symmetric pair. Then in each G-invariant Riemannian structure Q on G/K (such Q exist), the manifold G/K is a Riemannian globally symmetric space.*

Theorem 1.12 ([115], Chap. 4.3). *Let M be a Riemannian globally symmetric space, p_0 any point in M, and $T_{p_0}M$ the tangent space to M at p_0. Then the following are true.*

(i) *If $G = I_0(M)$ and if K is the subgroup of G which leaves p_0 fixed, then K is a compact subgroup of the connected group G, and G/K is analytically diffeomorphic to M under the mapping*

$$gK \to gp_0, \quad g \in G.$$

(ii) *The mapping*

$$\sigma : g \to s_{p_0} g s_{p_0}$$

is an involutive automorphism of G such that K lies between the closed group K_σ of all fixed points of σ and the identity component of K_σ. The group K contains no normal subgroup of G other than the identity $\{e\}$.

(iii) *Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively, and let $(\mathrm{d}\sigma)_e$ be the differential of σ at e. Then*

$$\mathfrak{k} = \left\{A \in \mathfrak{g} \colon (\mathrm{d}\sigma)_e A = A\right\},$$

and if $\mathfrak{p} = \{A \in \mathfrak{g} \colon (\mathrm{d}\sigma)_e A = -A\}$, we have

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} \quad (\textit{direct sum}).$$

The differential $(\mathrm{d}\pi)_e$, *where* $\pi g = gp_0$, $g \in G$, *maps* $\mathfrak{k}$ *into* $\{0\}$ *and* $\mathfrak{p}$ *isomorphically onto* $T_{p_0}M$.

If $\mathfrak{g}$ is compact and semisimple (see [115], Chap. 2, Sects. 5, 6), M is said to be of *compact type*. If $\mathfrak{g}$ is noncompact and semisimple, then M is said to be of *noncompact type*. If $\mathfrak{p}$ is an abelian ideal in $\mathfrak{g}$, then M is said to be of *Euclidean type*.

The symmetric spaces corresponding to these have positive, negative, and zero sectional curvature, respectively (see [115], Chap. 5, Sect. 3).

The *rank* of the symmetric space M is an important invariant; it is defined as the dimension of any maximal abelian subalgebra of $\mathfrak{p}$. For any two points in M, one can speak of their *complex distance*; this is an l-tuple $(r_1, \dots, r_l)$ of real numbers (l = rank of M) and has the property that two point-pairs in M are congruent under an isometry of M if and only if their complex distance is the same (see [115], Chap. 5, Sect. 6).

In Theorem 1.12 we have seen that a Riemannian globally symmetric space gives rise to a pair $(\mathfrak{l}, s)$ where

(a) $\mathfrak{l}$ is a Lie algebra over $\mathbb{R}$,
(b) s is an involutive automorphism of $\mathfrak{l}$,
(c) $\mathfrak{u}$, the set of fixed points of s, is a compactly embedded subalgebra of $\mathfrak{l}$ (see [115], Chap. 2, Sect. 5),
(d) $\mathfrak{u} \cap \mathfrak{z} = \{0\}$ if $\mathfrak{z}$ denotes the center of $\mathfrak{g}$.

A pair $(\mathfrak{l}, s)$ with the properties (a), (b), (c) is called an *orthogonal symmetric Lie algebra*. It is said to be *effective* if, in addition, (d) holds. A pair $(\mathcal{L}, U)$, where $\mathcal{L}$ is a connected Lie group with Lie algebra $\mathfrak{l}$, and U is a Lie subgroup of $\mathcal{L}$ with Lie algebra $\mathfrak{u}$, is said to be *associated* with the orthogonal symmetric Lie algebra $(\mathfrak{l}, s)$.

The orthogonal symmetric Lie algebra $(\mathfrak{l}, s)$ is called *irreducible* if the two following conditions are satisfied:

(a) $\mathfrak{l}$ is semisimple, and $\mathfrak{u}$ contains no ideal $\neq \{0\}$ of $\mathfrak{l}$;
(b) $\mathfrak{u}$ is a maximal proper subalgebra of $\mathfrak{l}$.
Let $(\mathcal{L}, U)$ be a pair associated with $(\mathfrak{l}, s)$. Then $(\mathcal{L}, U)$ is said to be *irreducible* if $(\mathfrak{l}, s)$ is irreducible.

Theorem 1.13. *Let* $(\mathcal{L}, U)$ *be an irreducible Riemannian symmetric pair. Then all* $\mathcal{L}$*-invariant Riemannian structures on* $\mathcal{L}/U$ *coincide except for a constant factor.*

For the proof, we refer the reader to [115], Chap. 8, Sect. 5.

A Riemannian globally symmetric space M is called *irreducible* if the pair $(I_0(M), K)$ is irreducible, K being the isotropy subgroup of $I_0(M)$ at some point in M.

In [47], É. Cartan accomplished a complete classification of irreducible Riemannian globally symmetric spaces in terms of the classical and exceptional simple Lie groups. The list of Cartan shows, in particular, that Riemannian globally symmetric spaces of rank one consist of

(1) *the real hyperbolic spaces*, $\mathrm{SO}_0(n,1)/\mathrm{SO}(n)$;
(2) *the complex hyperbolic spaces*, $\mathrm{SU}(n,1)/\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(1))$;
(3) *the quaternionic hyperbolic spaces*, $\mathrm{Sp}(n,1)/\mathrm{Sp}(n)\times\mathrm{Sp}(1)$;
(4) *the Cayley hyperbolic plane*, $F_4^*/\mathrm{Spin}(9)$;
(5) *the spheres*, $\mathrm{SO}(n+1)/\mathrm{SO}(n)$;
(6) *the real projective spaces*, $\mathrm{SO}(n+1)/\mathrm{O}(n)$;
(7) *the complex projective spaces*, $\mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(1))$;
(8) *the quaternionic projective spaces*, $\mathrm{Sp}(n+1)/\mathrm{Sp}(n)\times\mathrm{Sp}(1)$;
(9) *the Cayley projective plane*, $F_4/\mathrm{Spin}(9)$.

The spaces $\mathrm{SO}_0(n,1)/\mathrm{SO}(n)$, $\mathrm{SO}(n+1)/\mathrm{SO}(n)$, and $\mathrm{SO}(n+1)/\mathrm{O}(n)$ have constant sectional curvature (see, for example, Wolf [263], Chap. 2, Sect. 2.4). In addition, $\mathrm{SU}(n,1)/\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(1))$ and $\mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(1))$ are *Hermitian symmetric spaces* of constant holomorphic sectional curvature (see [137], Vol. 2, Chap. 11, Sects. 9, 10). Realizations of all Riemannian globally symmetric spaces of rank one as domains of Euclidean spaces are given in Chaps. 2 and 3 below.

We close this section by considering the *two-point homogeneous spaces*. These are defined as Riemannian manifolds with pairwise transitive groups of isometries. This means that for any two pairs points (x_1,x_2) and (y_1,y_2) satisfying

$$d(x_1,x_2)=d(y_1,y_2),$$

there exists an isometry mapping x_1 to y_1 and x_2 to y_2. The two-point homogeneous spaces have all been determined. A Riemannian manifold is two-point homogeneous if and only if it is either a Euclidean space or a Riemannian globally symmetric space of rank one. The proof of this result can be found in Wolf [263], Chap. 8, Sect. 8.12.

1.4 Convolution, Invariant Differential Operators and Spherical Functions

Let G be a separable unimodular Lie group with Haar measure $\mathrm{d}g$. The *convolution*

$$(f_1*f_2)(\mathbf{g})=\int_G f_1\big(\mathbf{g}g^{-1}\big)f_2(g)\,\mathrm{d}g=\int_G f_1(g)f_2\big(g^{-1}\mathbf{g}\big)\,\mathrm{d}g$$

is well defined for $f_1,f_2\in C(G)$ if at least one of them has compact support. More generally, we define the convolution t_1*t_2 of two distributions on G, at least one of compact support, as the distribution

$$\langle t_1*t_2,\psi\rangle=\big\langle t_2(g),\langle t_1(\mathbf{g}),\psi(\mathbf{g}g)\rangle\big\rangle,\quad \psi\in\mathcal{D}(G). \tag{1.53}$$

Then we have the associativity

$$(t_1*t_2)*t_3=t_1*(t_2*t_3)$$

if at least two of the t_i have compact support (see Helgason [122], Chap. 2, Sect. 5.1).

A locally integrable function f on G can be viewed as the distribution

$$\psi \to \int_G \psi(g) f(g)\,\mathrm{d}g, \quad \psi \in \mathcal{D}(G),$$

on G. With this identification we have for $f \in \mathcal{E}(G)$, $t \in \mathcal{D}'(G)$, at least one of compact support,

$$(f * t)(\mathbf{g}) = \langle t(g), f(\mathbf{g}g^{-1})\rangle, \quad (t * f)(\mathbf{g}) = \langle t(g), f(g^{-1}\mathbf{g})\rangle.$$

Let $K \subset G$ be a compact subgroup with Haar measure $\mathrm{d}k$ normalized by

$$\int_K \mathrm{d}k = 1. \tag{1.54}$$

Denote by $\mathrm{d}x$ the measure on G/K defined by the relation

$$\int_{G/K} f(x)\,\mathrm{d}x = \int_G (f \circ \pi)(g)\,\mathrm{d}g, \quad f \in C_c(G/K), \tag{1.55}$$

where $\pi : G \to G/K$ is the natural projection. Let $L^{1,\mathrm{loc}}(G/K)$ be the class of complex-valued functions on G/K that are locally integrable with respect to $\mathrm{d}x$. If $f \in L^{1,\mathrm{loc}}(G/K)$ is regarded as a distribution, then

$$\langle f, \psi\rangle = \int_{G/K} \psi(x) f(x)\,\mathrm{d}x, \quad \psi \in \mathcal{D}(G/K).$$

Each $T \in \mathcal{D}'(G/K)$ lifts to the distribution t_T on G given by

$$\langle t_T, \psi\rangle = \left\langle T(gK), \int_K \psi(gk)\,\mathrm{d}k\right\rangle, \quad \psi \in \mathcal{D}(G).$$

The lifting induces the notion of convolution in $\mathcal{D}'(G/K)$ by

$$\langle T_1 \times T_2, \psi\rangle = \langle t_{T_1} * t_{T_2}, \psi \circ \pi\rangle, \quad \psi \in \mathcal{D}(G/K). \tag{1.56}$$

In this case,

$$\begin{aligned}(f \times T)(\mathbf{g}K) &= \langle t_T(g), f(\mathbf{g}g^{-1}K)\rangle,\\ (T \times f)(\mathbf{g}K) &= \langle t_T(g), f(g^{-1}\mathbf{g}K)\rangle,\end{aligned} \tag{1.57}$$

for $f \in \mathcal{E}(G/K)$, $T \in \mathcal{E}'(G/K)$. Furthermore,

$$(f_1 \times f_2)(\mathbf{g}K) = \int_G f_1(gK) f_2(g^{-1}\mathbf{g}K)\,\mathrm{d}g$$

for $f_1 \in L^{1,\mathrm{loc}}(G/K)$ and $f_2 \in (L^{1,\mathrm{loc}} \cap \mathcal{E}')(G/K)$.

The convolution on G/K is associative and satisfies $T \times \delta_o = T$, where δ_o is the delta distribution $\psi \to \psi(o)$ at the origin $o = \{K\}$ in G/K. Also,

$$\langle \delta_o \times T, \psi \rangle = \int_K \langle T, \psi \circ \tau(k) \rangle \, \mathrm{d}k, \quad \psi \in \mathcal{D}(G/K),$$

where $\tau(k)$ is the translation $gK \to kgK$, $g \in G$.

In addition to our assumption that G is a separable unimodular Lie group and K compact, we assume now that (G, K) is a symmetric pair. Let $\mathfrak{W}(G)$ be a given class of distributions on G. Denote by $\mathfrak{W}_\natural(G)$ the set of K-*bi-invariant* distributions in $\mathfrak{W}(G)$, i.e.,

$$\mathfrak{W}_\natural(G) = \big\{ t \in \mathfrak{W}(G) \colon \langle t(g), \psi(k_1 g k_2) \rangle = \langle t, \psi \rangle \ \forall \, k_1, k_2 \in K, \ \psi \in \mathcal{D}(G) \big\}. \tag{1.58}$$

Theorem 1.14. *$\mathcal{E}'_\natural(G)$ and $(C_c)_\natural(G)$ are commutative algebras under convolution.*

For the proof, we refer to Helgason [122], Chap. 2, Sect. 5.1.

A distribution $T \in \mathcal{D}'(G/K)$ is said to be K-*invariant* if

$$\langle T, \psi \circ \tau(k) \rangle = \langle T, \psi \rangle \quad \text{for all } k \in K, \psi \in \mathcal{D}(G/K).$$

By analogy with (1.58), given a class $\mathfrak{W}(G/K) \subset \mathcal{D}'(G/K)$, we define $\mathfrak{W}_\natural(G/K)$ as the set of all K-invariant distributions in $\mathfrak{W}(G/K)$. Then, in view of (1.56) and Theorem 1.14,

$$T_1 \times T_2 = T_2 \times T_1$$

if $T_1 \in \mathcal{E}'_\natural(G/K)$, $T_2 \in \mathcal{D}'_\natural(G/K)$.

Let $\mathbf{D}(G/K)$ be the algebra of differential operators on G/K invariant under all translations

$$\tau(g) \colon \mathbf{g}K \to g\mathbf{g}K, \quad g, \mathbf{g} \in G.$$

The algebra $\mathbf{D}(G/K)$ is commutative. If $D \in \mathbf{D}(G/K)$, we have

$$D(T_1 \times T_2) = DT_1 \times T_2 = T_1 \times DT_2$$

for all $T_1, T_2 \in \mathcal{D}'(G/K)$, at least one of compact support (see [122], Chap. 2, Sect. 5.1).

A function $\varphi \in \mathcal{E}(G/K)$ is called a *spherical function* if the following conditions are satisfied:

(a) $\varphi(o) = 1$;
(b) $\varphi \circ \tau(k) = \varphi$ for all $k \in K$;
(c) for every $D \in \mathbf{D}(G/K)$, there exists $\lambda_D \in \mathbb{C}$ such that $D\varphi = \lambda_D \varphi$.

A *joint eigenfunction* on G/K is an eigenfunction of each of the operators $D \in \mathbf{D}(G/K)$. Let $\mu \colon \mathbf{D}(G/K) \to \mathbb{C}$ be a homomorphism, and let E_μ denote the corresponding joint eigenspace; i.e.,

$$E_\mu = \big\{ f \in \mathcal{E}(G/K) \colon Df = \mu(D) f \text{ for all } D \in \mathbf{D}(G/K) \big\}.$$

Every spherical function belongs to a unique E_μ.

Proposition 1.3. *Each joint eigenspace $E_\mu \neq 0$ contains exactly one spherical function φ. The members f of E_μ are characterized by the mean value property*

$$\int_K f(gk\mathbf{g}K)\,\mathrm{d}k = f(gK)\varphi(\mathbf{g}K), \quad g, \mathbf{g} \in G. \tag{1.59}$$

The proof of Proposition 1.3 can be found in [122], Chap. 4, Sect. 2.2.

Let Φ be the set of spherical functions on G/K. The *spherical transform* of a distribution $T \in \mathcal{E}'_\natural(G/K)$ is defined by

$$\widetilde{T}(\varphi) = \langle T, \check{\varphi} \rangle, \quad \varphi \in \Phi,$$

where

$$\check{\varphi}(gK) = \varphi\big(g^{-1}K\big), \quad g \in G.$$

With our notation in Proposition 1.3, we have

$$f \times T = \widetilde{T}(\varphi) f. \tag{1.60}$$

The proof of (1.60) follows from (1.57) and (1.59).

Let G/K be a two-point homogeneous space. Then $\mathbf{D}(G/K)$ is generated by the Laplace–Beltrami operator L (see [122], Chap. 2, Sect. 4.3). The radial part of L has the form

$$L_0 = \frac{\partial^2}{\partial r^2} + \frac{A'(r)}{A(r)}\frac{\partial}{\partial r}, \tag{1.61}$$

where $A(r)$ is the area of the sphere $S_r(o) \subset G/K$. (If $\dim G/K = 1$, $A(r)$ is understood to be the number of points in $S_r(o)$.)

In the case of the real n-dimensional Euclidean space,

$$A(r) = \omega_{n-1} r^{n-1}, \quad \text{where } \omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \tag{1.62}$$

and Γ denotes the gamma function. The spherical functions on $\mathbb{R}^n$ are of the form

$$2^{n/2-1}\Gamma\left(\frac{n}{2}\right)\frac{J_{n/2-1}(\lambda|x|)}{(\lambda|x|)^{n/2-1}}, \quad \lambda \in \mathbb{C},$$

where J_ν is the Bessel function of the first kind of order ν.

For rank one symmetric spaces of noncompact type,

$$A(r) = \omega_{N-1}\left(\frac{\sinh kr}{k}\right)^{2\alpha+1}(\cosh kr)^{2\beta+1}. \tag{1.63}$$

Here, N is the real dimension of G/K; $\alpha = N/2 - 1$; $\beta = N/2 - 1$, 0, 1, or 3 as $G/K = \mathrm{SO}_0(n,1)/\mathrm{SO}(n)$, $\mathrm{SU}(n,1)/\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$, $\mathrm{Sp}(n,1)/\mathrm{Sp}(n) \times \mathrm{Sp}(1)$, or $F_4^*/\mathrm{Spin}(9)$, respectively; and k is a real parameter whose dependence on the metric of G/K is given by $m = -k^2$, where m is the maximum sectional curvature of G/K. Ordinarily, we may take $k = 1$. The spherical functions on G/K are given by

$$\varphi_\lambda(x) = F\left(\frac{\alpha+\beta+1-\mathrm{i}\lambda}{2}, \frac{\alpha+\beta+1+\mathrm{i}\lambda}{2}; \alpha+1; -\sinh^2 kt\right), \quad \lambda \in \mathbb{C},$$

where t is the distance between x and $o \in G/K$, and $F(a, b; c; z)$ denotes the usual hypergeometric function.

For rank one symmetric spaces of compact type,

$$A(r) = \omega_{N-1}\left(\frac{\sin kr}{k}\right)^{2\alpha+1}(\cos kr)^{2\beta+1}. \tag{1.64}$$

Here, as before, N is the real dimension of G/K; $\alpha = N/2-1$; $\beta = N/2-1, -1/2$, 0, 1, or 3 as $G/K = \mathrm{SO}(n+1)/\mathrm{SO}(n)$, $\mathrm{SO}(n+1)/\mathrm{O}(n)$, $\mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$, $\mathrm{Sp}(n+1)/\mathrm{Sp}(n) \times \mathrm{Sp}(1)$, or $F_4/\mathrm{Spin}(9)$, respectively; k is a real parameter which may now be interpreted as $(2\,\mathrm{diam}\,(G/K))^{-1}\pi$, where $\mathrm{diam}\,(G/K)$ is the diameter (maximum distance between two points) of G/K. Finally, the corresponding spherical functions are given by

$$\varphi_j(x) = F\left(-j, j+\alpha+\beta+1; \alpha+1; \sin^2 kt\right), \quad j \in \mathbb{Z}_+.$$

Concerning the spherical functions on symmetric spaces of arbitrary rank, see [122], Chap. 4, Sects. 2–4.

1.5 Structure of Quasi-Regular Representations of Compact Groups

Let G be a locally compact group and V a locally convex topological vector space over $\mathbb{C}$. Denote by $\mathrm{Aut}\,(V)$ the group of all linear homeomorphisms of V onto itself. A *representation* π of the group G on the space V is a homomorphism from G into $\mathrm{Aut}\,(V)$ such that the mapping

$$(g, v) \to \pi(g)v$$

from $G \times V$ into V is continuous. We shall call $\dim V$ the *dimension* of π. We note that if V is a barreled topological vector space and $\pi : G \to \mathrm{Aut}\,(V)$ is a homomorphism such that for each $v \in V$, the mapping $g \to \pi(g)v$ is continuous from G to V, then π is a representation of G on V (see Helgason [122], Chap. 4, Sect. 1).

Let V' denote the dual space of V. If $v \in V$ and $\lambda \in V'$, the function

$$g \to \lambda\bigl(\pi(g)v\bigr)$$

is called a *representation coefficient*. In harmonic analysis on G one investigates these functions as well as how they can serve as building blocks for "arbitrary" function spaces on G.

Let V' be given the strong topology (that is, the topology of uniform convergence on bounded subsets of V). The mapping which sends a vector $v \in V$ into the continuous linear form $\widetilde{v}\colon \lambda \to \lambda(v)$ on V' is an injective map of V into the second dual space $(V')'$. If this map is surjective, then V is called *semireflexive*. It is known that any closed subspace W of a semireflexive space V is again semireflexive (see Köthe [142]).

If A is a continuous linear transformation of V, its *transpose* ${}^tA\colon V' \to V'$ is defined by $({}^tA(\lambda))(v) = \lambda(Av)$; it is again continuous. If A is a linear homeomorphism of V onto itself, then $({}^tA)^{-1} = {}^t(A^{-1})$.

Let π be a representation of G on V. The mapping

$$\check{\pi} : g \to {}^t\big(\pi\big(g^{-1}\big)\big)$$

is a homomorphism of G into $\mathrm{Aut}\,(V')$. If V is semireflexive, it is known that the mapping $(g, \lambda) \to \check{\pi}(g)\lambda$ of $G \times V'$ into V' is continuous (see Bruhat [43], Sect. 2) so $\check{\pi}$ is a representation; it is called the representation *contragredient* to π.

A representation π is said to be *irreducible* if for each closed subspace $W \subset V$ that is invariant under $\pi(G)$, we have $W = \{0\}$ or $W = V$.

Two representations (π_1, V_1) and (π_2, V_2) of G are said to be *equivalent* if there exists a linear homeomorphism A of V_1 onto V_2 satisfying

$$A\pi_1(g) = \pi_2(g)A \quad \text{for all } g \in G. \tag{1.65}$$

A continuous linear mapping A of V_1 into V_2 satisfying (1.65) is called an *intertwining operator*.

If V is a Hilbert space and π is a representation of G on V with each $\pi(g)$ unitary, then π is called a *unitary representation*.

We will now give some more details about a compact group and its finite-dimensional representations.

Let K be a compact group and $\mathrm{d}k$ the Haar measure on K, normalized by (1.54). Denote by $\widehat{K}$ the set of equivalence classes of finite-dimensional unitary irreducible representations of K. For each $\delta \in \widehat{K}$, let V_δ be a vector space with inner product $\langle\ ,\ \rangle$ on which a representation of class δ is realized, and let this representation also be denoted by δ. Let $d(\delta)$ denote the dimension of δ. If $\delta_1, \delta_2 \in \widehat{K}$, then for $\xi_1, \eta_1 \in V_{\delta_1}, \xi_2, \eta_2 \in V_{\delta_2}$, we have the orthogonality relations of Schur

$$\begin{aligned}\int_K \langle\delta_1(k)\xi_1, \eta_1\rangle\overline{\langle\delta_2(k)\xi_2, \eta_2\rangle}\,\mathrm{d}k \\ = \begin{cases}(d_1(\delta))^{-1}\langle\xi_1, \xi_2\rangle\overline{\langle\eta_1, \eta_2\rangle} & \text{if } \delta_1 = \delta_2, \\ 0 & \text{if } \delta_1 \neq \delta_2.\end{cases}\end{aligned} \tag{1.66}$$

If $v_1, \ldots, v_{d(\delta)}$ is an orthonormal basis of V_δ, then (1.66) means that the functions

$$d(\delta)^{1/2}\langle\delta(k)v_j, v_i\rangle, \quad i, j \in \{1, \ldots, d(\delta)\}$$

are orthonormal in $L^2(K)$. According to the Peter–Weyl theorem, they form a complete orthonormal basis of $L^2(K)$.

For each $\delta \in \widehat{K}$, the contragredient $\check{\delta}$ operates on the dual space $V'_\delta = V_{\check{\delta}}$. The mapping which assigns to each $v \in V_\delta$ the linear form $v'\colon w \to \langle w, v\rangle$ is a conjugate linear bijection of V_δ onto V'_δ. We define the inner product $\langle\ ,\ \rangle$ on $V_{\check{\delta}}$ by

$$\langle v', w'\rangle = \langle w, v\rangle.$$

For a linear transform L on V_δ, let L' denote the linear transformation $v' \to (Lv)'$ of $V_{\check{\delta}}$. If L has a matrix expression $\|l_{ij}\|_{i,j=1}^{d(\delta)}$ in the basis $v_1, \dots, v_{d(\delta)}$, then L' has a matrix expression $\|\overline{l_{ij}}\|_{i,j=1}^{d(\delta)}$ in the basis $v'_1, \dots, v'_{d(\delta)}$. Since $\check{\delta}(k) = {}^t\delta(k^{-1})$, we find

$$\begin{aligned}\big(\check{\delta}(k)v'_j\big)(w) &= v'_j\big(\delta\big(k^{-1}\big)w\big) = \langle w, \delta(k)v_j\rangle = \big(\delta(k)v_j\big)'(w)\\ &= \sum_{i=1}^{d(\delta)}\big(\overline{\langle\delta(k)v_j, v_i\rangle}v'_i\big)(w).\end{aligned}$$

This means that $\check{\delta}(k) = \delta(k)'$, $\check{\delta}$ is unitary, and

$$\big\langle\check{\delta}(k)v'_j, v'_i\big\rangle = \overline{\langle\delta(k)v_j, v_i\rangle}. \tag{1.67}$$

Let M be a closed subgroup of K. We put

$$\begin{aligned}V_\delta^M &= \big\{v \in V_\delta\colon \delta(m)v = v \text{ for all } m \in M\big\},\\ \widehat{K}_M &= \big\{\delta \in \widehat{K}\colon V_\delta^M \neq 0\big\},\end{aligned} \tag{1.68}$$

and assume that V_δ^M is one-dimensional for each $\delta \in \widehat{K}_M$. We shall write the orthogonal Hilbert space decomposition for $L^2(K/M)$ viewed as the subspace of $L^2(K)$ consisting of functions right invariant under M. For $\delta \in \widehat{K}_M$, let $v_1, \dots, v_{d(\delta)}$ be a basis of V_δ such that v_1 spans V_δ^M. Denote by H_δ the subspace of $L^2(K)$ spanned by the functions

$$k \to \langle\delta(k)u, v\rangle, \quad u, v \in V_\delta.$$

Also let H_δ^M be the subspace of functions in H_δ which are right invariant under M. Then the functions

$$k \to \langle\delta(k)v_1, v_j\rangle, \quad 1 \leqslant j \leqslant d(\delta),$$

form a basis of H_δ^M, and we have the orthogonal Hilbert space decomposition

$$L^2(K/M) = \bigoplus_{\delta\in\widehat{K}_M} H_\delta^M \tag{1.69}$$

due to Weyl (see Helgason [122], Chap. 4, Sect. 1.2). Setting

$$Y_j^\delta(kM) = \langle\delta(k)v_1, v_j\rangle, \quad 1 \leqslant j \leqslant d(\delta), \tag{1.70}$$

by (1.70) and (1.68) we have

$$Y_1^\delta\big(k^{-1}M\big) = \overline{Y_1^\delta(kM)}, \quad Y_j^\delta(eM) = \delta_{1,j},$$

where e is the unity in K, and $\delta_{1,j}$ is the Kronecker delta.

Let F be any function space on K/M such that the mapping $(k, f) \to f \circ k^{-1}$ is continuous from $K \times F$ into F. The representation T of K on F defined by

$$T(k)f = f \circ k^{-1}$$

is called the *quasi-regular representation* of K on F. For $\delta \in \widehat{K}_M$, denote by T_δ the quasi-regular representation of K on H_δ^M. Let $\|t_{ij}^\delta(k)\|_{i,j=1}^{d(\delta)}$ be a representation matrix of T_δ, that is,

$$Y_j^\delta\big(k^{-1}\tau M\big) = \sum_{i=1}^{d(\delta)} t_{ij}^\delta(k) Y_i^\delta(\tau M), \quad k, \tau \in K. \tag{1.71}$$

Comparing (1.71) with (1.70), we see that

$$t_{ij}^\delta(k) = \overline{\langle \delta(k)v_j, v_i\rangle}, \quad i, j \in \{1, \dots, d(\delta)\}. \tag{1.72}$$

Thus, T_δ and $\check{\delta}$ are equivalent (see (1.67)).

Proposition 1.4.

(i) *For each $\delta \in \widehat{K}_M$, the representation T_δ is irreducible.*
(ii) *If $\delta_1, \delta_2 \in \widehat{K}_M$, $\delta_1 \neq \delta_2$, then T_{δ_1} and T_{δ_2} are not equivalent.*

Proof. Let W be a closed subspace of H_δ^M that is invariant under $T_\delta(K)$. Each $f \in W$ has the form $f(kM) = \langle \delta(k)v_1, u\rangle$ for some $u \in V_\delta$. We set

$$U = \big\{u \in V_\delta \colon f(kM) = \langle \delta(k)v_1, u\rangle \in W\big\}. \tag{1.73}$$

For any $\tau \in K$, we find

$$f\big(\tau^{-1}kM\big) = \big\langle \delta\big(\tau^{-1}k\big)v_1, u\big\rangle = \langle \delta(k)v_1, \delta(\tau)u\rangle.$$

Thus $\delta(\tau)u \in U$ for all $\tau \in K, u \in U$. Since δ is irreducible, we obtain $U = \{0\}$ or $U = V_\delta$. By (1.73) we have $W = \{0\}$ or $W = H_\delta^M$, so (i) follows.

For (ii), it suffices to prove that $\check{\delta}_1$ and $\check{\delta}_2$ are not equivalent. Otherwise there exists a linear homeomorphism $\widetilde{A}\colon V_{\check{\delta}_1} \to V_{\check{\delta}_2}$ satisfying

$$\widetilde{A}\check{\delta}_1(k) = \check{\delta}_2(k)\widetilde{A} \quad \text{for all } k \in K. \tag{1.74}$$

We shall use the notation $\langle\cdot,\cdot\rangle_1$, $\langle\cdot,\cdot\rangle_2$ for the inner products of V_{δ_1} and V_{δ_2}, respectively. Consider the linear operator $A\colon V_{\delta_1} \to V_{\delta_2}$ defined by

$$\big(\widetilde{A}v'\big)(u) = \langle u, Av\rangle_2, \quad u \in V_{\delta_2}, v \in V_{\delta_1}, \tag{1.75}$$

where $v' \in V_{\check{\delta}_1} = V'_{\delta_1}$. Since the map $\widetilde{A}$ is surjective, we conclude that A is a linear bijection. Let $1 \leqslant j \leqslant d(\delta)$. For each $u \in V_{\delta_2}$, by (1.75) we obtain

$$\langle u, A\delta_1(k)v_j\rangle_2 = \sum_{j=1}^{d(\delta)} \overline{\langle \delta_1(k)v_j, v_i\rangle}_1 \langle u, Av_i\rangle_2 = \widetilde{A}\big(\check{\delta}_1(k)v'_j\big)(u).$$

This together with (1.74) implies $\langle u, A\delta_1(k)v_j\rangle_2 = \langle u, \delta_2(k)Av_j\rangle_2$. So $A\delta_1(k) = \delta_2(k)A$ for all $k \in K$, which is in contradiction with the relation $\delta_1 \neq \delta_2$. The required result now follows. □

The decomposition (1.69), together with Proposition 1.4, means that the quasi-regular representation of K on $L^2(K/M)$ is a direct sum of pairwise nonequivalent irreducible unitary representations T_δ, $\delta \in \widehat{K}_M$.

From (1.66) and (1.72) it now follows that

$$\int_K t^{\delta_1}_{i_1,j_1}(k)\overline{t^{\delta_2}_{i_2,j_2}(k)}\,\mathrm{d}k = 0 \quad \text{if } (\delta_1, i_1, j_1) \neq (\delta_2, i_2, j_2) \tag{1.76}$$

and

$$\int_K \big|t^{\delta}_{ij}(k)\big|^2\,\mathrm{d}k = \frac{1}{d(\delta)}. \tag{1.77}$$

We associate with each function $f \in L^1(K/M)$ its Fourier series

$$f(kM) \sim \sum_{\delta\in\widehat{K}_M} \sum_{j=1}^{d(\delta)} f_{\delta,j} Y^{\delta}_j(kM), \quad k \in K, \tag{1.78}$$

where

$$f_{\delta,j} = d(\delta) \int_K f(kM)\overline{Y^{\delta}_j(kM)}\,\mathrm{d}k. \tag{1.79}$$

Let $f \in L^2(K/M)$, $\tau \in K$. It follows from the continuity of the representation operator and (1.71), (1.78) that

$$f\big(\tau^{-1}kM\big) = \sum_{\delta\in\widehat{K}_M} \sum_{j=1}^{d(\delta)} f_{\delta,j} T_\delta(\tau) Y^{\delta}_j(kM) = \sum_{\delta\in\widehat{K}_M} \sum_{i,j=1}^{d(\delta)} f_{\delta,j} Y^{\delta}_i(kM) t^{\delta}_{ij}(\tau).$$

Using (1.76), (1.77), from this we obtain

$$f_{\delta,j} Y^{\delta}_i(kM) = d(\delta) \int_K f\big(\tau^{-1}kM\big)\overline{t^{\delta}_{ij}(\tau)}\,\mathrm{d}\tau \tag{1.80}$$

for all $\delta \in \widehat{K}_M, i, j \in \{1, \ldots, d(\delta)\}$. Since the space $L^2(K/M)$ is dense in $L^1(K/M)$, by (1.79) we conclude that equality (1.80) holds for each $f \in L^1(K/M)$.

Chapter 2
Analogues of the Beltrami–Klein Model for Rank One Symmetric Spaces of Noncompact Type

It is well known that the hyperbolic plane $\mathbb{H}^2$ of constant sectional curvature -1 can be realized as the open disk $x^2 + y^2 < 1$ with the Riemannian metric

$$\mathrm{d}s^2 = \frac{(1-y^2)\,\mathrm{d}x^2 + 2xy\,\mathrm{d}x\,\mathrm{d}y + (1-x^2)\,\mathrm{d}y^2}{(1-x^2-y^2)^2}.$$

The realization is said to be the *Beltrami–Klein model* of $\mathbb{H}^2$. The purpose of Chap. 2 is to obtain similar models for all Riemannian symmetric spaces of rank one and negative curvature.

All this program is not too hard to carry out for real, complex, or quaternionic hyperbolic spaces. In these cases, the analog of the Beltrami–Klein model is the open ball $B^n_{\mathbb{K}} = \{u \in \mathbb{K}^n : |u| < 1\}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{Q}$ respectively) with the Riemannian structure

$$\mathrm{d}s^2 = \big(1-|u|^2\big)^{-1}\big(|\mathrm{d}u|^2 + \big(1-|u|^2\big)^{-1}|\langle \mathrm{d}u, u\rangle_{\mathbb{K}}|^2\big)$$

(see Sects. 2.1–2.3). However we encounter a prime difficulty when realizing $F_4^*/\mathrm{Spin}(9)$. The basic reason for this is the fact that the system of Cayley numbers $\mathbb{C}a$ is not associative. Nevertheless, in Sect. 2.4 we show that the Cayley hyperbolic plane $F_4^*/\mathrm{Spin}(9)$ of maximal sectional curvature -1 is isometric to the ball $B^{16}_{\mathbb{R}}$ with the metric tensor

$$g_{ij}(x) = \frac{\delta_{i,j}}{1-|x|^2} + \frac{1}{2(1-|x|^2)^2}\frac{\partial^2}{\partial y_i \partial y_j}\big(\Phi_{\mathbb{C}a}(x,y)\big)$$

(for the definition of $\Phi_{\mathbb{C}a}(x,y)$, see Sect. 1.1). In the course of our analysis, we also give explicit formulas for the Riemannian measure, the Laplace–Beltrami operator, the symmetries, and the distance. These results will be needed later.

V.V. Volchkov, V.V. Volchkov, *Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group,* Springer Monographs in Mathematics,

2.1 The Real Hyperbolic Space $\mathbf{SO_0}(n, 1)/\mathbf{SO}(n)$

In this section we construct the usual Beltrami–Klein model for the real hyperbolic space. For the rest of Chap. 2, we assume that $n \in \mathbb{N}\backslash\{1\}$.

Let $B^n_{\mathbb{R}} = \{x \in \mathbb{R}^n : \ |x| < 1\}$ be the open unit ball in $\mathbb{R}^n$ with the Riemannian structure

$$\mathrm{d}s^2 = \frac{|\mathrm{d}x|^2}{1-|x|^2} + \frac{\langle x, \mathrm{d}x\rangle^2_{\mathbb{R}}}{(1-|x|^2)^2}. \tag{2.1}$$

The metric (2.1) corresponds to the metric tensor

$$g_{ij}(x) = \frac{\delta_{i,j}}{1-|x|^2} + \frac{x_i x_j}{(1-|x|^2)^2}, \quad i, j \in \{1, \dots, n\}, \tag{2.2}$$

and it is easy to see that

$$g^{ij}(x) = \big(1-|x|^2\big)(\delta_{i,j} - x_i x_j), \quad i, j \in \{1, \dots, n\}. \tag{2.3}$$

We wish to compute the curvature tensor of metric (2.1). One obtains (see (1.37), (1.38))

$$\begin{aligned}
\frac{\partial g_{ij}}{\partial x_k} &= \frac{2x_k\delta_{i,j} + x_i\delta_{j,k} + x_j\delta_{i,k}}{(1-|x|^2)^2} + \frac{4x_i x_j x_k}{(1-|x|^2)^3},\\
\Gamma^k_{ij} &= \frac{x_i\delta_{k,j} + x_j\delta_{i,k}}{1-|x|^2},\\
\frac{\partial \Gamma^k_{ij}}{\partial x_l} &= \frac{\delta_{i,l}\delta_{k,j} + \delta_{j,l}\delta_{k,i}}{1-|x|^2} + \frac{2(x_i x_l\delta_{k,j} + x_l x_j\delta_{i,k})}{(1-|x|^2)^2},\\
R^l_{ijk} &= \frac{\delta_{i,j}\delta_{k,l} - \delta_{i,k}\delta_{j,l}}{1-|x|^2} + \frac{x_i x_j\delta_{k,l} - x_i x_k\delta_{j,l}}{(1-|x|^2)^2}.
\end{aligned} \tag{2.4}$$

Hence, by (1.39)

$$\begin{aligned}
R_{ijkl} &= \frac{\delta_{i,l}\delta_{j,k} - \delta_{i,k}\delta_{j,l}}{(1-|x|^2)^2} + \frac{x_i x_l\delta_{j,k} - x_i x_k\delta_{j,l} + x_j x_k\delta_{i,l} - x_j x_l\delta_{i,k}}{(1-|x|^2)^3}\\
&= -(g_{ik}g_{jl} - g_{jk}g_{il}).
\end{aligned} \tag{2.5}$$

The formula for R_{ijkl} shows that the ball $B^n_{\mathbb{R}}$ with the metric (2.1) is a space of constant sectional curvature -1 (see Sect. 1.2). We denote this space by $\mathbb{H}^n_{\mathbb{R}}$.

Let us establish some properties of $\mathbb{H}^n_{\mathbb{R}}$.

Proposition 2.1. *The Riemannian measure on $\mathbb{H}^n_{\mathbb{R}}$ has the form*

$$\mathrm{d}\mu(x) = \frac{\mathrm{d}x}{(1-|x|^2)^{\frac{n+1}{2}}}. \tag{2.6}$$

Proof. Let $g(x)$ be the determinant of the matrix (2.2). Using the formula

$$\frac{\partial}{\partial x_i}\big(\log g(x)\big) = \sum_{j,k=1}^{n} g^{jk}(x)\frac{\partial g_{jk}}{\partial x_i}(x)$$

(see Ahlfors [5], Chap. 4, Sect. 4.12), we find

$$\frac{\partial}{\partial x_i}\left(\log\big(\sqrt{g(x)}\big(1-|x|^2\big)^{\frac{n+1}{2}}\big)\right) = 0 \tag{2.7}$$

for $i \in \{1, \dots, n\}$. Bearing in mind that $g(0) = 1$, from (2.7) we obtain

$$\sqrt{g(x)} = \frac{1}{(1-|x|^2)^{\frac{n+1}{2}}}, \quad x \in B^n_{\mathbb{R}}. \tag{2.8}$$

Together, (2.8) and (1.36) give (2.6). □

Proposition 2.2. *The Laplace–Beltrami operator L on $\mathbb{H}^n_{\mathbb{R}}$ acts on a function $f \in C^2(B^n_{\mathbb{R}})$ as follows:*

$$(Lf)(x) = \big(1-|x|^2\big)\left((\Delta f)(x) - \sum_{i,j=1}^{n} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j}(x) - 2\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(x)\right). \tag{2.9}$$

In particular, if f has the form $f(x) = f_0(|x|)$, then

$$(Lf)(x) = \big(1-|x|^2\big)^2 f_0''(|x|) + \frac{1-|x|^2}{|x|}\big(n-1-2|x|^2\big) f_0'(|x|).$$

Proof. The assertions follow from (1.43), (1.40), (2.3), and (2.8). □

As usual, we denote by $I(\mathbb{H}^n_{\mathbb{R}})$ the isometry group of $\mathbb{H}^n_{\mathbb{R}}$.

Proposition 2.3. *The group $\mathrm{O}(n)$ is a subgroup of $I(\mathbb{H}^n_{\mathbb{R}})$.*

Proof. Let $\gamma(t) = (x_1(t), \dots, x_n(t))$, $0 \leqslant t \leqslant 1$, be an arbitrary piecewise smooth curve in $B^n_{\mathbb{R}}$, and let $\gamma'(t) = (x_1'(t), \dots, x_n'(t))$. It follows from (2.1) and (1.35) that the length $l(\gamma)$ of the curve γ can be calculated by the formula

$$l(\gamma) = \int_0^1 \left(\frac{|\gamma'(t)|^2}{1-|\gamma(t)|^2} + \frac{\langle\gamma(t), \gamma'(t)\rangle^2_{\mathbb{R}}}{(1-|\gamma(t)|^2)^2}\right)^{1/2} \mathrm{d}t. \tag{2.10}$$

We set $\gamma_\tau(t) = \tau(\gamma(t))$, $0 \leqslant t \leqslant 1$, for any $\tau \in \mathrm{O}(n)$. Since $(\gamma_\tau)'(t) = \tau(\gamma'(t))$, formula (2.10) implies that $l(\gamma_\tau) = l(\gamma)$. Therefore, $d(\tau x, \tau y) = d(x, y)$ for all $x, y \in B^n_{\mathbb{R}}$, where d is the distance on the space $\mathbb{H}^n_{\mathbb{R}}$. This concludes the proof. □

Now we construct an element $\varphi \in I(\mathbb{H}^n_{\mathbb{R}})$ such that $\varphi(0) \neq 0$. Let $a \in B^n_{\mathbb{R}}$. Put

$$\sigma_a(x) = \frac{a - P_a(x) - \mu_a Q_a(x)}{1 - \langle x, a\rangle_{\mathbb{R}}}, \quad x \in \mathrm{Cl}\big(B^n_{\mathbb{R}}\big), \tag{2.11}$$

where $\mu_a = \sqrt{1 - |a|^2}$,

$$P_a(x) = \begin{cases} \langle x, a\rangle_{\mathbb{R}} |a|^{-2} a & \text{if } a \neq 0, \\ 0 & \text{if } a = 0, \end{cases}$$
$$Q_a(x) = x - P_a(x).$$

Notice that

$$\tau \circ \sigma_a = \sigma_{\tau a} \circ \tau \tag{2.12}$$

for $\tau \in \mathrm{O}(n)$.

Proposition 2.4. *Let $a \in B^n_{\mathbb{R}}$ be fixed. Then*

(i) $\sigma_a(0) = a$ *and* $\sigma_a(a) = 0$.
(ii) *The identity*

$$1 - \big\langle \sigma_a(x), \sigma_a(y)\big\rangle_{\mathbb{R}} = \frac{(1 - |a|^2)(1 - \langle x, y\rangle_{\mathbb{R}})}{(1 - \langle x, a\rangle_{\mathbb{R}})(1 - \langle y, a\rangle_{\mathbb{R}})} \tag{2.13}$$

holds for all $x, y \in \mathrm{Cl}(B^n_{\mathbb{R}})$. In particular,

$$1 - |\sigma_a(x)|^2 = \frac{(1 - |a|^2)(1 - |x|^2)}{(1 - \langle x, a\rangle_{\mathbb{R}})^2}, \quad x \in \mathrm{Cl}\big(B^n_{\mathbb{R}}\big). \tag{2.14}$$

(iii) *For any $b \in B^n_{\mathbb{R}}$,*

$$\big\langle \psi(x), \psi(y)\big\rangle_{\mathbb{R}} = \langle x, y\rangle_{\mathbb{R}}, \quad x, y \in \mathrm{Cl}\big(B^n_{\mathbb{R}}\big), \tag{2.15}$$

where $\psi = \sigma_{\sigma_a(b)} \circ \sigma_a \circ \sigma_b$.
(iv) *For each $f \in C^2(B^n_{\mathbb{R}})$,*

$$(Lf)(a) = \Delta(f \circ \sigma_a)(0). \tag{2.16}$$

(v) σ_a *is an involution:* $\sigma_a(\sigma_a(x)) = x$.
(vi) σ_a *is a homeomorphism of* $\mathrm{Cl}(B^n_{\mathbb{R}})$ *onto* $\mathrm{Cl}(B^n_{\mathbb{R}})$, *and* $\sigma_a \in I(\mathbb{H}^n_{\mathbb{R}})$.
(vii) *The relation*

$$\sigma_a\left(\frac{a}{1 + \mu_a}\right) = \frac{a}{1 + \mu_a}$$

holds. Moreover, σ_a fixes exactly one point of $B^n_{\mathbb{R}}$ and no point of $\mathbb{S}^{n-1}$.

Proof. Assertion (i) follows immediately from (2.11). Let us prove (ii). We have

$$\big\langle a - P_a(x), Q_a(y)\big\rangle_{\mathbb{R}} = 0, \qquad \big\langle Q_a(x), a - P_a(y)\big\rangle_{\mathbb{R}} = 0.$$

Consequently,

$$1 - \big\langle \sigma_a(x), \sigma_a(y)\big\rangle_{\mathbb{R}} = 1 - \frac{\langle a - P_a(x), a - P_a(y)\rangle_{\mathbb{R}} + \mu_a^2 \langle Q_a(x), Q_a(y)\rangle_{\mathbb{R}}}{(1 - \langle x, a\rangle_{\mathbb{R}})(1 - \langle y, a\rangle_{\mathbb{R}})}. \tag{2.17}$$

It is straightforward to verify that

$$\big\langle a - P_a(x), a - P_a(y)\big\rangle_{\mathbb{R}} = |a|^2 - \langle y, a\rangle_{\mathbb{R}} - \langle x, a\rangle_{\mathbb{R}} + \frac{\langle x, a\rangle_{\mathbb{R}}\langle y, a\rangle_{\mathbb{R}}}{|a|^2},$$

$$\big\langle Q_a(x), Q_a(y)\big\rangle_{\mathbb{R}} = \langle x, y\rangle_{\mathbb{R}} - \frac{\langle x, a\rangle_{\mathbb{R}}\langle y, a\rangle_{\mathbb{R}}}{|a|^2}.$$

Combining these relations with (2.17), we deduce (2.13). Taking $x = y$ in (2.13) gives (2.14). As an obvious consequence of (2.14), note that $|\sigma_a(x)| < 1$ if and only if $|x| < 1$ and that $|\sigma_a(x)| = 1$ if and only if $|x| = 1$. Thus, σ_a maps $B^n_{\mathbb{R}}$ into $B^n_{\mathbb{R}}$ and $\mathbb{S}^{n-1}$ into $\mathbb{S}^{n-1}$.

Next, relations (2.13) and (2.14) yield

$$\begin{aligned}
1 - \big\langle \psi(x), \psi(y)\big\rangle_{\mathbb{R}} &= \frac{(1 - |\sigma_a(b)|^2)(1 - \langle \sigma_a(\sigma_b(x)), \sigma_a(\sigma_b(y))\rangle_{\mathbb{R}})}{(1 - \langle \sigma_a(\sigma_b(x)), \sigma_a(b)\rangle_{\mathbb{R}})(1 - \langle \sigma_a(\sigma_b(y)), \sigma_a(b)\rangle_{\mathbb{R}})} \\
&= \frac{(1 - |b|^2)(1 - \langle \sigma_b(x), \sigma_b(y)\rangle_{\mathbb{R}})}{(1 - \langle \sigma_b(x), \sigma_b(0)\rangle_{\mathbb{R}})(1 - \langle \sigma_b(y), \sigma_b(0)\rangle_{\mathbb{R}})} \\
&= 1 - \langle x, y\rangle_{\mathbb{R}},
\end{aligned}$$

whence assertion (iii) follows.

Let $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$. Denote by $f_1, \dots, f_n$ the coordinates of the mapping $\sigma_{|a|e_1}$. Then

$$\begin{aligned}
\Delta(f \circ \sigma_{|a|e_1})(0) = &\sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k}(|a|e_1) \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(0)\frac{\partial f_k}{\partial x_i}(0) \\
&+ \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(|a|e_1) \sum_{i=1}^{n} \frac{\partial^2 f_j}{\partial x_i^2}(0).
\end{aligned}$$

Because

$$f_1(x) = \frac{|a| - x_1}{1 - |a|x_1},$$

$$f_k(x) = -\mu_a \frac{x_k}{1 - |a|x_1}, \quad k \in \{2, \dots, n\},$$

we have

$$\frac{\partial f_1}{\partial x_i}(0) = -\mu_a^2 \delta_{1,i}, \qquad \frac{\partial f_k}{\partial x_i}(0) = -\mu_a \delta_{k,i},$$
$$\frac{\partial^2 f_1}{\partial x_i^2}(0) = -2|a|\mu_a^2 \delta_{1,i}, \qquad \frac{\partial^2 f_k}{\partial x_i^2}(0) = -2|a|\mu_a \delta_{1,i}\delta_{k,i}$$

for $i \in \{1, \ldots, n\}$, $k \in \{2, \ldots, n\}$. By that

$$\Delta(f \circ \sigma_{|a|e_1})(0) = \mu_a^2 \bigg((\Delta f)(|a|e_1) - |a|^2 \frac{\partial^2 f}{\partial x_1^2}(|a|e_1) - 2|a| \frac{\partial f}{\partial x_1}(|a|e_1) \bigg).$$

Using (2.9), we get

$$(Lf)(|a|e_1) = \Delta(f \circ \sigma_{|a|e_1})(0)$$

for any $f \in C^2(B_{\mathbb{R}}^n)$. This, together with (2.12), implies assertion (iv), since the operators Δ and L are invariant under the group $\mathrm{O}(n)$.

We now proceed to the proof of assertion (v). By the definition of σ_a,

$$\begin{aligned} \sigma_a\big(\sigma_a(x)\big) &= (1 - \langle \sigma_a(x), a\rangle_{\mathbb{R}})^{-1} \bigg(\bigg(1 - \frac{\langle \sigma_a(x), a\rangle_{\mathbb{R}}}{1 + \mu_a} \bigg) a - \mu_a \sigma_a(x) \bigg), \\ \langle \sigma_a(x), a\rangle_{\mathbb{R}} &= \frac{|a|^2 - \langle x, a\rangle_{\mathbb{R}}}{1 - \langle x, a\rangle_{\mathbb{R}}}. \end{aligned} \tag{2.18}$$

Inserting (2.18) into the expression for $\sigma_a(\sigma_a(x))$, after some computation, we obtain assertion (v).

Assertion (v) shows that σ_a is a one-to-one map of $\mathrm{Cl}(B_{\mathbb{R}}^n)$ onto $\mathrm{Cl}(B_{\mathbb{R}}^n)$ and that $\sigma_a^{-1} = \sigma_a$. We claim that $\sigma_a \in I(\mathbb{H}_{\mathbb{R}}^n)$. It suffices to prove that L commutes with the mapping σ_a (see Sect. 1.2). Let $f \in C^2(B_{\mathbb{R}}^n)$. In view of (2.16), we have

$$L(f \circ \sigma_a)(b) = \Delta(f \circ \sigma_a \circ \sigma_b)(0), \quad b \in B_{\mathbb{R}}^n.$$

It follows from (2.15) that the mapping ψ is the restriction of some element of the orthogonal group $\mathrm{O}(n)$ to $\mathrm{Cl}(B_{\mathbb{R}}^n)$. Therefore,

$$L(f \circ \sigma_a)(b) = \Delta(f \circ \sigma_{\sigma_a(b)} \circ \psi)(0) = \Delta(f \circ \sigma_{\sigma_a(b)})(0) = (Lf)(\sigma_a b),$$

i.e., L commutes with σ_a. Thus, $\sigma_a \in I(\mathbb{H}_{\mathbb{R}}^n)$.

It remains to prove assertion (vii). We may assume that $a \neq 0$. Let $x \in \mathrm{Cl}(B_{\mathbb{R}}^n)$ and $\sigma_a(x) = x$. Then $x = \lambda a$ for some $\lambda \in \mathbb{R}$. The equation $\sigma_a(\lambda a) = \lambda a$ has $\lambda = (1 \pm \mu_a)^{-1}$ as roots. Since $(1 - \mu_a)^{-1} a$ is not in $\mathrm{Cl}(B_{\mathbb{R}}^n)$, the point $(1 + \mu_a)^{-1} a$ is the only fixed point of σ_a in $\mathrm{Cl}(B_{\mathbb{R}}^n)$. This concludes the proof of Proposition 2.4. □

Corollary 2.1. *The group $I(\mathbb{H}_{\mathbb{R}}^n)$ acts transitively on $\mathbb{H}_{\mathbb{R}}^n$.*

Proof. Let $a, b \in \mathbb{H}_{\mathbb{R}}^n$. Then $\sigma_b \circ \sigma_a$ is an isometry of $\mathbb{H}_{\mathbb{R}}^n$ that takes a to b. □

Corollary 2.2. *The mapping $\sigma_{2a/(1+|a|^2)}$ is the symmetry of $\mathbb{H}^n_{\mathbb{R}}$ at the point a.*

Proof. This follows from Proposition 2.4 (v)–(vii). □

Remark 2.1. From Corollary 2.1 and Theorem 1.4 we see that $\mathbb{H}^n_{\mathbb{R}}$ is a complete Riemannian manifold. Moreover, Corollary 2.2 shows that the space $\mathbb{H}^n_{\mathbb{R}}$ is a Riemannian symmetric space.

Now we are ready to calculate the distance between two points $x, y \in \mathbb{H}^n_{\mathbb{R}}$.

Proposition 2.5. *The following equality is valid:*

$$d(x,y) = \frac{1}{2}\log\left(\frac{1-\langle x,y\rangle_{\mathbb{R}} + \sqrt{|x-y|^2 + \langle x,y\rangle^2_{\mathbb{R}} - |x|^2|y|^2}}{1-\langle x,y\rangle_{\mathbb{R}} - \sqrt{|x-y|^2 + \langle x,y\rangle^2_{\mathbb{R}} - |x|^2|y|^2}}\right). \tag{2.19}$$

Proof. Suppose first that $y = 0$. Since $O(n) \subset I(\mathbb{H}^n_{\mathbb{R}})$, the distance $d(0,x)$ is a radial function on the ball $B^n_{\mathbb{R}}$. Hence, $d(0,x) = d(0,|x|e_1)$. Put

$$\gamma(t) = \tanh\bigl(t \tanh^{-1}|x|\bigr)e_1 \quad \text{for } 0 \leqslant t \leqslant 1.$$

Then $\gamma(t)$ is a geodesic joining 0 and $|x|e_1$ (see (1.41) and (2.4)). In addition, by virtue of (2.10), the length of γ is equal to $\tanh^{-1}|x|$. Using Helgason [115], Chap. 1, Theorem 13.3 and Lemma 9.3, we obtain

$$d(0,x) = \frac{1}{2}\log\frac{1+|x|}{1-|x|}.$$

It now follows from assertions (i) and (vi) of Proposition 2.4 that

$$d(x,y) = d\bigl(0,\sigma_x(y)\bigr) = \frac{1}{2}\log\frac{1+|\sigma_x(y)|}{1-|\sigma_x(y)|}. \tag{2.20}$$

Together, (2.20) and (2.14) give (2.19). □

Theorem 2.1. *The real hyperbolic space* $SO_0(n,1)/SO(n)$ *of constant sectional curvature -1 is isometric to the space $\mathbb{H}^n_{\mathbb{R}}$.*

Proof. The space $\mathbb{H}^n_{\mathbb{R}}$ is a simply connected complete Riemannian manifold of constant curvature -1 (see (2.5) and Remark 2.1). Applying Theorem 1.6, we arrive at the desired assertion. □

We close this section by considering the Poincaré metric on $B^n_{\mathbb{R}}$. We now set

$$ds^2 = \frac{4|dx|^2}{(1-|x|^2)^2}. \tag{2.21}$$

Proposition 2.6. *The mapping*

$$\varphi\colon x \to \frac{2x}{1+|x|^2}, \quad x \in B^n_{\mathbb{R}},$$

is an isometry of $B^n_{\mathbb{R}}$ *with the metric* (2.21) *onto the space* $\mathbb{H}^n_{\mathbb{R}}$.

Proof. We write

$$y = \frac{2x}{1+|x|^2}, \quad x \in B^n_{\mathbb{R}}. \tag{2.22}$$

Relation (2.22) gives

$$x = \frac{y}{1+\sqrt{1-|y|^2}}.$$

In addition,

$$|\mathrm{d}y|^2 = \frac{4|\mathrm{d}x|^2}{(1+|x|^2)^2} - \frac{16}{(1+|x|^2)^4}\langle x, \mathrm{d}x\rangle^2_{\mathbb{R}},$$

$$\langle y, \mathrm{d}y\rangle_{\mathbb{R}} = \frac{4(1-|x|^2)}{(1+|x|^2)^3}\langle x, \mathrm{d}x\rangle_{\mathbb{R}}.$$

Hence,

$$\frac{|\mathrm{d}y|^2}{1-|y|^2} + \frac{\langle y, \mathrm{d}y\rangle^2_{\mathbb{R}}}{(1-|y|^2)^2} = \frac{4|\mathrm{d}x|^2}{(1-|x|^2)^2}.$$

This proves Proposition 2.6. □

Corollary 2.3. *For the Poincaré metric, we have the following results.*

(i) *The Riemannian measure has the form*

$$\mathrm{d}\mu(x) = \frac{2^n\,\mathrm{d}x}{(1-|x|^2)^n}.$$

(ii) *The Laplace–Beltrami operator L acts on a function $f \in C^2(B^n_{\mathbb{R}})$ as follows:*

$$(Lf)(x) = \frac{(1-|x|^2)^2}{4}\left((\Delta f)(x) + \frac{2(n-2)}{1-|x|^2}\sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(x)\right).$$

(iii) *The symmetry s_a at the point $a \in B^n_{\mathbb{R}}$ is defined by*

$$s_a(x) = \frac{2((1+|x|^2)(1+|a|^2) - 4\langle x, a\rangle_{\mathbb{R}})a - (1-|a|^2)^2 x}{(1+|a|^2)^2 - 4(1+|a|^2)\langle x, a\rangle_{\mathbb{R}} + 4|x|^2|a|^2}.$$

(iv) *The distance between two points $x, y \in B^n_{\mathbb{R}}$ is given by the formula*

$$d(x, y) = \log\left(\frac{\sqrt{1-2\langle x, y\rangle_{\mathbb{R}} + |x|^2|y|^2} + |x-y|}{\sqrt{1-2\langle x, y\rangle_{\mathbb{R}} + |x|^2|y|^2} - |x-y|}\right).$$

Proof. This is a standard change of variables argument, and we shall not stop to reproduce the details here. □

2.2 The Complex Hyperbolic Space SU(n, 1)/ S(U(n) × U(1))

In this section we consider an analogue of the Beltrami–Klein model for the space SU(n, 1)/ S(U(n) × U(1)).

Let $B^n_{\mathbb{C}} = \{z \in \mathbb{C}^n : \ |z| < 1\}$ be the open unit ball in $\mathbb{C}^n$ with the standard structure of a complex manifold. For $z = (z_1, \dots, z_n) \in B^n_{\mathbb{C}}$, we put

$$h_{ij}(z) = \frac{(1-|z|^2)\delta_{i,j} + \overline{z}_i z_j}{(1-|z|^2)^2}, \quad i, j \in \{1, \dots, n\}. \tag{2.23}$$

The matrix (2.23) is Hermitian symmetric, and

$$\sum_{i,j=1}^{n} h_{ij}(z) w_i \overline{w}_j = \frac{|w|^2}{1-|z|^2} + \frac{|\langle z, w\rangle_{\mathbb{C}}|^2}{(1-|z|^2)^2}, \tag{2.24}$$

where $w = (w_1, \dots, w_n) \in \mathbb{C}^n$. Hence, $\|h_{ij}\|_{i,j=1}^n$ defines a Hermitian metric $\mathrm{d}s^2$ on $B^n_{\mathbb{C}}$. In particular, $\|h_{ij}\|_{i,j=1}^n$ induces the structure of a Riemannian manifold on $B^n_{\mathbb{C}}$.

With respect to the coordinate system $z_1, \dots, z_n$, the metric $\mathrm{d}s^2$ may be written as follows:

$$\mathrm{d}s^2 = 2 \sum_{i,j=1}^{n} g_{i\overline{j}}(z)\, \mathrm{d}z_i\, \mathrm{d}\overline{z}_j,$$

where

$$g_{i\overline{j}}(z) = \frac{\partial^2}{\partial z_i \partial \overline{z}_j}\left(\frac{1}{2} \log \frac{1}{1-|z|^2}\right). \tag{2.25}$$

Therefore, $\mathrm{d}s^2$ is a Kaehler metric (see Sect. 1.2). Bearing in mind that

$$g^{i\overline{j}}(z) = 2\big(1-|z|^2\big)(\delta_{i,j} - z_i \overline{z}_j) \tag{2.26}$$

and using (1.51), we find

$$R_{i\overline{j}k\overline{l}} = 2(g_{i\overline{j}} g_{k\overline{l}} + g_{i\overline{l}} g_{k\overline{j}}). \tag{2.27}$$

By (2.27) and (1.52), the ball $B^n_{\mathbb{C}}$ with the metric (2.23) is a space of constant holomorphic sectional curvature -4. We denote this space by $\mathbb{H}^n_{\mathbb{C}}$.

Maximal sectional curvature of $\mathbb{H}^n_{\mathbb{C}}$ is equal to -1 (see Yano and Bochner [264], Chap. 8, Theorem 8.3). Now we study other properties of the space $\mathbb{H}^n_{\mathbb{C}}$.

Proposition 2.7. *The Riemannian measure on* $\mathbb{H}^n_{\mathbb{C}}$ *has the form*

$$d\mu(z) = \frac{dm_n(z)}{(1-|z|^2)^{n+1}},$$

where $dm_n(z)$ *is the Lebesgue measure on* $\mathbb{C}^n$.

Proof. We know that $d\mu(z) = h(z)\,dm_n(z)$, where $h(z)$ is the determinant of the matrix (2.23) (see (1.47)). To calculate $h(z)$, we use the formula

$$\frac{\partial}{\partial z_i}(\log h) = \sum_{j,k=1}^{n} g^{j\overline{k}} \frac{\partial g_{j\overline{k}}}{\partial z_i}$$

(see Kobayashi and Nomizu [137], Vol. 2, Chap. 9, Sect. 5). Taking (2.25) and (2.26) into account, we find

$$\frac{\partial}{\partial z_i}\left(\log\left(h(z)\left(1-|z|^2\right)^{n+1}\right)\right) = 0$$

for $i = 1, \dots, n$. It follows that

$$h(z) = \frac{h(0)}{(1-|z|^2)^{n+1}}, \quad z \in B^n_{\mathbb{C}}. \tag{2.28}$$

Since $h(0) = 1$, we arrive at the desired assertion. □

Let L be the Laplace–Beltrami operator on $\mathbb{H}^n_{\mathbb{C}}$.

Proposition 2.8. *The relation*

$$(Lf)(z) = 4\left(1-|z|^2\right) \sum_{i,j=1}^{n} (\delta_{i,j} - z_i \overline{z}_j) \frac{\partial^2 f}{\partial z_i \partial \overline{z}_j}(z)$$

holds for an arbitrary function $f \in C^2(B^n_{\mathbb{C}})$. *In particular, if* f *has the form* $f(z) = f_0(|z|)$, *then*

$$(Lf)(z) = \left(1-|z|^2\right)^2 f_0''(|z|) + \frac{1-|z|^2}{|z|}\left(2n-1-|z|^2\right) f_0'(|z|).$$

Proof. As usual, we denote by $\|h^{ij}\|_{i,j=1}^n$ the inverse matrix of $\|h_{ij}\|_{i,j=1}^n$. Then

$$h^{ij}(z) = \frac{\overline{g^{i\overline{j}}(z)}}{2}. \tag{2.29}$$

Representing L in the form (1.48), we deduce the required statement from (2.28) and (2.29). □

Proposition 2.9. *The group* $\mathrm{O}_{\mathbb{C}}(n)$ *is a subgroup of* $I(\mathbb{H}^n_{\mathbb{C}})$. *In particular,* $\mathrm{U}(n) \subset I(\mathbb{H}^n_{\mathbb{C}})$.

Proof. It is enough to use the arguments in the proof of Proposition 2.3 with substituting $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ for $\langle\cdot,\cdot\rangle_{\mathbb{R}}$. □

Next let $a \in B^n_{\mathbb{C}}$. By analogy with (2.11) we set

$$\sigma_a(z) = \frac{a - P_a(z) - \mu_a Q_a(z)}{1 - \langle z, a\rangle_{\mathbb{C}}}, \quad z \in \mathrm{Cl}\big(B^n_{\mathbb{C}}\big), \tag{2.30}$$

where $\mu_a = \sqrt{1 - |a|^2}$,

$$P_a(z) = \begin{cases} \langle z, a\rangle_{\mathbb{C}} |a|^{-2} a & \text{if } a \neq 0, \\ 0 & \text{if } a = 0, \end{cases}$$
$$Q_a(z) = z - P_a(z).$$

It is obvious that the mapping $\sigma_a : B^n_{\mathbb{C}} \to \mathbb{C}^n$ is holomorphic.

Proposition 2.10. *For* $a \in B^n_{\mathbb{C}}$, *the following are true.*

(i) $\sigma_a(0) = a$ *and* $\sigma_a(a) = 0$.
(ii) *The identity*

$$1 - \big\langle \sigma_a(z), \sigma_a(w)\big\rangle_{\mathbb{C}} = \frac{(1 - |a|^2)(1 - \langle z, w\rangle_{\mathbb{C}})}{(1 - \langle z, a\rangle_{\mathbb{C}})(1 - \langle a, w\rangle_{\mathbb{C}})} \tag{2.31}$$

holds for all $z, w \in \mathrm{Cl}(B^n_{\mathbb{C}})$. *In particular,*

$$1 - |\sigma_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \langle z, a\rangle_{\mathbb{C}}|^2}, \quad z \in \mathrm{Cl}\big(B^n_{\mathbb{C}}\big). \tag{2.32}$$

(iii) *For any* $b \in B^n_{\mathbb{C}}$, *we have* $\langle \psi(z), \psi(w)\rangle_{\mathbb{C}} = \langle z, w\rangle_{\mathbb{C}}$, *where* $\psi = \sigma_{\sigma_a(b)} \circ \sigma_a \circ \sigma_b$, $z, w \in \mathrm{Cl}(B^n_{\mathbb{C}})$.
(iv) *For* $f \in C^2(B^n_{\mathbb{C}})$, *we have*

$$(Lf)(a) = \Delta(f \circ \sigma_a)(0). \tag{2.33}$$

(v) σ_a *is an involutory isometry of* $\mathbb{H}^n_{\mathbb{C}}$.
(vi) *The relation*

$$\sigma_a\left(\frac{a}{1 + \mu_a}\right) = \frac{a}{1 + \mu_a}$$

holds. Moreover, σ_a *fixes exactly one point of* $B^n_{\mathbb{C}}$ *and no point of* $\mathbb{S}^{2n-1}$.

Proof. The argument is similar to that of Proposition 2.4. As an example, let us prove formula (2.33). Denote by $f_1, \ldots, f_n$ the coordinates of the mapping σ_a. Using the chain rule and bearing in mind that the functions $f_1, \ldots, f_n$ are holomorphic, we obtain

$$\Delta(f \circ \sigma_a)(0) = 4 \sum_{i,k=1}^{n} \frac{\partial^2 f}{\partial z_i \partial \overline{z}_k}(a) \sum_{m=1}^{n} \frac{\partial f_i}{\partial z_m}(0) \overline{\frac{\partial f_k}{\partial z_m}}(0). \tag{2.34}$$

It follows from (2.30) that

$$\begin{aligned}\sigma_a(z) &= \left(\sum_{k=0}^{\infty} \langle z, a\rangle_{\mathbb{C}}^k\right)\big(a - P_a(z) - \mu_a Q_a(z)\big)\\ &= a - \mu_a z + \frac{\mu_a}{1+\mu_a}\langle z, a\rangle_{\mathbb{C}} a + \cdots ,\end{aligned}$$

where the missing terms have z-degree 2 or more. Therefore,

$$\frac{\partial f_i}{\partial z_m}(0) = -\mu_a \delta_{i,m} + \frac{\mu_a}{1+\mu_a} a_i \overline{a}_m. \tag{2.35}$$

Relations (2.34) and (2.35) and Proposition 2.8 imply (2.33). □

Remark 2.2. From assertions (i) and (v) it follows that we now have enough isometries to map any point of the space $\mathbb{H}_{\mathbb{C}}^n$ to any other (see the proof of Corollary 2.1). In particular, $\mathbb{H}_{\mathbb{C}}^n$ is a complete space (see Theorem 1.4). Moreover, part (vi) shows that the space $\mathbb{H}_{\mathbb{C}}^n$ is a Hermitian symmetric space (see Helgason [115], Chap. 8, Sect. 4), since $\sigma_{2a/(1+|a|^2)}$ is the holomorphic symmetry of $\mathbb{H}_{\mathbb{C}}^n$ at the point a.

Next, we wish to compute the distance between two points $z, w \in \mathbb{H}_{\mathbb{C}}^n$.

Proposition 2.11. *The relation*

$$d(z, w) = \frac{1}{2}\log\left(\frac{|1 - \langle z, w\rangle_{\mathbb{C}}| + \sqrt{|z-w|^2 + |\langle z, w\rangle_{\mathbb{C}}|^2 - |z|^2|w|^2}}{|1 - \langle z, w\rangle_{\mathbb{C}}| - \sqrt{|z-w|^2 + |\langle z, w\rangle_{\mathbb{C}}|^2 - |z|^2|w|^2}}\right)$$

holds for all $z, w \in \mathbb{H}_{\mathbb{C}}^n$.

Proof. Let $z = (r, 0, \ldots, 0)$, where $0 < r < 1$, and let $\gamma(t)$, $0 \leqslant t \leqslant 1$, be an arbitrary piecewise smooth curve joining the points 0 and z in $B_{\mathbb{C}}^n$. Relation (2.24) implies that

$$\begin{aligned}l(\gamma) &= \int_0^1 \left(\frac{|\gamma'(t)|^2}{1 - |\gamma(t)|^2} + \frac{|\langle \gamma(t), \gamma'(t)\rangle_{\mathbb{C}}|^2}{(1 - |\gamma(t)|^2)^2}\right)^{1/2} \mathrm{d}t\\ &\geqslant \int_0^1 \left(\frac{|\gamma'(t)|^2}{1 - |\gamma(t)|^2} + \frac{\langle \gamma(t), \gamma'(t)\rangle_{\mathbb{R}}^2}{(1 - |\gamma(t)|^2)^2}\right)^{1/2} \mathrm{d}t\\ &\geqslant d_1(0, z),\end{aligned}$$

where $d_1(0, z)$ is the distance between 0 and z on the space $\mathbb{H}^{2n}_{\mathbb{R}}$ (see Sect. 2.1). We deduce the inequality

$$l(\gamma) \geqslant \frac{1}{2} \log \frac{1+|z|}{1-|z|},$$

which becomes an equality for the line segment joining 0 and z. Hence,

$$d(0, z) = \frac{1}{2} \log \frac{1+|z|}{1-|z|}. \tag{2.36}$$

In addition, Proposition 2.9 implies that (2.36) holds for any $z \in B^n_{\mathbb{C}}$. The rest follows from (2.32), since $\sigma_a \in I(\mathbb{H}^n_{\mathbb{C}})$ for $a \in B^n_{\mathbb{C}}$. □

Theorem 2.2. *The complex hyperbolic space* SU(n, 1)/ S(U(n) × U(1)) *of maximal sectional curvature* -1 *is holomorphically isometric to the space* $\mathbb{H}^n_{\mathbb{C}}$.

Proof. The complex hyperbolic space SU(n, 1)/ S(U(n) × U(1)) of maximal sectional curvature -1 is a simply connected complete Kaehler manifold of constant holomorphic sectional curvature -4 (see Sect. 1.3 and Yano and Bochner [264], Chap. 8, Theorem 8.3). This, together with Theorem 1.7, implies the required assertion. □

2.3 The Quaternionic Hyperbolic Space Sp(n, 1)/ Sp(n) × Sp(1)

The construction in Sect. 2.2 generalizes to the case of the space Sp(n, 1)/ Sp(n) × Sp(1) straightforwardly.

Let $B^n_{\mathbb{Q}} = \{q \in \mathbb{Q}^n : \ |q| < 1\}$. For $q = (q_1, \dots, q_n) \in B^n_{\mathbb{Q}}$, we put

$$Q_{ij}(q) = \frac{(1-|q|^2)\delta_{i,j} + \overline{q}_i q_j}{(1-|q|^2)^2}, \quad i, j \in \{1, \dots, n\}.$$

As usual, we identify $\mathbb{Q}^n$ with $\mathbb{C}^{2n}$ via the correspondence

$$q = (q_1, \dots, q_n) \leftrightarrow z = (z_1, \dots, z_{2n}), \tag{2.37}$$

where $q_i = z_i + z_{n+i}\mathbf{i}_2$ for $1 \leqslant i \leqslant n$ (see (1.14)). Then $B^n_{\mathbb{Q}} = \{z \in \mathbb{C}^{2n} : \ |z| < 1\}$, and the matrix $\|Q_{ij}(q)\|^n_{i,j=1}$ passes into the matrix $\|h_{kl}(z)\|^{2n}_{k,l=1}$, where

$$\begin{aligned} h_{ij}(z) &= \frac{(1-|z|^2)\delta_{i,j} + \overline{z}_i z_j + z_{n+i}\overline{z}_{n+j}}{(1-|z|^2)^2}, \\ h_{i,n+j}(z) &= \frac{\overline{z}_i z_{n+j} - z_{n+i}\overline{z}_j}{(1-|z|^2)^2}, \\ h_{n+i,j}(z) &= \frac{\overline{z}_{n+i} z_j - z_i\overline{z}_{n+j}}{(1-|z|^2)^2}, \end{aligned} \tag{2.38}$$

$$h_{n+i,n+j}(z) = \frac{(1-|z|^2)\delta_{i,j} + z_i\overline{z}_j + \overline{z}_{n+i}z_{n+j}}{(1-|z|^2)^2}$$

for $i, j \in \{1, \ldots, n\}$ (see (1.18)).

For any $w = (w_1, \ldots, w_{2n}) \in \mathbb{C}^{2n}$, we have the relation

$$\sum_{k,l=1}^{2n} h_{kl}(z) w_k \overline{w}_l = \frac{|w|^2}{1-|z|^2} + \frac{|\langle z, w\rangle_{\mathbb{Q}}|^2}{(1-|z|^2)^2}, \tag{2.39}$$

that is, the matrix $\|h_{kl}(z)\|_{k,l=1}^{2n}$ induces the structure of a Riemannian manifold on $B^n_{\mathbb{Q}}$. Denote this manifold by $\mathbb{H}^n_{\mathbb{Q}}$.

We shall now obtain analogues of the results from Sects. 2.1 and 2.2 for the space $\mathbb{H}^n_{\mathbb{Q}}$.

Proposition 2.12. *The group* $\mathrm{O}_{\mathbb{Q}}(n)$ *is a subgroup of* $I(\mathbb{H}^n_{\mathbb{Q}})$. *In particular,* $\mathrm{Sp}(n) \subset I(\mathbb{H}^n_{\mathbb{Q}})$.

We omit the proof of Proposition 2.12, because it is entirely analogous to that of Proposition 2.3.

Proposition 2.13. *The Riemannian measure on* $\mathbb{H}^n_{\mathbb{Q}}$ *has the form*

$$\mathrm{d}\mu(z) = \frac{\mathrm{d}m_{2n}(z)}{(1-|z|^2)^{2n+2}}.$$

Proof. Let $h(z)$ be the determinant of the matrix (2.38). The invariance of μ with respect to $I(\mathbb{H}^n_{\mathbb{Q}})$ and Proposition 2.12 imply that h is a radial function on the ball $B^n_{\mathbb{Q}}$. It follows that

$$h(z) = h(|z|, 0, \ldots, 0) = \frac{1}{(1-|z|^2)^{2n+2}}, \tag{2.40}$$

which brings us to the required assertion. □

Proposition 2.14. *The Laplace–Beltrami operator* L *on* $\mathbb{H}^n_{\mathbb{Q}}$ *acts on a function* $f \in C^2(B^n_{\mathbb{Q}})$ *as follows:*

$$\begin{aligned}(Lf)(z) = 4\big(1-|z|^2\big)\Bigg(&\sum_{i,j=1}^{n}\bigg((\delta_{i,j} - z_i\overline{z}_j - \overline{z}_{n+i}z_{n+j})\frac{\partial^2 f}{\partial z_i \partial \overline{z}_j}(z)\\ &+ (\overline{z}_i z_{n+j} - z_{n+i}\overline{z}_j)\frac{\partial^2 f}{\partial z_{n+i}\partial \overline{z}_j}(z) + (\overline{z}_{n+i}z_j - z_i\overline{z}_{n+j})\frac{\partial^2 f}{\partial z_i \partial \overline{z}_{n+j}}(z)\\ &+ (\delta_{i,j} - \overline{z}_i z_j - z_{n+i}\overline{z}_{n+j})\frac{\partial^2 f}{\partial z_{n+i}\partial \overline{z}_{n+j}}(z)\bigg)\\ &+ \sum_{k=1}^{2n}\bigg(z_k\frac{\partial f}{\partial z_k}(z) + \overline{z}_k\frac{\partial f}{\partial \overline{z}_k}(z)\bigg)\Bigg).\end{aligned} \tag{2.41}$$

In particular, if f has the form $f(z) = f_0(|z|)$, then

$$(Lf)(z) = \big(1 - |z|^2\big)^2 f_0''(|z|) + \frac{1 - |z|^2}{|z|}\big(4n - 1 + |z|^2\big) f_0'(|z|). \tag{2.42}$$

Proof. As above, let $\|h^{kl}\|_{k,l=1}^{2n}$ be the inverse matrix of $\|h_{kl}\|_{k,l=1}^{2n}$. Then

$$\begin{aligned}
h^{ij}(z) &= (\delta_{i,j} - \overline{z}_i z_j - z_{n+i}\overline{z}_{n+j})\big(1 - |z|^2\big),\\
h^{i,n+j}(z) &= (z_{n+i}\overline{z}_j - \overline{z}_i z_{n+j})\big(1 - |z|^2\big),\\
h^{n+i,j}(z) &= (z_i\overline{z}_{n+j} - \overline{z}_{n+i} z_j)\big(1 - |z|^2\big),\\
h^{n+i,n+j}(z) &= (\delta_{i,j} - z_i\overline{z}_j - \overline{z}_{n+i} z_{n+j})\big(1 - |z|^2\big)
\end{aligned}$$

for $i, j \in \{1, \dots, n\}$. Hence, we deduce (2.41) from (1.48) and (2.40). □

Our further purpose is to find an analog of the mapping (2.30) for the space $\mathbb{H}^n_{\mathbb{Q}}$. Take $A \in \mathbb{Q}^n$ such that $|A| < 1$. Let

$$\sigma_A(q) = (1 - \langle q, A\rangle_{\mathbb{Q}})^{-1}\big(A - P_A(q) - \mu_A Q_A(q)\big), \quad q \in \mathrm{Cl}\big(B^n_{\mathbb{Q}}\big),$$

where $\mu_A = \sqrt{1 - |A|^2}$,

$$\begin{aligned}
P_A(q) &= \begin{cases} \langle q, A\rangle_{\mathbb{Q}} |A|^{-2} A & \text{if } A \neq 0,\\ 0 & \text{if } A = 0,\end{cases}\\
Q_A(q) &= q - P_A(q).
\end{aligned}$$

Suppose that $A \leftrightarrow a$, $q \leftrightarrow z$, and $\sigma_A(q) \leftrightarrow w$ under identification (2.37). Then we put

$$\sigma_a(z) = w. \tag{2.43}$$

Proposition 2.15. *Let $a \in B^n_{\mathbb{Q}}$. Then*

(i) $\sigma_a(0) = a$ *and* $\sigma_a(a) = 0$.
(ii) *The identity*

$$\begin{aligned}
1 - &\big\langle \sigma_a(z), \sigma_a(w)\big\rangle_{\mathbb{Q}}\\
&= \big(1 - \langle z, a\rangle_{\mathbb{Q}}\big)^{-1}\big(1 - \langle z, w\rangle_{\mathbb{Q}}\big)\big(1 - \langle a, w\rangle_{\mathbb{Q}}\big)^{-1}\big(1 - |a|^2\big)
\end{aligned} \tag{2.44}$$

holds for all $z, w \in \mathrm{Cl}(B^n_{\mathbb{Q}})$. In particular,

$$1 - |\sigma_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \langle z, a\rangle_{\mathbb{Q}}|^2}, \quad z \in \mathrm{Cl}\big(B^n_{\mathbb{Q}}\big). \tag{2.45}$$

(iii) *For any* $b \in B^n_{\mathbb{Q}}$,

$$\big\langle \psi(z), \psi(w)\big\rangle_{\mathbb{Q}} = \big(1 - \langle b, a\rangle_{\mathbb{Q}}\big)^{-1} \langle z, w\rangle_{\mathbb{Q}} \big(1 - \langle b, a\rangle_{\mathbb{Q}}\big), \quad z, w \in \mathrm{Cl}\big(B^n_{\mathbb{Q}}\big),$$

where $\psi = \sigma_{\sigma_a(b)} \circ \sigma_a \circ \sigma_b$.

(iv) *For each* $f \in C^2(B^n_{\mathbb{Q}})$,

$$(Lf)(a) = \Delta(f \circ \sigma_a)(0). \tag{2.46}$$

(v) σ_a *is an involutory isometry of* $\mathbb{H}^n_{\mathbb{Q}}$.

(vi) *The relation*

$$\sigma_a\left(\frac{a}{1 + \sqrt{1 - |a|^2}}\right) = \frac{a}{1 + \sqrt{1 - |a|^2}}$$

holds. Moreover, σ_a *fixes exactly one point of* $B^n_{\mathbb{Q}}$ *and no point of* $\mathbb{S}^{4n-1}$.

Proof. The argument is similar to that of Proposition 2.4. We restrict ourselves here only to verification of formula (2.46).

Denote by $f_1, \ldots, f_{2n}$ the coordinates of the mapping σ_a. The chain rule gives

$$\begin{aligned}
\frac{1}{4}\Delta(f \circ \sigma_a)(0) = {} & \sum_{k,l,m=1}^{2n} \left(\frac{\partial^2 f}{\partial z_k \partial z_l}(a) \frac{\partial f_l}{\partial \overline{z}_m}(0) + \frac{\partial^2 f}{\partial z_k \partial \overline{z}_l}(a) \frac{\partial \overline{f}_l}{\partial \overline{z}_m}(0)\right) \frac{\partial f_k}{\partial z_m}(0) \\
& + \sum_{k,l,m=1}^{2n} \left(\frac{\partial^2 f}{\partial \overline{z}_k \partial z_l}(a) \frac{\partial f_l}{\partial \overline{z}_m}(0) + \frac{\partial^2 f}{\partial \overline{z}_k \partial \overline{z}_l}(a) \frac{\partial \overline{f}_l}{\partial \overline{z}_m}(0)\right) \frac{\partial \overline{f}_k}{\partial z_m}(0) \\
& + \sum_{k,l=1}^{2n} \left(\frac{\partial f}{\partial z_k}(a) \frac{\partial^2 f_k}{\partial z_l \partial \overline{z}_l}(0) + \frac{\partial f}{\partial \overline{z}_k}(a) \frac{\partial^2 \overline{f}_k}{\partial z_l \partial \overline{z}_l}(0)\right).
\end{aligned} \tag{2.47}$$

The definition of σ_a shows that

$$\begin{aligned}
|1 - \langle z, a\rangle_{\mathbb{Q}}|^2 f_i(z) &= [a, z]_{\mathbb{C}}\big(\mu_a \overline{z}_{n+i} + (\eta_a - 1)\overline{a}_{n+i}\big) + \langle a, z\rangle_{\mathbb{C}}(\mu_a z_i - a_i) \\
&\quad - \langle z, a\rangle_{\mathbb{C}} \eta_a a_i + |\langle z, a\rangle_{\mathbb{Q}}|^2 \eta_a a_i + a_i - \mu_a z_i, \\
|1 - \langle z, a\rangle_{\mathbb{Q}}|^2 f_{n+i}(z) &= [a, z]_{\mathbb{C}}\big(-\mu_a \overline{z}_i + (1 - \eta_a)\overline{a}_i\big) + \langle a, z\rangle_{\mathbb{C}}(\mu_a z_{n+i} - a_{n+i}) \\
&\quad - \langle z, a\rangle_{\mathbb{C}} \eta_a a_{n+i} + |\langle z, a\rangle_{\mathbb{Q}}|^2 \eta_a a_{n+i} + a_{n+i} - \mu_a z_{n+i}
\end{aligned}$$

for $1 \leqslant i \leqslant n$, where $\mu_a = \sqrt{1 - |a|^2}$, $\eta_a = (1 + \mu_a)^{-1}$, and $a_1, \ldots, a_{2n}$ are the coordinates of a. Hence, for $k, l \in \{1, \ldots, 2n\}$ and $i, j \in \{1, \ldots, n\}$, we have

$$\frac{\partial f_k}{\partial \overline{z}_l}(0) = 0, \qquad \frac{\partial f_i}{\partial z_j}(0) = (1 - \eta_a)(a_i \overline{a}_j + \overline{a}_{n+i} a_{n+j}) - \mu_a \delta_{i,j},$$

$$\frac{\partial f_i}{\partial z_{n+j}}(0) = (1-\eta_a)(a_i\overline{a}_{n+j} - a_j\overline{a}_{n+i}),$$
$$\frac{\partial f_{n+i}}{\partial z_j}(0) = (1-\eta_a)(a_{n+i}\overline{a}_j - \overline{a}_i a_{n+j}),$$
$$\frac{\partial f_{n+i}}{\partial z_{n+j}}(0) = (1-\eta_a)(\overline{a}_i a_j + a_{n+i}\overline{a}_{n+j}) - \mu_a\delta_{i,j},$$
$$\frac{\partial^2 f_i}{\partial z_j\partial\overline{z}_j}(0) = (\eta_a-1)\big(a_i|a_{n+j}|^2 - a_j a_{n+j}\overline{a}_{n+i}\big),$$
$$\frac{\partial^2 f_i}{\partial z_{n+j}\partial\overline{z}_{n+j}}(0) = (\eta_a-1)\big(a_i|a_j|^2 + a_j a_{n+j}\overline{a}_{n+i}\big) + \mu_a\delta_{i,j}a_j,$$
$$\frac{\partial^2 f_{n+i}}{\partial z_j\partial\overline{z}_j}(0) = (\eta_a-1)\big(a_{n+i}|a_{n+j}|^2 + \overline{a}_i a_j a_{n+j}\big) + \mu_a\delta_{i,j}a_{n+j},$$
$$\frac{\partial^2 f_{n+i}}{\partial z_{n+j}\partial\overline{z}_{n+j}}(0) = (\eta_a-1)\big(a_{n+i}|a_j|^2 - \overline{a}_i a_j a_{n+j}\big).$$

Combining this with (2.47) and using (2.41), we complete the proof of (2.46). □

Proposition 2.16. *The distance between two points $z, w \in \mathbb{H}^n_{\mathbb{Q}}$ is given by the formula*

$$d(z,w) = \frac{1}{2}\log\left(\frac{|1-\langle z,w\rangle_{\mathbb{Q}}| + \sqrt{|z-w|^2 + |\langle z,w\rangle_{\mathbb{Q}}|^2 - |z|^2|w|^2}}{|1-\langle z,w\rangle_{\mathbb{Q}}| - \sqrt{|z-w|^2 + |\langle z,w\rangle_{\mathbb{Q}}|^2 - |z|^2|w|^2}}\right). \quad (2.48)$$

Proof. Using (2.39) and repeating the arguments in the proof of Proposition 2.11, we derive (2.48). □

Theorem 2.3. *The quaternionic hyperbolic space* Sp$(n,1)/$Sp$(n)\times$Sp(1) *of maximal sectional curvature -1 is isometric to the space $\mathbb{H}^n_{\mathbb{Q}}$.*

Proof. We first prove that the space $\mathbb{H}^n_{\mathbb{Q}}$ is a noncompact two-point homogeneous space. Let $x, y, z, w \in \mathbb{H}^n_{\mathbb{Q}}$ and assume that $d(x,y) = d(z,w)$. Then

$$d\big(0,\sigma_x(y)\big) = d\big(0,\sigma_z(w)\big), \quad (2.49)$$

since $\sigma_x, \sigma_z \in I(\mathbb{H}^n_{\mathbb{Q}})$. Relations (2.49) and (2.48) yield $|\sigma_x(y)| = |\sigma_z(w)|$. Therefore, $\sigma_x(y) = \tau\sigma_z(w)$ for some $\tau \in \mathrm{Sp}(n)$. We set

$$\varphi = \sigma_x \circ \tau \circ \sigma_z.$$

In view of Propositions 2.12 and 2.15, we have $\varphi \in I(\mathbb{H}^n_{\mathbb{Q}})$, $\varphi(z) = x$, and $\varphi(w) = y$. This together with (2.48) implies that $\mathbb{H}^n_{\mathbb{Q}}$ is a noncompact two-point homogeneous space. Hence, $\mathbb{H}^n_{\mathbb{Q}}$ is isometric to one of the spaces $\mathbb{R}^l$, $\mathrm{SO}_0(l,1)/\mathrm{SO}(l)$,

$\mathrm{SU}(l, 1)/\,\mathrm{S}(\mathrm{U}(l) \times \mathrm{U}(1))$, $\mathrm{Sp}(l, 1)/\,\mathrm{Sp}(l) \times \mathrm{Sp}(1)$, or $F_4^*/\,\mathrm{Spin}(9)$ (see Sect. 1.3). We conclude from (2.42) and (2.48) that the radial part L_0 of the Laplace–Beltrami operator L has the form

$$L_0 = \frac{\partial^2}{\partial t^2} + \big((4n-1)\coth t + 3\tanh t\big)\frac{\partial}{\partial t}.$$

From this result and from (1.61)–(1.63) it follows that the space $\mathbb{H}^n_{\mathbb{Q}}$ is a quaternionic hyperbolic space of real dimension $4n$ and maximal sectional curvature -1. □

2.4 The Cayley Hyperbolic Plane $F_4^*/\,\mathrm{Spin}(9)$

The foregoing constructions break down for the case $F_4^*/\,\mathrm{Spin}(9)$ because the algebra $\mathbb{C}a$ is nonassociative. Nevertheless, it turns out that there is an analogue of Theorems 2.1–2.3 for the space $F_4^*/\,\mathrm{Spin}(9)$.

Let $x = (x_1, \dots, x_{16}) \in \mathbb{R}^{16}$, $y = (y_1, \dots, y_{16}) \in \mathbb{R}^{16}$. Consider the form $\Phi_{\mathbb{C}a}(x, y)$ given by (1.16). For $i, j \in \{1, \dots, 16\}$, we set

$$a_{ij}(x) = \delta_{i,j}\big(1-|x|^2\big)+\frac{1}{2}\frac{\partial^2}{\partial y_i \partial y_j}\big(\Phi_{\mathbb{C}a}(x, y)\big), \qquad g_{ij}(x) = \frac{a_{ij}(x)}{(1-|x|^2)^2}. \quad (2.50)$$

We note that

$$\|a_{ij}\|_{i,j=1}^{16} = \left\| \begin{matrix} G_1 & G_2 \\ G_3 & G_4 \end{matrix} \right\|,$$

where G_1, G_2, G_3, G_4 are defined in Table 2.1 below. Since

$$\sum_{i,j=1}^{16} g_{ij}(x) y_i y_j = \frac{|y|^2}{1-|x|^2} + \frac{\Phi_{\mathbb{C}a}(x, y)}{(1-|x|^2)^2}, \quad (2.51)$$

it follows from (1.12) that the matrix $\|g_{ij}\|_{i,j=1}^{16}$ induces the structure of a Riemannian manifold on $B^{16}_{\mathbb{R}}$. Denote this manifold by $\mathbb{H}^2_{\mathbb{C}a}$. In this section we want to show that the space $\mathbb{H}^2_{\mathbb{C}a}$ is isometric to the Cayley hyperbolic plane. Our first purpose is to find the inverse matrix $\|g^{ij}\|_{i,j=1}^{16}$.

We introduce the matrix

$$b_{ij}(x) = \delta_{i,j}\big(2-|x|^2\big) - a_{ij}(x), \quad i, j \in \{1, \dots, 16\}.$$

Then

$$\|b_{ij}\|_{i,j=1}^{16} = \left\| \begin{matrix} G_5 & G_6 \\ G_7 & G_8 \end{matrix} \right\|,$$

$$G_1 = \begin{Vmatrix} 1-p_{10} & p_1 & 0 & p_2 & 0 & p_3 & 0 & p_4 \\ p_1 & 1-p_9 & p_2 & 0 & p_3 & 0 & p_4 & 0 \\ 0 & p_2 & 1-p_{10} & -p_1 & 0 & -p_4 & 0 & p_3 \\ p_2 & 0 & -p_1 & 1-p_9 & p_4 & 0 & -p_3 & 0 \\ 0 & p_3 & 0 & p_4 & 1-p_{10} & -p_1 & 0 & -p_2 \\ p_3 & 0 & -p_4 & 0 & -p_1 & 1-p_9 & p_2 & 0 \\ 0 & p_4 & 0 & -p_3 & 0 & p_2 & 1-p_{10} & -p_1 \\ p_4 & 0 & p_3 & 0 & -p_2 & 0 & -p_1 & 1-p_9 \end{Vmatrix}$$

$$G_2 = \begin{Vmatrix} 0 & p_5 & 0 & p_6 & 0 & p_7 & 0 & p_8 \\ p_5 & 0 & p_6 & 0 & p_7 & 0 & p_8 & 0 \\ 0 & -p_6 & 0 & p_5 & 0 & -p_8 & 0 & p_7 \\ p_6 & 0 & -p_5 & 0 & p_8 & 0 & -p_7 & 0 \\ 0 & -p_7 & 0 & p_8 & 0 & p_5 & 0 & -p_6 \\ p_7 & 0 & -p_8 & 0 & -p_5 & 0 & p_6 & 0 \\ 0 & p_8 & 0 & p_7 & 0 & -p_6 & 0 & -p_5 \\ -p_8 & 0 & -p_7 & 0 & p_6 & 0 & p_5 & 0 \end{Vmatrix}$$

$$G_3 = \begin{Vmatrix} 0 & p_5 & 0 & p_6 & 0 & p_7 & 0 & -p_8 \\ p_5 & 0 & -p_6 & 0 & -p_7 & 0 & p_8 & 0 \\ 0 & p_6 & 0 & -p_5 & 0 & -p_8 & 0 & -p_7 \\ p_6 & 0 & p_5 & 0 & p_8 & 0 & p_7 & 0 \\ 0 & p_7 & 0 & p_8 & 0 & -p_5 & 0 & p_6 \\ p_7 & 0 & -p_8 & 0 & p_5 & 0 & -p_6 & 0 \\ 0 & p_8 & 0 & -p_7 & 0 & p_6 & 0 & p_5 \\ p_8 & 0 & p_7 & 0 & -p_6 & 0 & -p_5 & 0 \end{Vmatrix}$$

$$G_4 = \begin{Vmatrix} 1-p_{10} & -p_1 & 0 & -p_2 & 0 & -p_3 & 0 & p_4 \\ -p_1 & 1-p_9 & p_2 & 0 & p_3 & 0 & -p_4 & 0 \\ 0 & p_2 & 1-p_{10} & -p_1 & 0 & p_4 & 0 & p_3 \\ -p_2 & 0 & -p_1 & 1-p_9 & -p_4 & 0 & -p_3 & 0 \\ 0 & p_3 & 0 & -p_4 & 1-p_{10} & -p_1 & 0 & -p_2 \\ -p_3 & 0 & p_4 & 0 & -p_1 & 1-p_9 & p_2 & 0 \\ 0 & -p_4 & 0 & -p_3 & 0 & p_2 & 1-p_{10} & -p_1 \\ p_4 & 0 & p_3 & 0 & -p_2 & 0 & -p_1 & 1-p_9 \end{Vmatrix}$$

Table 2.1 The matrices G_1, G_2, G_3, G_4

where G_5, G_6, G_7, G_8 are defined in Table 2.2. A direct check shows that

$$\sum_{j=1}^{16} x_j b_{ij}(x) = x_i\bigl(1-|x|^2\bigr), \tag{2.52}$$

$$\sum_{j=1}^{16} \frac{\partial b_{ij}}{\partial x_j}(x) = -10x_i \tag{2.53}$$

$$G_5 = \begin{Vmatrix} 1-p_9 & -p_1 & 0 & -p_2 & 0 & -p_3 & 0 & -p_4 \\ -p_1 & 1-p_{10} & -p_2 & 0 & -p_3 & 0 & -p_4 & 0 \\ 0 & -p_2 & 1-p_9 & p_1 & 0 & p_4 & 0 & -p_3 \\ -p_2 & 0 & p_1 & 1-p_{10} & -p_4 & 0 & p_3 & 0 \\ 0 & -p_3 & 0 & -p_4 & 1-p_9 & p_1 & 0 & p_2 \\ -p_3 & 0 & p_4 & 0 & p_1 & 1-p_{10} & -p_2 & 0 \\ 0 & -p_4 & 0 & p_3 & 0 & -p_2 & 1-p_9 & p_1 \\ -p_4 & 0 & -p_3 & 0 & p_2 & 0 & p_1 & 1-p_{10} \end{Vmatrix}$$

$$G_6 = \begin{Vmatrix} 0 & -p_5 & 0 & -p_6 & 0 & -p_7 & 0 & -p_8 \\ -p_5 & 0 & -p_6 & 0 & -p_7 & 0 & -p_8 & 0 \\ 0 & p_6 & 0 & -p_5 & 0 & p_8 & 0 & -p_7 \\ -p_6 & 0 & p_5 & 0 & -p_8 & 0 & p_7 & 0 \\ 0 & p_7 & 0 & -p_8 & 0 & -p_5 & 0 & p_6 \\ -p_7 & 0 & p_8 & 0 & p_5 & 0 & -p_6 & 0 \\ 0 & -p_8 & 0 & -p_7 & 0 & p_6 & 0 & p_5 \\ p_8 & 0 & p_7 & 0 & -p_6 & 0 & -p_5 & 0 \end{Vmatrix}$$

$$G_7 = \begin{Vmatrix} 0 & -p_5 & 0 & -p_6 & 0 & -p_7 & 0 & p_8 \\ -p_5 & 0 & p_6 & 0 & p_7 & 0 & -p_8 & 0 \\ 0 & -p_6 & 0 & p_5 & 0 & p_8 & 0 & p_7 \\ -p_6 & 0 & -p_5 & 0 & -p_8 & 0 & -p_7 & 0 \\ 0 & -p_7 & 0 & -p_8 & 0 & p_5 & 0 & -p_6 \\ -p_7 & 0 & p_8 & 0 & -p_5 & 0 & p_6 & 0 \\ 0 & -p_8 & 0 & p_7 & 0 & -p_6 & 0 & -p_5 \\ -p_8 & 0 & -p_7 & 0 & p_6 & 0 & p_5 & 0 \end{Vmatrix}$$

$$G_8 = \begin{Vmatrix} 1-p_9 & p_1 & 0 & p_2 & 0 & p_3 & 0 & -p_4 \\ p_1 & 1-p_{10} & -p_2 & 0 & -p_3 & 0 & p_4 & 0 \\ 0 & -p_2 & 1-p_9 & p_1 & 0 & -p_4 & 0 & -p_3 \\ p_2 & 0 & p_1 & 1-p_{10} & p_4 & 0 & p_3 & 0 \\ 0 & -p_3 & 0 & p_4 & 1-p_9 & p_1 & 0 & p_2 \\ p_3 & 0 & -p_4 & 0 & p_1 & 1-p_{10} & -p_2 & 0 \\ 0 & p_4 & 0 & p_3 & 0 & -p_2 & 1-p_9 & p_1 \\ -p_4 & 0 & -p_3 & 0 & p_2 & 0 & p_1 & 1-p_{10} \end{Vmatrix}$$

Table 2.2 The matrices G_5, G_6, G_7, G_8

for $i \in \{1, \dots, 16\}$. In addition,

$$\sum_{i,j=1}^{16} b_{ij}(x) y_i y_j = |y|^2 - \Phi_{\mathbb{C}a}(x, y). \tag{2.54}$$

Proposition 2.17. *The following equality is valid:*

$$g^{ij}(x) = \big(1 - |x|^2\big) b_{ij}(x), \quad i, j \in \{1, \dots, 16\}. \tag{2.55}$$

Proof. Let $i, j \in \{1, \dots, 16\}$ and assume that $i \neq j$. Then using Tables 2.1 and 2.2, we find

$$\sum_{k=1}^{16} a_{ik} b_{kj} = 0. \tag{2.56}$$

Next, under the identification of $\mathbb{R}^{16}$ with $\mathbb{C}a^2$ via the correspondence (1.15), we have

$$\begin{aligned} z_1 z_2 = p_1(x) + p_5(x)\mathbf{i}_1 + p_3(x)\mathbf{i}_2 + p_7(x)\mathbf{i}_3 + p_2(x)\mathbf{i}_4 + p_6(x)\mathbf{i}_5 \\ + p_4(x)\mathbf{i}_6 + p_8(x)\mathbf{i}_7. \end{aligned} \tag{2.57}$$

It follows from (2.57) that

$$\sum_{k=1}^{8} p_k^2 = p_9 p_{10}. \tag{2.58}$$

Therefore,

$$\sum_{k=1}^{16} a_{ik}(x) b_{ki}(x) = \big(1 - p_9(x)\big)\big(1 - p_{10}(x)\big) - \sum_{k=1}^{8} p_k^2(x) = 1 - |x|^2. \tag{2.59}$$

Combining (2.56) with (2.59), we obtain (2.55). □

Corollary 2.4. *The relation*

$$\sum_{i,j=1}^{16} g_{ij}(x) g^{ij}(y) = \frac{8(1-|y|^2)}{(1-|x|^2)^2}\Big(\big(1-|x|^2\big)\big(1-|y|^2\big) + 1 - \Phi_{\mathbb{C}a}(x,y)\Big) \tag{2.60}$$

holds.

Proof. Let $i \in \{1, 3, \dots, 15\}$. Then

$$\begin{aligned} \sum_{j=1}^{16} a_{ij}(x) b_{ij}(y) &= \big(1 - p_{10}(x)\big)\big(1 - p_9(y)\big) - \sum_{k=1}^{8} p_k(x) p_k(y) \\ &= \big(1 - p_{10}(x)\big)\big(1 - p_9(y)\big) \\ &\quad + \frac{1}{2}\big(p_9(x)p_9(y) + p_{10}(x)p_{10}(y) - \Phi_{\mathbb{C}a}(x,y)\big). \end{aligned} \tag{2.61}$$

Analogously, for $i \in \{2, 4, \dots, 16\}$, we have

$$\begin{aligned} \sum_{j=1}^{16} a_{ij}(x) b_{ij}(y) &= \big(1 - p_9(x)\big)\big(1 - p_{10}(y)\big) \\ &\quad + \frac{1}{2}\big(p_9(x)p_9(y) + p_{10}(x)p_{10}(y) - \Phi_{\mathbb{C}a}(x,y)\big). \end{aligned}$$

This together with (2.61) implies (2.60). □

We now proceed to the study of the isometry group $I(\mathbb{H}^2_{\mathbb{C}a})$.

Proposition 2.18. *The group* $\mathrm{O}_{\mathbb{C}a}(2)$ *is a subgroup of* $I(\mathbb{H}^2_{\mathbb{C}a})$.

The proof of this statement is the same as that of Proposition 2.3.

Next, let $a \in \mathbb{C}a^2$, $|a| < 1$. Define

$$\sigma_a = \begin{cases} \tau_{a/|a|} \circ \varkappa_a \circ \tau^{-1}_{a/|a|} & \text{if } a \neq 0, \\ \varkappa_a & \text{if } a = 0, \end{cases} \tag{2.62}$$

where $\tau_{a/|a|}$ is given by (1.23), and $\varkappa_a$ is the mapping acting by the formula

$$\varkappa_a(z_1, z_2) = \Big((z_1 - |a|)(|a|z_1 - 1)^{-1}, \sqrt{1 - |a|^2}(|a|\overline{z}_1 - 1)^{-1} z_2\Big), \tag{2.63}$$

$(z_1, z_2) \in \mathbb{C}a^2 \cap \mathrm{Cl}(B^{16}_{\mathbb{R}})$. We also put

$$\Psi_{\mathbb{C}a}(z, w) = \Phi_{\mathbb{C}a}(z, w) - 2\langle z, w\rangle_{\mathbb{R}} + 1, \quad z, w \in \mathbb{C}a^2,$$

where, as usual, $\langle z, w\rangle_{\mathbb{R}}$ is the Euclidean inner product of the vectors $z, w \in \mathbb{R}^{16}$ (see (1.15)). For $z = (z_1, z_2)$ and $w = (w_1, w_2)$, we have

$$\Psi_{\mathbb{C}a}(z, w) = \begin{cases} \big|1 - (\overline{z}_1 w_2)\big(w_2^{-1} w_1\big) - z_2 \overline{w}_2\big|^2 & \text{if } w_2 \neq 0, \\ |1 - \overline{z}_1 w_1|^2 & \text{if } w_2 = 0. \end{cases} \tag{2.64}$$

It follows from (2.64), (1.12), and (1.24) that

$$\Psi_{\mathbb{C}a}(z, w) > 0 \tag{2.65}$$

for $z \in \mathrm{Cl}(B^{16}_{\mathbb{R}})$ and $w \in B^{16}_{\mathbb{R}}$.

Proposition 2.19. *For every* $a \in B^{16}_{\mathbb{R}}$, *the mapping* σ_a *possesses the following properties.*

(i) $\sigma_a(0) = a$ *and* $\sigma_a(a) = 0$.

(ii) *The identity*

$$\big(1 - |a|^2\big)^2 \Psi_{\mathbb{C}a}(z, w) = \Psi_{\mathbb{C}a}(z, a)\Psi_{\mathbb{C}a}(w, a)\Psi_{\mathbb{C}a}\big(\sigma_a(z), \sigma_a(w)\big) \tag{2.66}$$

holds for all $z, w \in \mathrm{Cl}(B^{16}_{\mathbb{R}})$. *In particular,*

$$1 - |\sigma_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{\Psi_{\mathbb{C}a}(z, a)}, \quad z \in \mathrm{Cl}\big(B^{16}_{\mathbb{R}}\big). \tag{2.67}$$

(iii) σ_a *is an involution.*

(iv) σ_a *is an isometry of the space* $\mathbb{H}^2_{\mathbb{C}a}$.

(v) *The relation*

$$\sigma_a\left(\frac{a}{1+\sqrt{1-|a|^2}}\right)=\frac{a}{1+\sqrt{1-|a|^2}}$$

holds. Moreover, σ_a fixes exactly one point of $B^{16}_{\mathbb{R}}$ and no point of $\mathbb{S}^{15}$.

Proof. Assertions (i), (iii), and (v) follow from (2.62) and (2.63) with the help of simple transformations.

We turn to the proof of (ii). Put

$$\begin{aligned}p(a,z,w)&=\Psi_{\mathbb{C}a}(z,|a|e_1)\Psi_{\mathbb{C}a}(w,|a|e_1)\Big(\big(1-|a|^2\big)^2\Psi_{\mathbb{C}a}(z,w)\\&\quad+2\big\langle h_a(z),h_a(w)\big\rangle_{\mathbb{R}}-\Psi_{\mathbb{C}a}(z,|a|e_1)\Psi_{\mathbb{C}a}(w,|a|e_1)\Big)\\&\quad-\Phi_{\mathbb{C}a}\big(h_a(z),h_a(w)\big),\end{aligned}$$

where $e_1=(1,0)\in\mathbb{C}a^2$, and the mapping $h_a\colon\mathbb{C}a^2\to\mathbb{C}a^2$ is defined as follows:

$$h_a(\xi,\eta)=\Big((\xi-|a|)\big(|a|\overline{\xi}-1\big),\sqrt{1-|a|^2}(|a|\xi-1)\eta\Big),\quad \xi,\eta\in\mathbb{C}a.$$

We write $p(a,z,w)$ in the form

$$p(a,z,w)=\sum_{k=0}^{8}c_k(z,w)|a|^k,$$

where $c_k(z,w)$ do not depend on a. By a direct calculation using (1.2)–(1.8) we find that $c_k(z,w)=0$ for $k=0,1,\dots,8$. Therefore,

$$\big(1-|a|^2\big)^2\Psi_{\mathbb{C}a}(z,w)=\Psi_{\mathbb{C}a}(z,|a|e_1)\Psi_{\mathbb{C}a}(w,|a|e_1)\Psi_{\mathbb{C}a}\big(\varkappa_a(z),\varkappa_a(w)\big).\quad(2.68)$$

Now we deduce (2.66) from (2.68) and (2.62), since $\tau_{a/|a|}\in\mathrm{O}_{\mathbb{C}a}(2)$ (see (1.23) and Example 1.3).

Let us prove assertion (iv). Under the identification (1.15), we have

$$\rho_a(x)\varkappa_a(x)=\big(\theta_{a,1}(x),\dots,\theta_{a,16}(x)\big),$$

where

$$\begin{aligned}\theta_{a,1}(x)&=|a|\big(1+p_9(x)\big)-\big(1+|a|^2\big)x_1,\\\theta_{a,2k-1}(x)&=-\big(1-|a|^2\big)x_{2k-1},\quad k\in\{2,3,\dots,8\},\\\theta_{a,2k}(x)&=\sqrt{1-|a|^2}\big(|a|p_k(x)-x_{2k}\big),\quad k\in\{1,2,\dots,8\},\\\rho_a(x)&=1-2|a|x_1+|a|^2p_9(x).\end{aligned}$$

For $i, j \in \{1, \dots, 16\}$ such that $i \neq j$ and $k \in \{1, \dots, 8\}$, we put

$$A_{ij}(x) = a_{ij}\big(\rho_a(x)\varkappa_a(x)\big),$$

$$A_{2k-1,2k-1}(x) = \rho_a^2(x) - \sum_{m=1}^{8} \theta_{a,2m}^2(x),$$

$$A_{2k,2k}(x) = \rho_a^2(x) - \sum_{m=1}^{8} \theta_{a,2m-1}^2(x).$$

Using Maple (see Heal, Hansen, and Rickard [114]), we obtain

$$\big(1 - |a|^2\big)^2 \rho_a^4(x) a_{kl}(x) = \sum_{i,j=1}^{16} A_{ij}(x)\left(\frac{\partial\theta_{a,i}}{\partial x_k}(x)\rho_a(x) - \theta_{a,i}(x)\frac{\partial\rho_a}{\partial x_k}(x)\right)$$
$$\times\left(\frac{\partial\theta_{a,j}}{\partial x_l}(x)\rho_a(x) - \theta_{a,j}(x)\frac{\partial\rho_a}{\partial x_l}(x)\right)$$

for $k, l \in \{1, \dots, 16\}$. Therefore,

$$g_{kl}(x) = \sum_{i,j=1}^{16} g_{ij}\big(\varkappa_a(x)\big)\frac{\partial\varkappa_{a,i}}{\partial x_k}(x)\frac{\partial\varkappa_{a,j}}{\partial x_l}(x), \quad x \in B_{\mathbb{R}}^{16},$$

where $\varkappa_{a,i}$ are the coordinates of the mapping $\varkappa_a$. This together with assertion (iii) implies that $\varkappa_a \in I(\mathbb{H}_{\mathbb{C}a}^2)$ (see Miščenko and Fomenko [154], Chap. 4, Sect. 3.1). Hence, we derive our result, because $\tau_{a/|a|} \in I(\mathbb{H}_{\mathbb{C}a}^2)$. □

Remark 2.3. Another proof of the fourth assertion of Proposition 2.19 will be obtained in Sect. 5.4.

Remark 2.4. Assertions (iii)–(v) show that the mapping $\sigma_{2a/(1+|a|^2)}$ is the symmetry of $\mathbb{H}_{\mathbb{C}a}^2$ at the point a. Thus, $\mathbb{H}_{\mathbb{C}a}^2$ is a Riemannian symmetric space.

Proposition 2.20. *The Riemannian measure on* $\mathbb{H}_{\mathbb{C}a}^2$ *has the form*

$$d\mu(x) = \frac{dx}{(1 - |x|^2)^{12}}. \tag{2.69}$$

Proof. Bearing in mind that $O_{\mathbb{C}a}(2)$ acts transitively on $\mathbb{S}^{15}$ (see Proposition 1.1) and repeating the arguments used in the proof of Proposition 2.13, we obtain

$$\det \|g_{ij}(x)\|_{i,j=1}^{16} = \frac{1}{(1 - |x|^2)^{24}}. \tag{2.70}$$

This proves (2.69). □

Now we establish some expressions for the Laplace–Beltrami operator L on $\mathbb{H}_{\mathbb{C}a}^2$.

Proposition 2.21. *Let* $f \in C^2(B_{\mathbb{R}}^{16})$. *Then*

$$(Lf)(x) = \sum_{i,j=1}^{16} g^{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + 12\big(1-|x|^2\big)\sum_{i=1}^{16} x_i \frac{\partial f}{\partial x_i}(x). \tag{2.71}$$

In particular, if f *has the form* $f(x) = f_0(|x|)$, *then*

$$(Lf)(x) = \big(1-|x|^2\big)^2 f_0''(|x|) + \frac{1-|x|^2}{|x|}\big(15+5|x|^2\big) f_0'(|x|). \tag{2.72}$$

Proof. The statements follow from (1.43), (2.52), (2.53), and (2.70). □

Corollary 2.5. *The relation*

$$(Lf)(a) = \Delta(f \circ \sigma_a)(0) \tag{2.73}$$

holds for every $a \in B_{\mathbb{R}}^{16}$.

Proof. Using (2.71) and (2.63), we find

$$(L\varphi)(|a|e_1) = \Delta(\varphi \circ \varkappa_a)(0) \tag{2.74}$$

for any $\varphi \in C^2(B_{\mathbb{R}}^{16})$. In particular, we have (2.73) for $a = 0$. Let $a \neq 0$. Because $\tau_{a/|a|} \in I(\mathbb{H}_{\mathbb{C}a}^2)$, from (2.74) we obtain

$$\begin{aligned}(Lf)(a) &= (Lf)\big(\tau_{a/|a|}(|a|e_1)\big)\\ &= \big(L(f\circ\tau_{a/|a|})\big)(|a|e_1)\\ &= \Delta(f\circ\tau_{a/|a|}\circ\varkappa_a)(0)\\ &= \Delta(f\circ\sigma_a)(0),\end{aligned}$$

which completes the proof. □

Proposition 2.22. *The distance between two points* $x, y \in \mathbb{H}_{\mathbb{C}a}^2$ *is given by the formula*

$$d(x,y) = \frac{1}{2}\log\left(\frac{\sqrt{\Psi_{\mathbb{C}a}(x,y)} + \sqrt{\Psi_{\mathbb{C}a}(x,y) - (1-|x|^2)(1-|y|^2)}}{\sqrt{\Psi_{\mathbb{C}a}(x,y)} - \sqrt{\Psi_{\mathbb{C}a}(x,y) - (1-|x|^2)(1-|y|^2)}}\right). \tag{2.75}$$

Proof. Let $0 \leqslant r < 1$. Put

$$\gamma(t) = \tanh\big(t\tanh^{-1} r\big)e_1 \quad \text{for } 0 \leqslant t \leqslant 1.$$

Using (1.41) and (2.50), it is easy to verify that $\gamma(t)$ is a geodesic. In addition, in view of (2.51),

$$l(\gamma) = \frac{1}{2}\log\frac{1+r}{1-r}, \tag{2.76}$$

where $l(\gamma)$ is the length of the curve γ. We claim that

$$d(0, re_1) = l(\gamma). \tag{2.77}$$

Consider the function $\varphi(r) = d(0, re_1)$. Since $O_{\mathbb{C}a}(2) \subset I(\mathbb{H}^2_{\mathbb{C}a})$ and the group $O_{\mathbb{C}a}(2)$ acts transitively on $\mathbb{S}^{15}$, we have

$$d(0, x) = \varphi(|x|), \quad x \in B^{16}_{\mathbb{R}}. \tag{2.78}$$

Next, for arbitrary points $x, y \in \mathbb{H}^2_{\mathbb{C}a}$, there exists a geodesic $\gamma_{x,y}$ joining x and y such that $d(x, y) = l(\gamma_{x,y})$ (see Remark 2.4 and Theorem 1.5). Hence, it follows from (2.78) that φ is an increasing function on $[0, 1)$. Repeating the arguments in the proof of Theorem 2.3, we infer that $\mathbb{H}^2_{\mathbb{C}a}$ is a noncompact two-point homogeneous space (see (2.76) and Helgason [115], Chap. 9, Proposition 5.3). This implies (2.77) (see Helgason [115], Chap. 6, Theorem 1.1 and Chap. 1, Lemma 9.3). Taking into account that $\sigma_a \in I(\mathbb{H}^2_{\mathbb{C}a})$ for all $a \in B^{16}_{\mathbb{R}}$ and using (2.67), we obtain our result. □

Theorem 2.4. *The Cayley hyperbolic plane* $F_4^*/\,\mathrm{Spin}(9)$ *of maximal sectional curvature* -1 *is isometric to the space* $\mathbb{H}^2_{\mathbb{C}a}$.

Proof. As we already know, $\mathbb{H}^2_{\mathbb{C}a}$ is a noncompact two-point homogeneous space. Furthermore, by (2.72) and (2.75), the radial part L_0 of the Laplace–Beltrami operator L on $\mathbb{H}^2_{\mathbb{C}a}$ has the form

$$L_0 = \frac{\partial^2}{\partial t^2} + (15 \coth t + 7 \tanh t)\frac{\partial}{\partial t}. \tag{2.79}$$

Using (2.79) and (1.61)–(1.63), we arrive at the desired assertion. □

Chapter 3
Realizations of Rank One Symmetric Spaces of Compact Type

This chapter aims to be the analogue for the compact case of Chap. 2. Section 3.1 treats the case of the sphere $\mathbb{S}^n$. Sections 3.2–3.5 are devoted to projective spaces.

In Sects. 3.2–3.5 we start with the very definition of the corresponding projective space. If $\mathbb{K} = \mathbb{R}, \mathbb{C}$, or $\mathbb{Q}$, then the projective space $\mathbb{P}(\mathbb{K}^{n+1})$ is the orbit space for the left action of the group $\mathbb{K}^* = \mathbb{K}\backslash\{0\}$ on $\mathbb{K}^{n+1}\backslash\{0\}$. Since Cayley numbers do not satisfy the associative law, this definition cannot be applied to them, and there are various other reasons which preclude direct extension. Concerning the Cayley plane, we use the algebraic approach of Freudenthal. Freudenthal takes particular idempotent matrices in $\mathbb{A}1$ as points and lines to define the Cayley plane $\mathbb{P}^2(\mathbb{C}a)$.

After these definitions we shall describe the Riemannian structure of $\mathbb{P}(\mathbb{K}^{n+1})$ and of $\mathbb{P}^2(\mathbb{C}a)$. Then we characterize them as symmetric spaces. The metric tensors for the projective spaces and the hyperbolic spaces correspond under the substitution $x \to \mathrm{i}x$ (see, for example, (2.2) and (3.8)). The duality for symmetric spaces gives a general explanation of this formal analogy (see Helgason [121], Chap. 5, Sect. 2).

3.1 The Space SO$(n+1)$/SO(n)

Throughout Chap. 3 we assume that $n \in \mathbb{N}\backslash\{1\}$. Let $\mathrm{d}s^2$ be the standard Riemannian metric on $\mathbb{S}^n$, i.e.,

$$\mathrm{d}s^2 = \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_{n+1}^2\big|_{\mathbb{S}^n}, \tag{3.1}$$

where, as usual, $x_1, \ldots, x_{n+1}$ are the Cartesian coordinates on $\mathbb{R}^{n+1}$. Denote by $r, \theta_1, \ldots, \theta_n$ the system of spherical coordinates on $\mathbb{R}^{n+1}$ in which

$$\begin{aligned}
x_1 &= r\sin\theta_n \sin\theta_{n-1}\cdots\sin\theta_2\sin\theta_1,\\
x_2 &= r\sin\theta_n \sin\theta_{n-1}\cdots\sin\theta_2\cos\theta_1,\\
x_3 &= r\sin\theta_n \sin\theta_{n-1}\cdots\sin\theta_3\cos\theta_2,\\
&\ \vdots
\end{aligned}$$

V.V. Volchkov, V.V. Volchkov, *Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group,* Springer Monographs in Mathematics,

$$\begin{aligned} x_{n-1} &= r \sin\theta_n \sin\theta_{n-1} \cos\theta_{n-2}, \\ x_n &= r \sin\theta_n \cos\theta_{n-1}, \\ x_{n+1} &= r \cos\theta_n, \end{aligned}$$

where $r > 0$, $0 < \theta_1 < 2\pi$, and $0 < \theta_k < \pi$ for $k \neq 1$. Then

$$ds^2 = d\theta_n^2 + \sin^2\theta_n\, d\theta_{n-1}^2 + \cdots + \sin^2\theta_n \cdots \sin^2\theta_2\, d\theta_1^2. \tag{3.2}$$

It follows from (3.2) and (1.36) that the Riemannian measure $d\mu$ on $\mathbb{S}^n$ is defined by the equality

$$d\mu = \sin^{n-1}\theta_n \sin^{n-2}\theta_{n-1} \cdots \sin\theta_2\, d\theta_1 \cdots d\theta_n.$$

Furthermore, relations (3.2), (1.40), and (1.43) show that the Laplace–Beltrami operator L on $\mathbb{S}^n$ has the expression

$$\begin{aligned} L = {} & \frac{1}{\sin^{n-1}\theta_n} \frac{\partial}{\partial\theta_n} \sin^{n-1}\theta_n \frac{\partial}{\partial\theta_n} \\ & + \frac{1}{\sin^2\theta_n \sin^{n-2}\theta_{n-1}} \frac{\partial}{\partial\theta_{n-1}} \sin^{n-2}\theta_{n-1} \frac{\partial}{\partial\theta_{n-1}} + \cdots \\ & + \frac{1}{\sin^2\theta_n \sin^2\theta_{n-1} \cdots \sin^2\theta_3 \sin\theta_2} \frac{\partial}{\partial\theta_2} \sin\theta_2 \frac{\partial}{\partial\theta_2} \\ & + \frac{1}{\sin^2\theta_n \sin^2\theta_{n-1} \cdots \sin^2\theta_3 \sin^2\theta_2} \frac{\partial^2}{\partial\theta_1^2}. \end{aligned}$$

The operator L is the spherical part of the Euclidean Laplacian Δ in $\mathbb{R}^{n+1}$, restricted to $\mathbb{S}^n$.

The isometry group $I(\mathbb{S}^n)$ coincides with the orthogonal group $O(n+1)$. The distance between two points $\xi, \eta \in \mathbb{S}^n$ in the metric (3.1) is given by the formula

$$d(\xi, \eta) = \arccos \langle \xi, \eta \rangle_{\mathbb{R}}.$$

If $a = (a_1, \dots, a_{n+1}) \in \mathbb{S}^n$, then the mapping

$$s_a : \xi \to 2\langle a, \xi \rangle_{\mathbb{R}} a - \xi, \quad \xi \in \mathbb{S}^n,$$

is the symmetry of $\mathbb{S}^n$ at the point a. In addition, for

$$b = \big(2(1 - a_{n+1})\big)^{-1/2} (a_1, \dots, a_n, a_{n+1} - 1), \quad a_{n+1} \neq 1,$$

we obtain $s_b(a) = -e_{n+1}$, $s_b(-e_{n+1}) = a$, where $e_{n+1} = (0, \dots, 0, -1)$.

We can look at the preceding construction from a different angle.

Consider $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ and put $U_1 = \mathbb{R}^n$, $U_2 = \overline{\mathbb{R}^n} \backslash \{0\}$. Obviously, $U_1 \cap U_2 = \mathbb{R}^n \backslash \{0\}$. Introduce the bijective mappings $\phi_k : U_k \to \mathbb{R}^n$, $k = 1, 2$, as follows:

$$\phi_1(p) = p, \qquad \phi_2(p) = \begin{cases} p/|p|^2, & p \in \mathbb{R}^n \backslash \{0\}, \\ 0, & p = \infty. \end{cases}$$

It is clear that $\phi_1(U_1 \cap U_2) = \phi_2(U_1 \cap U_2) = \mathbb{R}^n \backslash \{0\}$ and

$$\phi_2 \circ \phi_1^{-1}(x) = \phi_1 \circ \phi_2^{-1}(x) = \frac{x}{|x|^2}, \quad x \in \mathbb{R}^n \backslash \{0\}. \tag{3.3}$$

Thus, the atlas $\{(U_k, \phi_k)\}_{k=1}^2$ induces a real-analytic structure on $\overline{\mathbb{R}^n}$ (see Postnikov [168], Lecture 6).

Next, let

$$\begin{aligned} g_{ij}(x) &= \frac{4\delta_{i,j}}{(1+|x|^2)^2}, \quad x \in \mathbb{R}^n,\ i, j \in \{1, \dots, n\}, \\ G_{ij}^k(p) &= g_{ij}\big(\phi_k(p)\big), \quad p \in U_k,\ k = 1, 2. \end{aligned} \tag{3.4}$$

A simple calculation shows that

$$G_{lm}^1(p) = \sum_{i,j=1}^n G_{ij}^2(p) \frac{\partial v_i}{\partial x_l}\big(\phi_1(p)\big) \frac{\partial v_j}{\partial x_m}\big(\phi_1(p)\big), \quad p \in U_1 \cap U_2,$$

where $v_1, \dots, v_n$ are the coordinates of the mapping (3.3). Therefore, matrices (3.4) define a Riemannian metric on $\overline{\mathbb{R}^n}$.

It is easily verified that the stereographic projection $\pi : \overline{\mathbb{R}^n} \to \mathbb{S}^n$, where

$$\pi(p) = \begin{cases} \big(1+|p|^2\big)^{-1}\big(2p_1, \dots, 2p_n, |p|^2 - 1\big), & p \in \mathbb{R}^n, \\ (0, \dots, 0, 1), & p = \infty, \end{cases}$$

is an isometry of $\overline{\mathbb{R}^n}$ with the metric (3.4) onto $\mathbb{S}^n$ with the metric (3.1) (see the proof of Proposition 2.6). For the metric (3.4), we have

$$\begin{aligned} d\mu(x) &= \frac{2^n\, dx}{(1+|x|^2)^n}, \quad x \in \mathbb{R}^n, \\ (Lf)(x) &= \frac{(1+|x|^2)^2}{4} \left((\Delta f)(x) - \frac{2(n-2)}{1+|x|^2} \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(x) \right), \quad f \in C^2(\mathbb{R}^n), \\ d(x, y) &= \begin{cases} 2 \arctan\big(|x-y||x|/|x+|x|^2 y|\big), & x, y \in \mathbb{R}^n,\ x + |x|^2 y \neq 0, \\ \pi, & x \in \mathbb{R}^n \backslash \{0\},\ x + |x|^2 y = 0, \\ \pi - 2 \arctan |x|, & x \in \mathbb{R}^n,\ y = \infty, \end{cases} \\ s_a(x) &= \frac{2((1-|x|^2)(1-|a|^2) + 4\langle x, a\rangle_{\mathbb{R}})a - (1+|a|^2)^2 x}{(1-|a|^2)^2 + 4(1-|a|^2)\langle x, a\rangle_{\mathbb{R}} + 4|x|^2|a|^2}, \quad a \in \mathbb{R}^n. \end{aligned} \tag{3.5}$$

Note also that

$$s_{a/(1+\sqrt{1+|a|^2})}(x) = \frac{(1 + 2\langle x, a\rangle_{\mathbb{R}} - |x|^2)a - (1+|a|^2)x}{1 + |x|^2|a|^2 + 2\langle x, a\rangle_{\mathbb{R}}}$$

and in particular

$$s_{a/(1+\sqrt{1+|a|^2})}(a) = 0, \qquad s_{a/(1+\sqrt{1+|a|^2})}(0) = a.$$

Theorem 3.1. *The symmetric space* $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ *of constant sectional curvature* 1 *is isometric to the sphere* $\mathbb{S}^n$ *with the metric* (3.1)*. Equivalently, this space is isometric to* $\overline{\mathbb{R}^n}$ *with the metric* (3.4)*.*

Proof. Let us compute the curvature tensor of metric (3.4). Putting

$$\alpha(x) = \frac{2}{1+|x|^2}, \quad \beta(x) = \log\alpha(x), \quad \beta_i = \frac{\partial\beta}{\partial x_i}, \quad \beta_{ij} = \frac{\partial^2\beta}{\partial x_i \partial x_j},$$

we find (see (1.37), (1.38))

$$\frac{\partial g_{ij}}{\partial x_k} = 2\alpha^2\delta_{i,j}\beta_k, \qquad \Gamma_{ij}^k = \delta_{i,k}\beta_j + \delta_{j,k}\beta_i - \delta_{i,j}\beta_k,$$

$$\frac{\partial \Gamma_{ij}^k}{\partial x_l} = \delta_{i,k}\beta_{jl} + \delta_{j,k}\beta_{il} - \delta_{i,j}\beta_{kl},$$

$$R_{ijk}^l = \delta_{l,k}(\beta_{ij} - \beta_i\beta_j) - \delta_{i,k}(\beta_{lj} - \beta_l\beta_j) + \delta_{i,j}(\beta_{lk} - \beta_l\beta_k) - \delta_{l,j}(\beta_{ik} - \beta_i\beta_k) \\ + (\delta_{i,j}\delta_{l,k} - \delta_{i,k}\delta_{l,j})|\nabla\beta|^2,$$

where $\nabla\beta = (\beta_1, \dots, \beta_n)$. Since

$$\beta_i = -\frac{2x_i}{1+|x|^2}, \qquad \beta_{ij} = -\frac{2\delta_{i,j}}{1+|x|^2} + \frac{4x_i x_j}{(1+|x|^2)^2},$$

$$\beta_{ij} - \beta_i\beta_j = -\frac{2\delta_{i,j}}{1+|x|^2}, \qquad |\nabla\beta|^2 = \frac{4|x|^2}{(1+|x|^2)^2},$$

we obtain

$$R_{ijk}^l = \frac{4}{(1+|x|^2)^2}(\delta_{i,k}\delta_{l,j} - \delta_{l,k}\delta_{i,j}).$$

Hence,

$$R_{ijkl} = \frac{16}{(1+|x|^2)^4}(\delta_{i,k}\delta_{j,l} - \delta_{j,k}\delta_{i,l}) = g_{ik}g_{jl} - g_{jk}g_{il}. \tag{3.6}$$

Relations (3.6) and (1.42) show that the sectional curvature is constantly equal to 1. The desired conclusion now follows by Theorem 1.6, since the sphere $\mathbb{S}^n$ ($n \geqslant 2$) is a simply connected complete Riemannian manifold. □

3.2 The Real Projective Space SO$(n+1)/$O(n)

Let $\mathbb{R}_*^{n+1} = \mathbb{R}^{n+1}\backslash\{0\}$. We consider in the domain $\mathbb{R}_*^{n+1}$ the following equivalence relation:

$$\omega \sim \varkappa \quad \text{if } \omega = \lambda\varkappa$$

for some $\lambda \in \mathbb{R}_* = \mathbb{R}\backslash\{0\}$. The equivalence class $[\omega]$ of every point $\omega \in \mathbb{R}_*^{n+1}$ is the punctured line $\{\lambda\omega\colon \lambda \in \mathbb{R}_*\}$. The set of all these classes is denoted by $\mathbb{P}(\mathbb{R}^{n+1})$.

We give some interpretations of the set $\mathbb{P}(\mathbb{R}^{n+1})$.

Assigning to each class $[\omega]$ the line $\mathbb{R}\cdot\omega$, we see that $\mathbb{P}(\mathbb{R}^{n+1})$ is the set of all one-dimensional subspaces of $\mathbb{R}^{n+1}$. Furthermore, each class $[\omega]$ uniquely determines a set $\{\pm\omega/|\omega|\}$ on the unit sphere $\mathbb{S}^n$. Therefore, $\mathbb{P}(\mathbb{R}^{n+1})$ can be regarded as $\mathbb{S}^n/\{\pm\,\mathrm{Id}\}$. It follows in particular that $\mathbb{P}(\mathbb{R}^{n+1})$ is the hemisphere

$$\mathbb{S}^n_- = \big\{x = (x_1, \dots, x_{n+1}) \in \mathbb{S}^n \colon x_{n+1} \leqslant 0\big\}$$

for which diametrically opposite points of the edge $\{x \in \mathbb{S}^n_- \colon x_{n+1} = 0\}$ are identified. Equivalently, $\mathbb{P}(\mathbb{R}^{n+1})$ may be realized (via the stereographic projection) as the closed unit ball $\mathrm{Cl}\,(B^n_{\mathbb{R}})$ for which antipodal points of the boundary $\mathbb{S}^{n-1}$ are identified.

Next, let $[\omega] \in \mathbb{P}(\mathbb{R}^{n+1})$. The *homogeneous coordinates* of $[\omega]$ are the coordinates of an arbitrary point $\varkappa \in \mathbb{R}_*^{n+1}$ belonging to the class $[\omega]$. They are defined up to a common factor $\lambda \in \mathbb{R}_*$. Using homogeneous coordinates, we shall show how $\mathbb{P}(\mathbb{R}^{n+1})$ can be made into a real-analytic manifold.

Denote by $\omega_0, \dots, \omega_n$ the coordinates of $\omega \in \mathbb{R}_*^{n+1}$ (this enumeration is most convenient). Consider the sets

$$U_k = \big\{[\omega] \in \mathbb{P}\big(\mathbb{R}^{n+1}\big) \colon \omega_k \neq 0\big\}, \quad k = 0, \dots, n.$$

Evidently, $\mathbb{P}(\mathbb{R}^{n+1}) = \bigcup_{k=0}^n U_k$. Introduce the bijective mappings $\phi_k \colon U_k \to \mathbb{R}^n$ by putting

$$\phi_k([\omega]) = (x_1, \dots, x_n),$$

where

$$x_m = \begin{cases} \omega_{m-1}/\omega_k & \text{if } m \leqslant k, \\ \omega_m/\omega_k & \text{if } m > k, \end{cases} \quad m = 1, \dots, n.$$

Note that both U_k and ϕ_k are well defined by the relation between different choices of homogeneous coordinates. Moreover, it is easy to see that

$$\phi_k^{-1}(x_1, \dots, x_n) = [\omega],$$

where

$$\omega_l = \begin{cases} x_{l+1} & \text{if } l < k, \\ 1 & \text{if } l = k, \\ x_l & \text{if } l > k, \end{cases} \quad l = 0, \dots, n.$$

To continue, let us assume that $k, l \in \{0, \dots, n\}$ and $k < l$. We have

$$\phi_k(U_k \cap U_l) = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_l \neq 0\},$$
$$\phi_l(U_k \cap U_l) = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_{k+1} \neq 0\}.$$

In addition,

$$\phi_l \circ \phi_k^{-1}(x_1, \dots, x_n) = (v_1, .., v_n), \quad x_l \neq 0,$$

where

$$v_m = \begin{cases} x_m/x_l & \text{if } m \leqslant k \text{ or } m > l, \\ 1/x_l & \text{if } m = k+1, \\ x_{m-1}/x_l & \text{if } k+1 < m \leqslant l, \end{cases} \qquad m = 1, \dots, n.$$

It follows that the map $\phi_l \circ \phi_k^{-1} : \phi_k(U_k \cap U_l) \to \phi_l(U_k \cap U_l)$ is a diffeomorphism. Thus, the charts (U_k, ϕ_k) form an atlas and define on $\mathbb{P}(\mathbb{R}^{n+1})$ a real-analytic structure.

Every coordinate neighborhood U_k is diffeomorphic to $\mathbb{R}^n$. We identify U_0 with $\mathbb{R}^n$ and call the mapping

$$\phi_0^{-1} : (x_1, \dots, x_n) \to [(1, x_1, \dots, x_n)]$$

the *transition* from affine coordinates (in $\mathbb{R}^n$) to homogeneous coordinates. The complement of $\mathbb{R}^n$ in $\mathbb{P}(\mathbb{R}^{n+1})$ consists of the points $[(0, \omega_1, \dots, \omega_n)]$, where $(\omega_1, \dots, \omega_n) \in \mathbb{R}^n_*$; hence, is identified with $\mathbb{P}(\mathbb{R}^n)$. By that

$$\mathbb{P}(\mathbb{R}^{n+1}) = \mathbb{R}^n \cup \mathbb{P}(\mathbb{R}^n). \tag{3.7}$$

Now we endow $\mathbb{P}(\mathbb{R}^{n+1})$ with a Riemannian structure. Put

$$g_{ij}(x) = \frac{\delta_{i,j}}{1+|x|^2} - \frac{x_i x_j}{(1+|x|^2)^2}, \quad x \in \mathbb{R}^n, \ i, j \in \{1, \dots, n\}, \tag{3.8}$$

$$G^k_{ij}(p) = g_{ij}(\phi_k(p)), \quad p \in U_k, \ k \in \{0, \dots, n\}. \tag{3.9}$$

It is clear that $\|g_{ij}\|^n_{i,j=1}$ is a symmetric matrix. For $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, we have

$$\sum_{i,j=1}^n g_{ij}(x) y_i y_j = \frac{|y|^2 + |y|^2|x|^2 - \langle x, y\rangle^2_{\mathbb{R}}}{(1+|x|^2)^2}.$$

Applying the Schwarz inequality, we obtain

$$\sum_{i,j=1}^n g_{ij}(x) y_i y_j \geqslant \frac{|y|^2}{(1+|x|^2)^2}. \tag{3.10}$$

Hence, $\|g_{ij}\|^n_{i,j=1}$ is positive definite. Next, as above, assume that $k, l \in \{0, \dots, n\}$ and $k < l$. We write

$$v(x) = \big(v_1(x), \dots, v_n(x)\big) = \phi_l \circ \phi_k^{-1}(x), \quad x \in \phi_k(U_k \cap U_l).$$

A direct check shows that

$$g_{ms}(x) = \sum_{i,j=1}^{n} g_{ij}\big(v(x)\big) \frac{\partial v_i}{\partial x_m}(x) \frac{\partial v_j}{\partial x_s}(x), \quad x \in \phi_k(U_k \cap U_l)$$

for $m, s \in \{1, \dots, n\}$. This implies

$$G^k_{ms}(p) = \sum_{i,j=1}^{n} G^l_{ij}(p) \frac{\partial v_i}{\partial x_m}\big(\phi_k(p)\big) \frac{\partial v_j}{\partial x_s}\big(\phi_k(p)\big), \quad p \in U_k \cap U_l.$$

So matrices (3.9) define a Riemannian metric on $\mathbb{P}(\mathbb{R}^{n+1})$.

Let us compute the curvature tensor of metric (3.9). One finds

$$\begin{aligned}
g^{ij}(x) &= \big(1+|x|^2\big)(\delta_{i,j} + x_i x_j), \\
\frac{\partial g_{ij}}{\partial x_m} &= -\frac{2x_m\delta_{i,j} + x_i\delta_{j,m} + x_j\delta_{i,m}}{(1+|x|^2)^2} + \frac{4x_i x_j x_m}{(1+|x|^2)^3}, \\
\Gamma^m_{ij} &= -\frac{x_i\delta_{m,j} + x_j\delta_{i,m}}{1+|x|^2}, \\
\frac{\partial \Gamma^m_{ij}}{\partial x_s} &= -\frac{\delta_{i,s}\delta_{m,j} + \delta_{j,s}\delta_{m,i}}{1+|x|^2} + \frac{2(x_i x_s\delta_{m,j} + x_s x_j\delta_{i,m})}{(1+|x|^2)^2}, \\
R^s_{ijm} &= \frac{\delta_{i,m}\delta_{j,s} - \delta_{i,j}\delta_{m,s}}{1+|x|^2} + \frac{x_i x_j\delta_{m,s} - x_i x_m\delta_{j,s}}{(1+|x|^2)^2}.
\end{aligned} \tag{3.11}$$

Hence,

$$\begin{aligned}
R_{ijms} &= \frac{\delta_{i,m}\delta_{j,s} - \delta_{i,s}\delta_{j,m}}{(1+|x|^2)^2} + \frac{x_i x_s\delta_{j,m} - x_i x_m\delta_{j,s} + x_j x_m\delta_{i,s} - x_j x_s\delta_{i,m}}{(1+|x|^2)^3} \\
&= g_{im}g_{js} - g_{jm}g_{is}.
\end{aligned}$$

This means (see (1.42)) that $\mathbb{P}(\mathbb{R}^{n+1})$ with the metric (3.9) is a space of constant sectional curvature 1. We denote this space by $\mathbb{P}^n_{\mathbb{R}}$.

We shall now determine a form of the Riemannian measure $\mathrm{d}\mu$ and the Laplace–Beltrami operator L on $\mathbb{P}^n_{\mathbb{R}}$.

If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, put $\mathrm{i}x = (\mathrm{i}x_1, \dots, \mathrm{i}x_n)$, where i denotes the imaginary unit. By (3.8) we have

$$g_{ij}(x) = \mathbf{g}_{ij}(\mathrm{i}x), \quad x \in \mathbb{R}^n, \ i, j \in \{1, \dots, n\}, \tag{3.12}$$

where

$$\mathbf{g}_{ij}(z_1, \dots, z_n) = \frac{\delta_{i,j}}{1 - z_1^2 - \dots - z_n^2} + \frac{z_i z_j}{(1 - z_1^2 - \dots - z_n^2)^2}$$

for $(z_1, \dots, z_n) \in \mathbb{C}^n : z_1^2 + \dots + z_n^2 \neq 1$. This together with (2.2) and (2.8) implies

$$\det \|g_{ij}(x)\|_{i,j=1}^n = \frac{1}{(1+|x|^2)^{n+1}}, \quad x \in \mathbb{R}^n. \tag{3.13}$$

Hence,

$$\mathrm{d}\mu(x) = \frac{\mathrm{d}x}{(1+|x|^2)^{\frac{n+1}{2}}}, \quad x \in \mathbb{R}^n. \tag{3.14}$$

Furthermore, using (1.43), (3.11), and (3.13), we find

$$(Lf)(x) = \big(1+|x|^2\big)\left((\Delta f)(x) + \sum_{i,j=1}^n x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + 2\sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(x)\right)$$

for an arbitrary function $f \in C^2(\mathbb{R}^n)$. In particular, if f has the form $f(x) = f_0(|x|)$, then

$$(Lf)(x) = \big(1+|x|^2\big)^2 f_0''(|x|) + \frac{1+|x|^2}{|x|}\big(n-1+2|x|^2\big) f_0'(|x|).$$

We turn to the isometry group of the space $\mathbb{P}^n_{\mathbb{R}}$. Introduce an analog of involution (2.11). Let $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, $[\omega] = [(\omega_0, \dots, \omega_n)] \in \mathbb{P}^n_{\mathbb{R}}$. Consider the vector $\varkappa = (\varkappa_0, \dots, \varkappa_n) \in \mathbb{R}^{n+1}$ with

$$\varkappa_l = \begin{cases} \omega_0 + \langle '\omega, a\rangle_{\mathbb{R}} & \text{if } l = 0, \\ \big(\omega_0 + (1+\nu_a)^{-1}\langle '\omega, a\rangle_{\mathbb{R}}\big) a_l - \nu_a \omega_l & \text{if } 1 \leqslant l \leqslant n, \end{cases} \tag{3.15}$$

where $'\omega = (\omega_1, \dots, \omega_n)$ and $\nu_a = \sqrt{1+|a|^2}$. We claim that $\varkappa \in \mathbb{R}^{n+1}_*$. For otherwise suppose that $\varkappa = 0$. Then

$$\omega_0 + \langle '\omega, a\rangle_{\mathbb{R}} = 0, \qquad \omega_0 |a|^2 - \langle '\omega, a\rangle_{\mathbb{R}} = 0.$$

Therefore, $\omega_0 = 0$ and $\langle '\omega, a\rangle_{\mathbb{R}} = 0$. Hence, it follows from (3.15) that $\omega = 0$. But this is impossible since $[\omega] \in \mathbb{P}^n_{\mathbb{R}}$. Now we define the mapping $\sigma_a : \mathbb{P}^n_{\mathbb{R}} \to \mathbb{P}^n_{\mathbb{R}}$ by setting

$$\sigma_a([\omega]) = [\varkappa]. \tag{3.16}$$

The transformation σ_a is clearly well defined. In affine coordinates in $\mathbb{R}^n = U_0$, this transformation can be viewed as

$$\begin{aligned} x &\to [(1, x_1, \dots, x_n)] \\ &\to \left[\left(1 + \langle x, a\rangle_{\mathbb{R}}, \left(1 + \frac{\langle x, a\rangle_{\mathbb{R}}}{1+\nu_a}\right)a_1 - \nu_a x_1, \dots, \left(1 + \frac{\langle x, a\rangle_{\mathbb{R}}}{1+\nu_a}\right)a_n - \nu_a x_n\right)\right] \\ &\to \big(1 + \langle x, a\rangle_{\mathbb{R}}\big)^{-1}\left(\left(1 + \frac{\langle x, a\rangle_{\mathbb{R}}}{1+\nu_a}\right)a_1 - \nu_a x_1, \dots, \left(1 + \frac{\langle x, a\rangle_{\mathbb{R}}}{1+\nu_a}\right)a_n - \nu_a x_n\right) \end{aligned}$$

(the first arrow denotes transition to homogeneous coordinates, the second denotes transition to a map in homogeneous coordinates, and the third denotes return to affine coordinates). In other words,

$$\phi_0 \circ \sigma_a \circ \phi_0^{-1}(x) = \sigma_a^0(x),$$

where

$$\sigma_a^0(x) = \big(1 + \langle x, a\rangle_{\mathbb{R}}\big)^{-1}\left(\left(1 + \frac{\langle x, a\rangle_{\mathbb{R}}}{1+\nu_a}\right)a - \nu_a x\right).$$

Analogously, putting

$$\sigma_\infty([\omega]) = [(\omega_1, \omega_0, \omega_2, \dots, \omega_n)], \quad [\omega] \in \mathbb{P}^n_{\mathbb{R}}, \tag{3.17}$$

we see that

$$\phi_0 \circ \sigma_\infty \circ \phi_0^{-1}(x) = \sigma_\infty^0(x),$$

where

$$\sigma_\infty^0(x) = \left(\frac{1}{x_1}, \frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}\right).$$

Proposition 3.1. *The mappings σ_a and σ_∞ are involutory isometries of $\mathbb{P}^n_{\mathbb{R}}$.*

Proof. The relations

$$\sigma_a \circ \sigma_a([\omega]) = [\omega] \quad \text{and} \quad \sigma_\infty \circ \sigma_\infty([\omega]) = [\omega], \quad [\omega] \in \mathbb{P}^n_{\mathbb{R}},$$

follow directly from (3.16) and (3.17). We shall prove that $\sigma_a, \sigma_\infty \in I(\mathbb{P}^n_{\mathbb{R}})$. Let

$$(z, w) = z_1 w_1 + \dots + z_n w_n \quad \text{for } z = (z_1, \dots, z_n) \in \mathbb{C}^n,\ w = (w_1, \dots, w_n) \in \mathbb{C}^n.$$

If $(z, w) \neq 1$, $|w| < 1$, define $\psi_w(z)$ by

$$\big(1-(z,w)\big)\psi_w(z) = \left(1 - \frac{(z,w)}{1+\sqrt{1-w_1^2-\dots-w_n^2}}\right)w - \sqrt{1-w_1^2-\dots-w_n^2}\,z,$$

where we consider the branch of $\sqrt{\cdot}$ that takes positive values on $(0, +\infty)$. Observe that

$$\begin{aligned}\big(1 - &\langle x, b\rangle_{\mathbb{R}}\big)\psi_b(x)\\ &= \begin{cases} b - \langle x, b\rangle_{\mathbb{R}}|b|^{-2}b - \sqrt{1-|b|^2}\big(x - \langle x, b\rangle_{\mathbb{R}}|b|^{-2}b\big), & b \neq 0,\\ -x, & b = 0\end{cases}\end{aligned} \tag{3.18}$$

for $b, x \in B^n_{\mathbb{R}}$. Denote the components of $\psi_w(z)$ by $\psi_{w,1}(z), \dots, \psi_{w,n}(z)$. Then

$$\mathbf{g}_{ms}(x) = \sum_{i,j=1}^n \mathbf{g}_{ij}(\psi_b(x))\frac{\partial \psi_{b,i}}{\partial z_m}(x)\frac{\partial \psi_{b,j}}{\partial z_s}(x), \quad b, x \in B^n_{\mathbb{R}}, \tag{3.19}$$

for $m, s \in \{1, \ldots, n\}$, since $\psi_b|_{B^n_{\mathbb{R}}} \in I(\mathbb{H}^n_{\mathbb{R}})$ (see (3.18) and Proposition 2.4(vi)). Next,

$$\psi_{\mathrm{i}b}(\mathrm{i}x) = \mathrm{i}\sigma^0_b(x), \tag{3.20}$$

$$\frac{\partial \psi_{\mathrm{i}b,i}}{\partial z_m}(\mathrm{i}x) = \frac{\partial \sigma^0_{b,i}}{\partial x_m}(x), \tag{3.21}$$

where $\sigma^0_{b,1}, \ldots, \sigma^0_{b,n}$ are the coordinates of the mapping σ^0_b. By analytic continuation, using (3.12) and (3.19)–(3.21), we obtain

$$g_{ms}(x) = \sum_{i,j=1}^{n} g_{ij}\big(\sigma^0_a(x)\big) \frac{\partial \sigma^0_{a,i}}{\partial x_m}(x) \frac{\partial \sigma^0_{a,j}}{\partial x_s}(x), \quad a \in \mathbb{R}^n, \tag{3.22}$$

for $x \in \mathbb{R}^n \colon (x, a) \neq -1$. In particular, equality (3.22) is valid for $x \in U$, where $U = \{x \in \mathbb{R}^n \colon |x||a| < 1\}$. It follows that σ_a is an isometry of $\phi_0^{-1}U$ onto $\sigma_a(\phi_0^{-1}U)$. Therefore,

$$L(f \circ \sigma_a)([\omega]) = (Lf)\big(\sigma_a([\omega])\big), \quad [\omega] \in \phi_0^{-1}U, \tag{3.23}$$

for an arbitrary real-analytic function f on $\mathbb{P}^n_{\mathbb{R}}$ (see Sect. 1.2). Because σ_a is an analytic diffeomorphism of $\mathbb{P}^n_{\mathbb{R}}$, relation (3.23) holds for every $[\omega] \in \mathbb{P}^n_{\mathbb{R}}$. Hence, $\sigma_a \in I(\mathbb{P}^n_{\mathbb{R}})$.

It remains to prove that $\sigma_\infty \in I(\mathbb{P}^n_{\mathbb{R}})$. Setting $a = (\alpha, 0, \ldots, 0)$ in (3.22) and letting $\alpha \to -\infty$, we find

$$g_{ms}(x) = \sum_{i,j=1}^{n} g_{ij}\big(\sigma^0_\infty(x)\big) \frac{\partial \sigma^0_{\infty,i}}{\partial x_m}(x) \frac{\partial \sigma^0_{\infty,j}}{\partial x_s}(x), \quad x_1 \neq 0, \tag{3.24}$$

where $\sigma^0_{\infty,1}, \ldots, \sigma^0_{\infty,n}$ are the coordinates of the mapping σ^0_∞. Now repeating the above argument with σ_∞ instead of σ_a, we arrive at the desired assertion. □

Corollary 3.1. *The group $I(\mathbb{P}^n_{\mathbb{R}})$ acts transitively on $\mathbb{P}^n_{\mathbb{R}}$.*

Proof. Every orthogonal transformation $\tau \in \mathrm{O}(n)$ induce an isometry of $\mathbb{P}^n_{\mathbb{R}}$ by the correspondence

$$[\omega] \to \big[\big(\omega_0, \tau_1('\omega), \ldots, \tau_n('\omega)\big)\big],$$

where $\tau_1, \ldots, \tau_n$ are the components of τ. It follows in particular that the group $I(\mathbb{P}^n_{\mathbb{R}})$ acts transitively on $\mathbb{P}(\mathbb{R}^n)$ (see (3.7)). Since

$$\sigma_a\big([(1, 0, \ldots, 0)]\big) = [(1, a_1, \ldots, a_n)]$$

and $\sigma_\infty([(1, 0, \ldots, 0)]) \in \mathbb{P}(\mathbb{R}^n)$, this together with Proposition 3.1 concludes the proof. □

We now prove the following:

Proposition 3.2. *The distance d on $\mathbb{P}^n_{\mathbb{R}}$ is defined by*

$$d(0,x)=\begin{cases}\arctan|x|, & x\in\mathbb{R}^n,\\ \pi/2, & x\in\mathbb{P}(\mathbb{R}^n),\end{cases} \tag{3.25}$$

and the condition of invariance under the group $I(\mathbb{P}^n_{\mathbb{R}})$.

Proof. Let

$$K=\{\tau\in I(\mathbb{P}^n_{\mathbb{R}})\colon \tau 0=0\}.$$

Since K acts transitively on $\mathbb{S}^{n-1}$, the distance $d(0,\cdot)$ is a radial function on $\mathbb{R}^n$. Now using (3.10) and (3.5), we obtain the required result (see the proof of Proposition 2.11). □

We close this section with the following:

Theorem 3.2. *The real projective space* SO$(n+1)/$O(n) *of constant sectional curvature* 1 *is isometric to the space $\mathbb{P}^n_{\mathbb{R}}$.*

Proof. Using (3.25) and repeating the arguments in the proof of Theorem 2.3, we see that $\mathbb{P}^n_{\mathbb{R}}$ is a compact two-point homogeneous space. In addition we know that the sectional curvature of $\mathbb{P}^n_{\mathbb{R}}$ is constantly equal to 1. Therefore, $\mathbb{P}^n_{\mathbb{R}}$ is isometric to one of the spaces SO$(n+1)/$SO(n), SO$(n+1)/$O(n) (see Sect. 1.3). Unlike the space $\mathbb{P}^n_{\mathbb{R}}$, however, the sphere SO$(n+1)/SO(n)$ is simply connected for $n\geqslant 2$. Thus, the theorem is proved. □

3.3 The Complex Projective Space SU$(n+1)/$S(U$(n)\times$U(1))

By analogy with Sect. 3.2, consider in the domain $\mathbb{C}^{n+1}_*=\mathbb{C}^{n+1}\backslash\{0\}$ the following equivalence relation:

$$\omega\sim\varkappa \quad \text{if } \omega=\lambda\varkappa$$

for some $\lambda\in\mathbb{C}_*=\mathbb{C}\backslash\{0\}$. The equivalence class $[\omega]$ of any point $\omega\in\mathbb{C}^{n+1}_*$ is the punctured complex line $\{\lambda\omega\colon \lambda\in\mathbb{C}_*\}$. The set of these classes is denoted by $\mathbb{P}(\mathbb{C}^{n+1})$.

It is clear that $\mathbb{P}(\mathbb{C}^{n+1})$ can be regarded as the set of all complex lines in $\mathbb{C}^{n+1}$ passing through the coordinate origin. Next, each class $[\omega]$ uniquely determines a set

$$\gamma_\omega=\left\{e^{i\theta}\frac{\omega}{|\omega|}: 0\leqslant\theta<2\pi\right\}$$

on the unit sphere $\mathbb{S}^{2n+1}$ in $\mathbb{C}^{n+1}$; hence, $\mathbb{P}(\mathbb{C}^{n+1})$ is the set of all circles γ_ω on $\mathbb{S}^{2n+1}$.

We can make $\mathbb{P}(\mathbb{C}^{n+1})$ into a complex-analytic manifold as follows.

Let $\omega_0, \dots, \omega_n$ be coordinates in $\mathbb{C}^{n+1}$. We define

$$U_k = \big\{[(\omega_0, \dots, \omega_n)] \colon \omega_k \neq 0\big\}, \quad k = 0, \dots, n.$$

Introduce the bijective mappings $\phi_k \colon U_k \to \mathbb{C}^n$ by putting

$$\phi_k\big([(\omega_0, \dots, \omega_n)]\big) = (z_1, \dots, z_n),$$

where

$$z_m = \begin{cases} \omega_{m-1}/\omega_k & \text{if } m \leqslant k, \\ \omega_m/\omega_k & \text{if } m > k, \end{cases} \quad m = 1, \dots, n.$$

For $k, l \in \{0, \dots, n\}$ such that $k < l$, one has

$$\begin{aligned} \phi_k(U_k \cap U_l) &= \big\{(z_1, \dots, z_n) \in \mathbb{C}^n \colon z_l \neq 0\big\}, \\ \phi_l(U_k \cap U_l) &= \big\{(z_1, \dots, z_n) \in \mathbb{C}^n \colon z_{k+1} \neq 0\big\}. \end{aligned}$$

In addition,

$$\phi_l \circ \phi_k^{-1}(z_1, \dots, z_n) = (v_1, \dots, v_n), \quad z_l \neq 0,$$

where

$$v_m = \begin{cases} z_m/z_l & \text{if } m \leqslant k \text{ or } m > l, \\ 1/z_l & \text{if } m = k+1, \\ z_{m-1}/z_l & \text{if } k+1 < m \leqslant l, \end{cases} \quad m = 1, \dots, n. \tag{3.26}$$

It follows that the neighboring relations are holomorphic. Thus, the charts (U_k, ϕ_k) form a complex atlas and define on $\mathbb{P}(\mathbb{C}^{n+1})$ a complex-analytic structure (see Postnikov [168], Lecture 11).

Every coordinate neighborhood U_k is biholomorphic to $\mathbb{C}^n$. We identify U_0 with $\mathbb{C}^n$ and call the map

$$\phi_0^{-1} \colon (z_1, \dots, z_n) \to [(1, z_1, \dots, z_n)]$$

the *canonical imbedding* of $\mathbb{C}^n$ in $\mathbb{P}(\mathbb{C}^{n+1})$. The complement of $\mathbb{C}^n$ in $\mathbb{P}(\mathbb{C}^{n+1})$ is a complex submanifold in $\mathbb{P}(\mathbb{C}^{n+1})$ of codimension 1. It consists of the points $[(0, \omega_1, \dots, \omega_n)]$, where $(\omega_1, \dots, \omega_n) \in \mathbb{C}^n_*$, and hence is biholomorphic to $\mathbb{P}(\mathbb{C}^n)$. As a result, one gets

$$\mathbb{P}\big(\mathbb{C}^{n+1}\big) = \mathbb{C}^n \cup \mathbb{P}\big(\mathbb{C}^n\big). \tag{3.27}$$

Let us come to the Hermitian structure of $\mathbb{P}(\mathbb{C}^{n+1})$. Put

$$h_{ij}(z) = \frac{\delta_{i,j}}{1 + |z|^2} - \frac{\overline{z}_i z_j}{(1 + |z|^2)^2}, \quad z = (z_1, \dots, z_n) \in \mathbb{C}^n, \ i, j \in \{1, \dots, n\}, \tag{3.28}$$

$$H^k_{ij}(p) = h_{ij}\big(\phi_k(p)\big), \quad p \in U_k, \ k \in \{0, \dots, n\}. \tag{3.29}$$

It is obvious that $\|h_{ij}\|^n_{i,j=1}$ is a Hermitian symmetric matrix. For $w = (w_1, \dots, w_n) \in \mathbb{C}^n$, we have

$$\sum_{i,j=1}^n h_{ij}(z) w_i \overline{w}_j = \frac{|w|^2 + |w|^2|z|^2 - |\langle z, w\rangle_{\mathbb{C}}|^2}{(1 + |z|^2)^2} \geqslant \frac{|w|^2}{(1 + |z|^2)^2}, \tag{3.30}$$

and hence $\|h_{ij}\|_{i,j=1}^{n}$ is positive definite. Furthermore, from (3.28) and (3.26) we find

$$h_{ij}(z)=\sum_{m,s=1}^{n}h_{ms}\big(v_1(z),\dots,v_n(z)\big)\frac{\partial v_m}{\partial z_i}(z)\overline{\frac{\partial v_s}{\partial z_j}}(z),\quad z\in\phi_k(U_k\cap U_l).$$

This yields

$$H_{ij}^{k}(p)=\sum_{m,s=1}^{n}H_{ms}^{l}(p)\frac{\partial v_m}{\partial z_i}\big(\phi_k(p)\big)\overline{\frac{\partial v_s}{\partial z_j}}\big(\phi_k(p)\big),\quad p\in U_k\cap U_l.$$

So matrices (3.29) define a Hermitian metric ds^2 on $\mathbb{P}(\mathbb{C}^{n+1})$ (which is classically called the *Fubini–Study metric*).

Since

$$H_{ij}^{k}(p)=\frac{\partial^2(f_k\circ\phi_k^{-1})}{\partial z_i\partial\overline{z}_j}\big(\phi_k(p)\big),\quad p\in U_k,$$

where

$$f_k(p)=\log\big(1+|\phi_k(p)|^2\big),\quad p\in U_k,$$

we see that ds^2 is, in fact, a Kaehler metric. As usual, we write

$$g_{i\overline{j}}(z)=h_{ij}(z)/2.$$

Taking into account that $g^{i\overline{j}}(z)=2(1+|z|^2)(\delta_{i,j}+z_i\overline{z}_j)$ and using (1.51), we derive

$$R_{i\overline{j}m\overline{s}}=-2(g_{i\overline{j}}g_{m\overline{s}}+g_{i\overline{s}}g_{m\overline{j}}).\tag{3.31}$$

By (3.31) and (1.52), $\mathbb{P}(\mathbb{C}^{n+1})$ with the metric (3.29) is a space of constant holomorphic sectional curvature 4. Denote this space by $\mathbb{P}_{\mathbb{C}}^{n}$. Minimal sectional curvature of $\mathbb{P}_{\mathbb{C}}^{n}$ is equal to 1 (see Yano and Bochner [264], Chap. 8, Theorem 8.3).

We present explicit expressions for the Riemannian measure and the Laplace–Beltrami operator on $\mathbb{P}_{\mathbb{C}}^{n}$ that will be used later.

By analytic continuation using (2.23), (2.28), and (3.28), we find

$$\det\|h_{ij}(z)\|_{i,j=1}^{n}=\frac{1}{(1+|z|^2)^{n+1}},\quad z\in\mathbb{C}^n.\tag{3.32}$$

Consequently, the Riemannian measure on U_0 has the form

$$\mathrm{d}\mu(z)=\frac{\mathrm{d}m_n(z)}{(1+|z|^2)^{n+1}}.\tag{3.33}$$

Next, let $\|h^{ij}\|_{i,j=1}^{n}$ be the inverse matrix of $\|h_{ij}\|_{i,j=1}^{n}$. Then

$$h^{ij}(z)=\big(1+|z|^2\big)(\delta_{i,j}+\overline{z}_iz_j),\quad i,j\in\{1,\dots,n\}.\tag{3.34}$$

From (1.48), (3.32), and (3.34) it follows that the Laplace–Beltrami operator on $\mathbb{P}^n_{\mathbb{C}}$ acts on a function $f \in C^2(\mathbb{C}^n)$ as follows:

$$(Lf)(z) = 4\big(1+|z|^2\big)\sum_{i,j=1}^{n}(\delta_{i,j}+z_i\overline{z}_j)\frac{\partial^2 f}{\partial z_i \partial \overline{z}_j}(z).$$

Hence,

$$(Lf)(z) = \big(1+|z|^2\big)^2 f_0''(|z|) + \frac{1+|z|^2}{|z|}\big(2n-1+|z|^2\big) f_0'(|z|)$$

if f has the form $f(z) = f_0(|z|)$.

We discuss briefly some properties of the isometry group of the space $\mathbb{P}^n_{\mathbb{C}}$.

Observe first that unitary transformations of $\mathbb{C}^n$ induce isometric mappings of $\mathbb{P}^n_{\mathbb{C}}$ onto itself (see (3.30)). In particular, the isometry group $I(\mathbb{P}^n_{\mathbb{C}})$ acts transitively on the unit sphere $\mathbb{S}^{2n-1}$ and $\mathbb{P}(\mathbb{C}^n)$ (see (3.27)).

Next, let $a = (a_1, \dots, a_n) \in \mathbb{C}^n$. For $[\omega] = [(\omega_0, \dots, \omega_n)] \in \mathbb{P}^n_{\mathbb{C}}$, we set

$$\sigma_a([\omega]) = [(\varkappa_0, \dots, \varkappa_n)], \tag{3.35}$$

where

$$\varkappa_l = \begin{cases} \omega_0 + \omega_1\overline{a}_1 + \dots + \omega_n\overline{a}_n & \text{if } l = 0, \\ \left(\omega_0 + \dfrac{\omega_1\overline{a}_1 + \dots + \omega_n\overline{a}_n}{1+\sqrt{1+|a|^2}}\right)a_l - \sqrt{1+|a|^2}\,\omega_l & \text{if } 1 \leqslant l \leqslant n. \end{cases}$$

Also define

$$\sigma_\infty([\omega]) = [(\omega_1, \omega_0, \omega_2, \dots, \omega_n)]. \tag{3.36}$$

It follows from (3.35) and (3.36) that

$$\sigma_a\big([(1,0,\dots,0)]\big) = [(1, a_1, \dots, a_n)], \qquad \sigma_\infty\big([(1,0,\dots,0)]\big) \in \mathbb{P}(\mathbb{C}^n).$$

In addition,

$$\big(1+\langle z,a\rangle_{\mathbb{C}}\big)\phi_0\circ\sigma_a\circ\phi_0^{-1}(z) = \left(1+\frac{\langle z,a\rangle_{\mathbb{C}}}{1+\sqrt{1+|a|^2}}\right)a - \sqrt{1+|a|^2}z$$

and

$$\phi_0\circ\sigma_\infty\circ\phi_0^{-1}(z) = \left(\frac{1}{z_1}, \frac{z_2}{z_1}, \dots, \frac{z_n}{z_1}\right).$$

Repeating the arguments used in the proof of Proposition 3.1, we see that σ_a and σ_∞ are involutory isometries of the space $\mathbb{P}^n_{\mathbb{C}}$.

We now have enough isometries to map any point of $\mathbb{P}^n_{\mathbb{C}}$ to any other. This means that $\mathbb{P}^n_{\mathbb{C}}$ is a homogeneous space. The distance d on $\mathbb{P}^n_{\mathbb{C}}$ is defined by

$$d(0,z) = \begin{cases} \arctan|z|, & z \in \mathbb{C}^n, \\ \pi/2, & z \in \mathbb{P}(\mathbb{C}^n), \end{cases}$$

and the condition of invariance under the group $I(\mathbb{P}^n_{\mathbb{C}})$ (see (3.30) and the proof of Proposition 3.2). By according to the foregoing, given two pairs of points $[\omega]$, $[\eta]$ and $[\xi]$, $[\zeta]$ in $\mathbb{P}^n_{\mathbb{C}}$ such that $d([\omega],[\eta]) = d([\xi],[\zeta])$, there exists an isometry of $\mathbb{P}^n_{\mathbb{C}}$ that takes $[\omega]$ into $[\xi]$ and $[\eta]$ into $[\zeta]$. Thus, $\mathbb{P}^n_{\mathbb{C}}$ possesses pairwise transitive group of isometries.

As a final result, we establish the following analogue to Theorem 3.2.

Theorem 3.3. *The complex projective space* $\mathrm{SU}(n+1)/\,\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(1))$ *of minimal sectional curvature* 1 *is holomorphically isometric to the space* $\mathbb{P}^n_{\mathbb{C}}$.

Proof. Both of these spaces are simply connected complete Kaehler manifolds of constant holomorphic sectional curvature 4. This, together with Theorem 1.7, brings us to the desired assertion. □

3.4 The Quaternionic Projective Space Sp(n + 1)/ Sp(n) × Sp(1)

As usual, let $\mathbb{Q}^n$ be the n-dimensional left quaternionic Euclidean space. In the same way as in Sects. 3.2 and 3.3 we see that the action of $\mathbb{Q}\backslash\{0\}$ on $\mathbb{Q}^{n+1}\backslash\{0\}$ on the left gives rise to a set $\mathbb{P}(\mathbb{Q}^{n+1})$ which can be regarded as the set of all left quaternionic lines in $\mathbb{Q}^{n+1}$ passing through the coordinate origin. Because of identification (2.37), the points of $\mathbb{P}(\mathbb{Q}^{n+1})$ are the classes $[\omega] = [(\omega_0,\dots,\omega_{2n+1})]$, other than $[(0,\dots,0)]$, of $(2n+2)$-tuples $\omega = (\omega_0,\dots,\omega_{2n+1})$ in $\mathbb{C}$ where two $(2n+2)$-tuples ω and $\varkappa = (\varkappa_0,\dots,\varkappa_{2n+1})$ belong to the same class if and only if there exists $(\lambda,\mu)\in\mathbb{C}^2_*$ such that

$$\lambda\varkappa_k - \mu\overline{\varkappa}_{n+1+k} = \omega_k, \quad \lambda\varkappa_{n+1+k} + \mu\overline{\varkappa}_k = \omega_{n+1+k}, \quad k = 0,\dots,n.$$

We can also make $\mathbb{P}(\mathbb{Q}^{n+1})$ into a differentiable manifold in a way similar to that used for $\mathbb{P}(\mathbb{R}^{n+1})$ and $\mathbb{P}(\mathbb{C}^{n+1})$.

Let

$$U_k = \left\{[(\omega_0,\dots,\omega_{2n+1})]\colon |\omega_k|^2 + |\omega_{n+1+k}|^2 \neq 0\right\}, \quad k = 0,\dots,n.$$

Introduce the bijective mappings $\phi_k\colon U_k \to \mathbb{C}^{2n}$ by setting

$$\phi_k\big([(\omega_0,\dots,\omega_{2n+1})]\big) = (z_1,\dots,z_{2n}),$$

where

$$z_m = \begin{cases} \big(\overline{\omega}_k\omega_{m-1} + \omega_{n+1+k}\overline{\omega}_{n+m}\big)\big/\big(|\omega_k|^2 + |\omega_{n+1+k}|^2\big) & \text{if } m \leqslant k, \\ \big(\overline{\omega}_k\omega_{m} + \omega_{n+1+k}\overline{\omega}_{n+1+m}\big)\big/\big(|\omega_k|^2 + |\omega_{n+1+k}|^2\big) & \text{if } m > k, \end{cases}$$

$$z_{n+m} = \begin{cases} \big(\overline{\omega}_k\omega_{n+m} - \omega_{n+1+k}\overline{\omega}_{m-1}\big)\big/\big(|\omega_k|^2 + |\omega_{n+1+k}|^2\big) & \text{if } m \leqslant k, \\ \big(\overline{\omega}_k\omega_{n+1+m} - \omega_{n+1+k}\overline{\omega}_{m}\big)\big/\big(|\omega_k|^2 + |\omega_{n+1+k}|^2\big) & \text{if } m > k, \end{cases}$$

for $m = 1, \dots, n$. Using quaternionic coordinates it is easy to make sure that both U_k and ϕ_k are well defined. For $k, l \in \{0, \dots, n\}$ such that $k < l$, we get

$$\phi_k(U_k \cap U_l) = \big\{(z_1, \dots, z_{2n}) \in \mathbb{C}^{2n} \colon |z_l|^2 + |z_{n+l}|^2 \neq 0\big\},$$
$$\phi_l(U_k \cap U_l) = \big\{(z_1, \dots, z_{2n}) \in \mathbb{C}^{2n} \colon |z_{k+1}|^2 + |z_{n+1+k}|^2 \neq 0\big\}.$$

In addition,

$$\phi_l \circ \phi_k^{-1}(z_1, \dots, z_{2n}) = (v_1, \dots, v_{2n}), \quad |z_l|^2 + |z_{n+l}|^2 \neq 0,$$

where

$$v_m = \begin{cases} \big(\overline{z}_l z_m + z_{n+l}\overline{z}_{n+m}\big)/\big(|z_l|^2 + |z_{n+l}|^2\big) & \text{if } m \leqslant k \text{ or } m > l, \\ \overline{z}_l/\big(|z_l|^2 + |z_{n+l}|^2\big) & \text{if } m = k+1, \\ \big(\overline{z}_l z_{m-1} + z_{n+l}\overline{z}_{n+m-1}\big)/\big(|z_l|^2 + |z_{n+l}|^2\big) & \text{if } k+1 < m \leqslant l, \end{cases} \tag{3.37}$$

$$v_{n+m} = \begin{cases} \big(\overline{z}_l z_{n+m} - z_{n+l}\overline{z}_m\big)/\big(|z_l|^2 + |z_{n+l}|^2\big) & \text{if } m \leqslant k \text{ or } m > l, \\ -z_{n+l}/\big(|z_l|^2 + |z_{n+l}|^2\big) & \text{if } m = k+1, \\ \big(\overline{z}_l z_{n+m-1} - z_{n+l}\overline{z}_{m-1}\big)/\big(|z_l|^2 + |z_{n+l}|^2\big) & \text{if } k+1 < m \leqslant l, \end{cases} \tag{3.38}$$

for $m = 1, \dots, n$. Hence, the map

$$\phi_l \circ \phi_k^{-1} \colon \phi_k(U_k \cap U_l) \to \phi_l(U_k \cap U_l)$$

is a diffeomorphism. Thus, the charts (U_k, ϕ_k) form an atlas and define on $\mathbb{P}(\mathbb{Q}^{n+1})$ a real-analytic structure.

Every coordinate neighborhood U_k is diffeomorphic to $\mathbb{C}^{2n}$. Identifying U_0 with $\mathbb{C}^{2n}$ one has

$$\mathbb{P}\big(\mathbb{Q}^{n+1}\big) = \mathbb{C}^{2n} \cup \mathbb{P}\big(\mathbb{Q}^n\big). \tag{3.39}$$

The Riemannian metric on $\mathbb{P}(\mathbb{Q}^{n+1})$ is obtained as follows. For $z = (z_1, \dots, z_{2n}) \in \mathbb{C}^{2n}$ and $i, j \in \{1, \dots, n\}$, we put

$$h_{ij}(z) = \frac{(1 + |z|^2)\delta_{i,j} - \overline{z}_i z_j - z_{n+i}\overline{z}_{n+j}}{(1 + |z|^2)^2}, \tag{3.40}$$

$$h_{i,n+j}(z) = \frac{z_{n+i}\overline{z}_j - \overline{z}_i z_{n+j}}{(1 + |z|^2)^2}, \tag{3.41}$$

$$h_{n+i,j}(z) = \frac{z_i\overline{z}_{n+j} - \overline{z}_{n+i} z_j}{(1 + |z|^2)^2}, \tag{3.42}$$

$$h_{n+i,n+j}(z) = \frac{(1 + |z|^2)\delta_{i,j} - z_i\overline{z}_j - \overline{z}_{n+i} z_{n+j}}{(1 + |z|^2)^2}. \tag{3.43}$$

Now define

$$H_{rs}^k(p) = h_{rs}\big(\phi_k(p)\big), \quad p \in U_k, \quad r, s \in \{1, \dots, 2n\}. \tag{3.44}$$

For $w = (w_1, \dots, w_{2n}) \in \mathbb{C}^{2n}$, we have

$$\sum_{r,s=1}^{2n} h_{rs}(z) w_r \overline{w}_s = \frac{|w|^2 + |w|^2|z|^2 - |\langle z, w\rangle_{\mathbb{Q}}|^2}{(1+|z|^2)^2} \geqslant \frac{|w|^2}{(1+|z|^2)^2}, \tag{3.45}$$

whence $\|h_{rs}\|_{r,s=1}^{2n}$ is positive definite. Next,

$$h_{rs}(z) = \sum_{\alpha,\beta=1}^{2n} \biggl(h_{\alpha\beta}\bigl(v_1(z), \dots, v_{2n}(z)\bigr) \overline{\frac{\partial v_\beta}{\partial z_s}}(z) \biggl(\frac{\partial v_\alpha}{\partial z_r}(z) + \frac{\partial v_\alpha}{\partial \overline{z}_r}(z) \biggr) + \overline{h_{\alpha\beta}}\bigl(v_1(z), \dots, v_{2n}(z)\bigr) \frac{\partial v_\beta}{\partial \overline{z}_s}(z) \biggl(\frac{\partial \overline{v}_\alpha}{\partial z_r}(z) + \overline{\frac{\partial v_\alpha}{\partial z_r}}(z) \biggr)\biggr)$$

for $z \in \phi_k(U_k \cap U_l)$ (see (3.37), (3.38), (3.40)–(3.43), and the proof of relation (3.24)). This gives

$$H_{rs}^k(p) = \sum_{\alpha,\beta=1}^{2n} \biggl(H_{\alpha\beta}^l(p) \overline{\frac{\partial v_\beta}{\partial z_s}}\bigl(\phi_k(p)\bigr) \biggl(\frac{\partial v_\alpha}{\partial z_r}\bigl(\phi_k(p)\bigr) + \frac{\partial v_\alpha}{\partial \overline{z}_r}\bigl(\phi_k(p)\bigr) \biggr) + \overline{H_{\alpha\beta}^l}(p) \frac{\partial v_\beta}{\partial \overline{z}_s}\bigl(\phi_k(p)\bigr) \biggl(\frac{\partial \overline{v_\alpha}}{\partial z_r}\bigl(\phi_k(p)\bigr) + \overline{\frac{\partial v_\alpha}{\partial z_r}}\bigl(\phi_k(p)\bigr) \biggr)\biggr)$$

with $p \in U_k \cap U_l$. It follows that matrices (3.44) induce the structure of a Riemannian manifold on $\mathbb{P}(\mathbb{Q}^{n+1})$. Denote this manifold by $\mathbb{P}_{\mathbb{Q}}^n$.

By analogy with (3.14) and (3.33) the Riemannian measure on $\mathbb{P}_{\mathbb{Q}}^n$ has the form

$$\mathrm{d}\mu(z) = \frac{\mathrm{d}m_{2n}(z)}{(1+|z|^2)^{2n+2}}, \quad z \in \mathbb{C}^{2n}.$$

Since

$$\begin{aligned}
h^{ij}(z) &= (\delta_{i,j} + \overline{z}_i z_j + z_{n+i}\overline{z}_{n+j})\bigl(1+|z|^2\bigr),\\
h^{i,n+j}(z) &= (\overline{z}_i z_{n+j} - z_{n+i}\overline{z}_j)\bigl(1+|z|^2\bigr),\\
h^{n+i,j}(z) &= (\overline{z}_{n+i} z_j - z_i\overline{z}_{n+j})\bigl(1+|z|^2\bigr),\\
h^{n+i,n+j}(z) &= (\delta_{i,j} + z_i \overline{z}_j + \overline{z}_{n+i} z_{n+j})\bigl(1+|z|^2\bigr),
\end{aligned}$$

the Laplace–Beltrami operator L on $\mathbb{P}_{\mathbb{Q}}^n$ acts on a function $f \in C^2(\mathbb{C}^{2n})$ as follows:

$$(Lf)(z) = 4\bigl(1+|z|^2\bigr)\Biggl(\sum_{i,j=1}^{n} \biggl((\delta_{i,j} + z_i\overline{z}_j + \overline{z}_{n+i} z_{n+j}) \frac{\partial^2 f}{\partial z_i \partial \overline{z}_j}(z) + (z_i \overline{z}_{n+j} - \overline{z}_{n+i} z_j) \frac{\partial^2 f}{\partial z_i \partial \overline{z}_{n+j}}(z) + (z_{n+i}\overline{z}_j - \overline{z}_i z_{n+j}) \frac{\partial^2 f}{\partial z_{n+i} \partial \overline{z}_j}(z)$$

$$+ (\delta_{i,j} + \overline{z}_i z_j + z_{n+i}\overline{z}_{n+j})\frac{\partial^2 f}{\partial z_{n+i}\partial \overline{z}_{n+j}}(z)\Big)$$
$$-\sum_{q=1}^{2n}\Big(z_q\frac{\partial f}{\partial z_q}(z) + \overline{z}_q\frac{\partial f}{\partial \overline{z}_q}(z)\Big)\Big).$$

Hence,

$$(Lf)(z) = \big(1+|z|^2\big)^2 f_0''(|z|) + \frac{1+|z|^2}{|z|}\big(4n-1-|z|^2\big)f_0'(|z|) \tag{3.46}$$

if f has the form $f(z) = f_0(|z|)$.

Let us say a few words about the isometry group of the space $\mathbb{P}^n_{\mathbb{Q}}$.

In view of (3.45), symplectic unitary transformations of $\mathbb{C}^{2n}$ induce isometric mappings of $\mathbb{P}^n_{\mathbb{Q}}$ onto itself. In particular, the isometry group $I(\mathbb{P}^n_{\mathbb{Q}})$ acts transitively on $\mathbb{S}^{4n-1}$ and $\mathbb{P}(\mathbb{Q}^n)$ (see (3.39)).

Next, take $A = (A_1, \dots, A_n) \in \mathbb{Q}^n$ and $q = (q_0, \dots, q_n) \in \mathbb{Q}^{n+1}\backslash\{0\}$. Let $Q = (Q_0, \dots, Q_n)$, where

$$Q_l = \begin{cases} q_0 + q_1\overline{A}_1 + \dots + q_n\overline{A}_n & \text{if } l = 0, \\ \left(q_0 + \dfrac{q_1\overline{A}_1 + \dots + q_n\overline{A}_n}{1+\sqrt{1+|A|^2}}\right)A_l - \sqrt{1+|A|^2}\,q_l & \text{if } 1 \leqslant l \leqslant n. \end{cases}$$

If $A \leftrightarrow a$, $q \leftrightarrow \omega$, and $Q \leftrightarrow \varkappa$ under identification (2.37), put

$$\sigma_a([\omega]) = [\varkappa].$$

It is not hard to prove that σ_a is well defined. Denote by $a_1, \dots, a_{2n}$ the coordinates of $a \in \mathbb{C}^{2n}$. Then

$$\phi_0 \circ \sigma_a \circ \phi_0^{-1}(z) = \big(f_1(z), \dots, f_{2n}(z)\big)$$

with

$$|1+\langle z,a\rangle_{\mathbb{Q}}|^2 f_i(z) = -[a,z]_{\mathbb{C}}\big(\nu_a\overline{z}_{n+i} + (\gamma_a - 1)\overline{a}_{n+i}\big) + \langle a,z\rangle_{\mathbb{C}}(-\nu_a z_i + a_i)$$
$$+ \langle z,a\rangle_{\mathbb{C}}\gamma_a a_i + |\langle z,a\rangle_{\mathbb{Q}}|^2\gamma_a a_i + a_i - \nu_a z_i,$$
$$|1+\langle z,a\rangle_{\mathbb{Q}}|^2 f_{n+i}(z) = [a,z]_{\mathbb{C}}\big(\nu_a\overline{z}_i - (1-\gamma_a)\overline{a}_i\big) + \langle a,z\rangle_{\mathbb{C}}(-\nu_a z_{n+i} + a_{n+i})$$
$$+ \langle z,a\rangle_{\mathbb{C}}\gamma_a a_{n+i} + |\langle z,a\rangle_{\mathbb{Q}}|^2\gamma_a a_{n+i} + a_{n+i} - \nu_a z_{n+i}.$$

Here $i \in \{1, \dots, n\}$, $\nu_a = \sqrt{1+|a|^2}$, and $\gamma_a = (1+\nu_a)^{-1}$. Similarly, setting

$$\sigma_\infty\big([(\omega_0, \dots, \omega_{2n+1})]\big) = [(\omega_1, \omega_0, \omega_2, \dots, \omega_n, \omega_{n+2}, \omega_{n+1}, \omega_{n+3}, \dots, \omega_{2n+1})],$$

we find

$$\phi_0 \circ \sigma_\infty \circ \phi_0^{-1}(z) = \big(g_1(z), \dots, g_{2n}(z)\big),$$

where

$$g_i(z) = \begin{cases} \overline{z}_1/\big(|z_1|^2 + |z_{n+1}|^2\big) & \text{if } i = 1, \\ \big(\overline{z}_1 z_i + z_{n+1}\overline{z}_{n+i}\big)/\big(|z_1|^2 + |z_{n+1}|^2\big) & \text{if } i \in \{2, \dots, n\}, \end{cases}$$

$$g_{n+i}(z) = \begin{cases} -z_{n+1}/\big(|z_1|^2 + |z_{n+1}|^2\big) & \text{if } i = 1, \\ \big(\overline{z}_1 z_{n+i} - z_{n+1}\overline{z}_i\big)/\big(|z_1|^2 + |z_{n+1}|^2\big) & \text{if } i \in \{2, \dots, n\}. \end{cases}$$

The mappings σ_a and σ_∞ are involutory isometries of the space $\mathbb{P}^n_{\mathbb{Q}}$ (see the proof of Proposition 3.1). In addition,

$$\sigma_a\big([(1, 0, \dots, 0)]\big) = [(1, a_1, \dots, a_n, 0, a_{n+1}, \dots, a_{2n})],$$
$$\sigma_\infty\big([(1, 0, \dots, 0)]\big) \in \mathbb{P}(\mathbb{Q}^n).$$

In the same manner as in Sect. 3.2, we see that the distance on $\mathbb{P}^n_{\mathbb{Q}}$ from the origin is

$$d(0, z) = \begin{cases} \arctan|z|, & z \in \mathbb{C}^{2n}, \\ \pi/2, & z \in \mathbb{P}(\mathbb{Q}^n). \end{cases} \tag{3.47}$$

It again follows that $I(\mathbb{P}^n_{\mathbb{Q}})$ is pairwise transitive on $\mathbb{P}^n_{\mathbb{Q}}$.

Theorem 3.4. *The quaternionic projective space* $\operatorname{Sp}(n+1)/\operatorname{Sp}(n) \times \operatorname{Sp}(1)$ *of minimal sectional curvature* 1 *is isometric to the space* $\mathbb{P}^n_{\mathbb{Q}}$.

Proof. We already know that $\mathbb{P}^n_{\mathbb{Q}}$ is a compact two-point homogeneous space. Furthermore, by (3.46) and (3.47) one has

$$L_0 = \frac{\partial^2}{\partial r^2} + \big((4n-1)\cot r - 3\tan r\big)\frac{\partial}{\partial r}, \tag{3.48}$$

where L_0 is the radial part of the Laplace–Beltrami operator on $\mathbb{P}^n_{\mathbb{Q}}$. Using (3.48), (1.61), and (1.64), we obtain the required result. □

3.5 The Cayley Projective Plane $F_4/\operatorname{Spin}(9)$

In this section we wish to construct a suitable model for $F_4/\operatorname{Spin}(9)$. In this way we hope to contribute towards a better visualization and a better handling of this space. Note that there is no direct analogue to the realizations in real, complex, or quaternionic projective spaces, because the Cayley numbers do not obey the associative law.

Let $\mathbb{P}^2(\mathbb{C}a)$ be the set of all *primitive idempotents* of the algebra $\mathbb{A}1$, i.e.,

$$\mathbb{P}^2(\mathbb{C}a) = \big\{X \in \mathbb{A}1 : X^2 = X,\ \operatorname{Trace} X = 1\big\}.$$

We write $X \in \mathbb{A}1$ in the form

$$X = \begin{pmatrix} a_1 & \xi_3 & \overline{\xi}_2 \\ \overline{\xi}_3 & a_2 & \xi_1 \\ \xi_2 & \overline{\xi}_1 & a_3 \end{pmatrix}, \tag{3.49}$$

where $a_1, a_2, a_3 \in \mathbb{R}$ and $\xi_1, \xi_2, \xi_3 \in \mathbb{C}a$. It is not hard to prove that $X \in \mathbb{P}^2(\mathbb{C}a)$ if and only if $a_1 + a_2 + a_3 = 1$ and

$$a_k \overline{\xi}_k = \xi_{k+1}\xi_{k+2}, \quad |\xi_k|^2 = a_{k+1}a_{k+2}, \quad k = 1, 2, 3 \mod 3.$$

We endow $\mathbb{P}^2(\mathbb{C}a)$ with a real-analytic structure. Identifying matrix (3.49) with the vector $(\xi, a) = (\xi_1, \xi_2, \xi_3, a_1, a_2, a_3)$, put

$$U_k = \left\{(\xi, a) \in \mathbb{P}^2(\mathbb{C}a) : a_k \neq 0\right\}, \quad k = 1, 2, 3.$$

Introduce the bijective mappings $\phi_k : U_k \to \mathbb{C}a^2$ as follows:

$$\phi_k(\xi, a) = \left(\frac{\xi_{k+1}}{a_k}, \frac{\xi_{k+2}}{a_k}\right), \quad k = 1, 2, 3 \mod 3.$$

It is clear that $\phi_k(U_k \cap U_{k+1}) = \{(z_1, z_2) \in \mathbb{C}a^2 : z_2 \neq 0\}$ and $\phi_{k+1}(U_k \cap U_{k+1}) = \{(z_1, z_2) \in \mathbb{C}a^2 : z_1 \neq 0\}$, $k = 1, 2, 3 \mod 3$. For $(z_1, z_2) \in \mathbb{C}a^2$, we have

$$\begin{aligned} \phi_1^{-1}(z_1, z_2) &= (\xi_1, \xi_2, \xi_3, a_1, a_2, a_3), \\ \phi_2^{-1}(z_1, z_2) &= (\xi_3, \xi_1, \xi_2, a_3, a_1, a_2), \\ \phi_3^{-1}(z_1, z_2) &= (\xi_2, \xi_3, \xi_1, a_2, a_3, a_1), \end{aligned}$$

where

$$\begin{aligned} \xi_1 &= \frac{\overline{z}_2\,\overline{z}_1}{1 + |z_1|^2 + |z_2|^2}, \quad & \xi_2 &= \frac{z_1}{1 + |z_1|^2 + |z_2|^2}, \quad & \xi_3 &= \frac{z_2}{1 + |z_1|^2 + |z_2|^2}, \\ a_1 &= \frac{1}{1 + |z_1|^2 + |z_2|^2}, \quad & a_2 &= \frac{|z_2|^2}{1 + |z_1|^2 + |z_2|^2}, \quad & a_3 &= \frac{|z_1|^2}{1 + |z_1|^2 + |z_2|^2}. \end{aligned}$$

In addition,

$$\phi_{k+1} \circ \phi_k^{-1}(z_1, z_2) = \left((\overline{z}_2)^{-1}, z_2^{-1}\overline{z}_1\right), \quad k = 1, 2, 3 \mod 3$$

for $z_2 \neq 0$. Hence, the map $\phi_{k+1} \circ \phi_k^{-1}$ is a diffeomorphism between open sets in $\mathbb{C}a^2$ (here and in the sequel, we identify $\mathbb{C}a^2$ with $\mathbb{R}^{16}$ via (1.15)). Thus, $\mathbb{P}^2(\mathbb{C}a)$ is a real-analytic manifold of dimension 16.

Every coordinate neighborhood U_k is diffeomorphic to $\mathbb{R}^{16}$. Considering U_3 as $\mathbb{R}^{16}$, we see that

$$\mathbb{P}^2(\mathbb{C}a) = \mathbb{R}^{16} \cup S^8, \tag{3.50}$$

where

$$S^8 = \left\{(0, 0, \xi_3, a_1, 1 - a_1, 0) \in \mathbb{P}^2(\mathbb{C}a)\colon |\xi_3|^2 + \left(a_1 - \frac{1}{2}\right)^2 = \frac{1}{4}\right\}.$$

Let $\Phi_{\mathbb{C}a}(x, y)$ be the function defined in Sect. 1.1. Set

$$\begin{aligned} g_{ij}(x) &= \frac{\delta_{i,j}}{1+|x|^2} - \frac{1}{2(1+|x|^2)^2}\frac{\partial^2}{\partial y_i \partial y_j}\big(\Phi_{\mathbb{C}a}(x, y)\big), \quad i, j \in \{1, \dots, 16\}, \\ G^k_{ij}(p) &= g_{ij}\big(\phi_k(p)\big), \quad p \in U_k, k = 1, 2, 3. \end{aligned} \tag{3.51}$$

Applying (1.24), we obtain

$$\sum_{i,j=1}^{16} g_{ij}(x) y_i y_j = \frac{|y|^2}{1+|x|^2} - \frac{\Phi_{\mathbb{C}a}(x, y)}{(1+|x|^2)^2} \geqslant \frac{|y|^2}{(1+|x|^2)^2}.$$

Consequently, $\|g_{ij}\|^{16}_{i,j=1}$ is positive definite. Next, for $x_2^2 + x_4^2 + \dots + x_{16}^2 \neq 0$, we have

$$g_{ms}(x) = \sum_{i,j=1}^{16} g_{ij}\big(v_1(x), \dots, v_{16}(x)\big)\frac{\partial v_i}{\partial x_m}(x)\frac{\partial v_j}{\partial x_s}(x), \quad m, s \in \{1, \dots, 16\},$$

where $v_1(x), \dots, v_{16}(x)$ are the components of $\phi_{k+1} \circ \phi_k^{-1}$ (see the proof of (3.24)). This relation implies

$$G^k_{ms}(p) = \sum_{i,j=1}^{16} G^{k+1}_{ij}(p)\frac{\partial v_i}{\partial x_m}\big(\phi_k(p)\big)\frac{\partial v_j}{\partial x_s}\big(\phi_k(p)\big), \quad p \in U_k \cap U_{k+1},$$

with $k = 1, 2, 3 \bmod 3$. By that matrices (3.51) define the structure of a Riemannian manifold on $\mathbb{P}^2(\mathbb{C}a)$. We denote this manifold by $\mathbb{P}^2_{\mathbb{C}a}$.

Basic properties of $\mathbb{P}^2_{\mathbb{C}a}$ are obtained by the method we already know (see Sect. 2.4 and the proof of Proposition 3.1). Accordingly, we shall content ourselves with a brief sketch and statement of the results.

The Riemannian measure on $\mathbb{P}^2_{\mathbb{C}a}$ has the form

$$\mathrm{d}\mu(x) = \frac{\mathrm{d}x}{(1+|x|^2)^{12}}, \quad x \in \mathbb{R}^{16}.$$

The Laplace–Beltrami operator L on $\mathbb{P}^2_{\mathbb{C}a}$ acts on a function $f \in C^2(\mathbb{R}^{16})$ as follows:

$$(Lf)(x) = \sum_{i,j=1}^{16} g^{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) - 12\big(1+|x|^2\big)\sum_{i=1}^{16} x_i \frac{\partial f}{\partial x_i}(x),$$

where

$$g^{ij}(x) = \big(1+|x|^2\big)\big(2+|x|^2\big)\delta_{i,j} - \big(1+|x|^2\big)^3 g_{ij}(x).$$

In particular, if f has the form $f(x) = f_0(|x|)$, then

$$(Lf)(x) = \big(1+|x|^2\big)^2 f_0''(|x|) + \frac{1+|x|^2}{|x|}\big(15-5|x|^2\big) f_0'(|x|).$$

Take $t \in \mathbb{R}$ and $\zeta \in \mathbb{C}a$ such that $t^2 + |\zeta|^2 = 1$. Let $u = (t, \zeta)$. Introduce the mapping $\mathcal{R}_u : \mathbb{P}^2_{\mathbb{C}a} \to \mathbb{P}^2_{\mathbb{C}a}$ by putting

$$\mathcal{R}_u(\xi_1, \xi_2, \xi_3, a_1, a_2, a_3) = (\eta_1, \eta_2, \eta_3, b_1, b_2, b_3),$$

where

$$\eta_1 = -t\xi_1 + \overline{\zeta}\,\overline{\xi}_2, \quad \eta_2 = t\xi_2 + \overline{\xi}_1\,\overline{\zeta}, \quad \eta_3 = -t^2\xi_3 + ta_1\zeta - ta_2\zeta + \zeta\overline{\xi}_3\zeta,$$
$$b_1 = t^2 a_1 + |\zeta|^2 a_2 + 2t\,\mathrm{Re}\,(\xi_3\overline{\zeta}), \quad b_2 = t^2 a_2 + |\zeta|^2 a_1 - 2t\,\mathrm{Re}\,(\xi_3\overline{\zeta}), \quad b_3 = a_3.$$

Using (1.2)–(1.8), one finds

$$\mathcal{R}_u|_{U_3} = \phi_3^{-1} \circ R_u \circ \phi_3,$$

where R_u is the involution defined in Example 1.3. Hence, $\mathcal{R}_u \in I(\mathbb{P}^2_{\mathbb{C}a})$. Furthermore, taking the relation

$$\mathcal{R}_u(0,0,0,0,1,0) = \big(0, 0, -t\zeta, |\zeta|^2, 1-|\zeta|^2, 0\big)$$

into account, we obtain that the group generated by all $\mathcal{R}_u$ acts transitively on $\mathbb{S}^{15}$ and S^8 (see the proof of Proposition 1.1).

Let $\alpha \in \mathbb{R}^1$ and $\nu_\alpha = \sqrt{1+\alpha^2}$. Set

$$\sigma_\alpha(\xi, a) = (\theta_1, \theta_2, \theta_3, c_1, c_2, c_3),$$

where

$$\theta_1 = \frac{\alpha a_3 + \alpha^2\overline{\xi}_1 - \xi_1 - \alpha a_2}{\nu_\alpha^2}, \quad \theta_2 = -\frac{\xi_2 + \alpha\overline{\xi}_3}{\nu_\alpha}, \quad \theta_3 = \frac{-\alpha\overline{\xi}_2 + \xi_3}{\nu_\alpha},$$
$$c_1 = a_1, \quad c_2 = \frac{a_2 + \alpha^2 a_3 - 2\alpha\,\mathrm{Re}\,\xi_1}{\nu_\alpha^2}, \quad c_3 = \frac{a_3 + \alpha^2 a_2 + 2\alpha\,\mathrm{Re}\,\xi_1}{\nu_\alpha^2}.$$

We also define

$$\sigma_\infty(\xi, a) = \big(\overline{\xi}_1, -\overline{\xi}_3, -\overline{\xi}_2, a_1, a_3, a_2\big).$$

Note that

$$\phi_3 \circ \sigma_\alpha \circ \phi_3^{-1}(z_1, z_2) = \big((\alpha - z_1)(\alpha z_1 + 1)^{-1}, -\nu_\alpha(\alpha\overline{z}_1 + 1)^{-1} z_2\big),$$

$$\phi_3 \circ \sigma_\infty \circ \phi_3^{-1}(z_1, z_2) = \big(z_1^{-1}, -(\overline{z}_1)^{-1} z_2\big).$$

The mappings σ_α and σ_∞ are involutory isometries of the space $\mathbb{P}^2_{\mathbb{C}a}$. In addition,

$$\phi_3 \circ \sigma_\alpha \circ \phi_3^{-1}(0,0) = (\alpha, 0), \qquad \sigma_\infty(0,0,0,0,0,1) \in S^8.$$

In view of what has been said above, the group $I(\mathbb{P}^2_{\mathbb{C}a})$ acts transitively on $\mathbb{P}^2_{\mathbb{C}a}$.

The distance d on $\mathbb{P}^2_{\mathbb{C}a}$ is defined by

$$d(0,x) = \begin{cases} \arctan|x|, & x \in \mathbb{R}^{16}, \\ \pi/2, & x \in S^8, \end{cases}$$

and the condition of invariance under the group $I\big(\mathbb{P}^2_{\mathbb{C}a}\big)$. As before, we see that the manifold $\mathbb{P}^2_{\mathbb{C}a}$ is a compact two-point homogeneous space. Moreover, the following result is valid.

Theorem 3.5. *The Cayley projective plane* $F_4/\,\mathrm{Spin}(9)$ *of minimal sectional curvature* 1 *is isometric to the space* $\mathbb{P}^2_{\mathbb{C}a}$.

The proof of Theorem 3.5 is analogous to that of a similar result about the space $\mathbb{P}^n_{\mathbb{Q}}$ (see Sect. 3.4).

Chapter 4
Realizations of the Irreducible Components of the Quasi-Regular Representation of Groups Transitive on Spheres. Invariant Subspaces

In this chapter we show how the techniques of Chaps. 2 and 3 can be applied to harmonic analysis on spheres. Thus, in contrast to harmonic analysis on general compact homogeneous spaces (see Sect. 1.5), our point of view here is to place the models of two-point homogeneous spaces in the foreground. The significance of such an approach for us is as follows: firstly, the obtained results play an important role in the theory of transmutation operators on rank one compact symmetric spaces (see Part II later); secondly, the treatment is accessible to a wider audience as its use of Lie theory is minimal.

From the Fourier analysis on $\mathbb{S}^1$ we know that every $f \in L^2(\mathbb{S}^1)$ has an expansion of the form

$$f\left(\mathrm{e}^{\mathrm{i}t}\right) = \sum_{k\in\mathbb{Z}} c_k \mathrm{e}^{\mathrm{i}kt},$$

where the sum converges in $L^2(\mathbb{S}^1)$. In Sect. 4.1 we will see that an analogous expansion is valid for functions $f \in L^2(\mathbb{S}^{n-1})$ when $n \geqslant 3$, with objects known as spherical harmonics playing the roles of the exponentials $\mathrm{e}^{\mathrm{i}kt}$. We prove that $\mathrm{SO}(n)$ acts irreducibly on each space $\mathcal{H}_1^{n,k}$ of spherical harmonics of degree k. Our purpose then is to give an explicit description of the decomposition of the spherical harmonics into irreducible submodules under the action of the following groups: $\mathrm{U}(n), \mathrm{O}_{\mathbb{C}}(n), \mathrm{Sp}(n), \mathrm{O}_{\mathbb{Q}}(n),$ and $\mathrm{O}_{\mathbb{C}a}(2)$. The corresponding results are presented in Sects. 4.2–4.6. We characterize the components of the decomposition as eigenspaces of some differential operators and determine explicitly the zonal harmonics in these spaces. In all cases except $\mathrm{Sp}(n)$, $n \geqslant 2$, the zonal is unique (see the proofs of Theorems 4.1, 4.3, 4.5, 4.8, and 4.9). The case of $\mathrm{Sp}(n)$ is rather different. It is taken up in Theorem 4.6. Finally, as a consequence of these results, we give explicit representations for the spaces on spheres which are invariant under the above mentioned groups (see Theorems 4.2, 4.4, 4.7, and 4.10 and Remark 4.3).

V.V. Volchkov, V.V. Volchkov, *Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group,* Springer Monographs in Mathematics,

4.1 The Groups SO(n) and O(n)

Let $K = \mathrm{SO}(n)$ or $K = \mathrm{O}(n)$, $n \geqslant 2$, and let $T_1^n(\tau)$, $\tau \in K$, be the quasi-regular representation of the group K on $L^2(\mathbb{S}^{n-1})$. For $k \in \mathbb{Z}_+$, $\mathcal{P}_k^n$ denotes the space of all homogeneous complex-valued polynomials on $\mathbb{R}^n$ of degree k, and $\mathcal{H}_1^{n,k}$ is the space of all $f \in \mathcal{P}_k^n$ that satisfy $\Delta f = 0$. Each $f \in \mathcal{H}_1^{n,k}$ is uniquely determined by its restriction to $\mathbb{S}^{n-1}$. These restrictions are called *spherical harmonics* of degree k. Identifying $\mathcal{H}_1^{n,k}$ with $\mathcal{H}_1^{n,k}|_{\mathbb{S}^{n-1}}$, we obtain that $\mathcal{H}_1^{n,k}$ is an invariant subspace of $T_1^n(\tau)$, $\tau \in K$; hence, it defines the representation

$$T_1^{n,k}(\tau) = T_1^n(\tau)\big|_{\mathcal{H}_1^{n,k}}, \quad \tau \in K.$$

It is easy to see that $\mathcal{H}_1^{2,k}$ is the complex linear span of $\{(x_1+\mathrm{i}x_2)^k, (x_1-\mathrm{i}x_2)^k\}$. Therefore, $\mathcal{H}_1^{2,k}|_{\mathbb{S}^1}$, as a space of functions of the variable $\mathrm{e}^{\mathrm{i}\varphi}$, $-\pi < \varphi \leqslant \pi$, is the complex linear span of $\{\mathrm{e}^{\mathrm{i}k\varphi}, \mathrm{e}^{-\mathrm{i}k\varphi}\}$. It follows from the theory of Fourier series on the unit circle that $L^2(\mathbb{S}^1)$ is the orthogonal direct sum of the spaces $\mathcal{H}_1^{2,k}$. The representations $T_1^{2,k}(\tau)$, $\tau \in K$, are pairwise nonequivalent. In addition, $T_1^{2,k}(\tau)$, $\tau \in \mathrm{O}(2)$, are irreducible, whereas $T_1^{2,k}(\tau)$, $\tau \in \mathrm{SO}(2)$, is the direct sum of two one-dimensional representations.

In the next statements of this section we shall assume that $n \geqslant 3$.

Theorem 4.1. *The quasi-regular representation $T_1^n(\tau)$, $\tau \in K$, is the orthogonal direct sum of the pairwise nonequivalent irreducible unitary representations*

$$T_1^{n,k}(\tau), \quad \tau \in K, k \in \mathbb{Z}_+.$$

To prove Theorem 4.1 we require several auxiliary results.

Lemma 4.1.

(i) *Let $f \in \mathcal{P}_{k-2m}^n$, where $m \in \{0, \dots, [k/2]\}$. Then*

$$\Delta\big(|x|^{2m} f(x)\big) = 2m(n+2k-2m-2)|x|^{2m-2} f(x) + |x|^{2m}(\Delta f)(x). \quad (4.1)$$

(ii) *If $f \in \mathcal{P}_k^n$, $k \geqslant 2$, then the polynomial*

$$f(x) + \sum_{m=1}^{[k/2]} \frac{(-1)^m |x|^{2m} \Delta^m f(x)}{2^m m!(n+2k-4)(n+2k-6)\cdots(n+2k-2m-2)}$$

belongs to the space $\mathcal{H}_1^{n,k}$.

Proof. Equality (4.1) can be obtained by induction on m using Euler's formula for homogeneous functions. Part (ii) follows from (4.1) by a direct calculation. □

Corollary 4.1. *For $f \in \mathcal{P}_k^n$, we have*

$$f(x) = \sum_{m=0}^{[k/2]} |x|^{2m} f_{k-2m}(x), \tag{4.2}$$

where $f_{k-2m} \in \mathcal{H}_1^{n,k-2m}$.

Proof. Any polynomial of degree less than 2 is harmonic. Hence we may assume that $k \geqslant 2$. From Lemma 4.1(ii) we see that

$$f(x) = f_k(x) + |x|^2 g_{k-2}(x),$$

where $f_k \in \mathcal{H}_1^{n,k}$, $g_{k-2} \in \mathcal{P}_{k-2}^n$. Now the assertion follows by induction. □

Corollary 4.2. *The linear span of $\{\mathcal{H}_1^{n,k},\ k \in \mathbb{Z}_+\}$ is dense in $C(\mathbb{S}^{n-1})$.*

Proof. The restriction to $\mathbb{S}^{n-1}$ of any polynomial of n variables is a sum of restrictions to $\mathbb{S}^{n-1}$ of harmonic polynomials (see (4.2)). Applying the Weierstrass approximation theorem, we arrive at the desired assertion. □

Lemma 4.2. *The space $L^2(\mathbb{S}^{n-1})$ is the orthogonal direct sum of the spaces $\mathcal{H}_1^{n,k}$, $k \in \mathbb{Z}_+$.*

Proof. Let $f \in \mathcal{H}_1^{n,k}$, $g \in \mathcal{H}_1^{n,l}$, where $k \neq l$. Denote by L the Laplace–Beltrami operator on $\mathbb{S}^{n-1}$ (see Sect. 3.1). Then

$$Lf = -k(k+n-2)f \quad \text{and} \quad Lg = -l(l+n-2)g.$$

Since $-k(k+n-2) \neq -l(l+n-2)$, relation (1.44) gives

$$\int_{\mathbb{S}^{n-1}} f(\xi)\overline{g(\xi)}\, \mathrm{d}\omega(\xi) = 0,$$

where $\mathrm{d}\omega$ is a surface element of $\mathbb{S}^{n-1}$. Hence, the spaces $\mathcal{H}_1^{n,k}$ and $\mathcal{H}_1^{n,l}$ are orthogonal. In view of Corollary 4.2, this concludes the proof. □

Let us fix a point $x \in \mathbb{S}^{n-1}$ and consider the linear functional on $\mathcal{H}_1^{n,k}$ that assigns to each f in $\mathcal{H}_1^{n,k}$ the value $f(x)$. By the self-duality of the finite-dimensional inner product space $\mathcal{H}_1^{n,k}$ there exists a unique function $P_x \in \mathcal{H}_1^{n,k}$ such that

$$f(x) = \int_{\mathbb{S}^{n-1}} f(\xi)\overline{P_x(\xi)}\, \mathrm{d}\omega(\xi), \qquad f \in \mathcal{H}_1^{n,k}. \tag{4.3}$$

This function P_x is called the *zonal harmonic* of degree k with pole x. To simplify notation, we shall sometimes use $\langle f, P_x \rangle$ to denote the integral on the right-hand side of (4.3).

Lemma 4.3.

(i) *For any function* $f \in L^2(\mathbb{S}^{n-1})$*, we have*

$$(\pi_k f)(x) = \langle f, P_x \rangle,$$

where π_k *is the orthogonal projection of* $L^2(\mathbb{S}^{n-1})$ *onto* $\mathcal{H}_1^{n,k}$.

(ii)
$$P_x(y) = \overline{P_y(x)}, \quad x, y \in \mathbb{S}^{n-1}. \tag{4.4}$$

(iii)
$$P_{\tau x} = P_x \circ \tau^{-1}, \quad \tau \in \mathrm{O}(n). \tag{4.5}$$

(iv) $P_x = P_x \circ \tau$ *for any* $\tau \in \mathrm{O}(n)$ *such that* $\tau x = x$.

(v)
$$P_x(x) = P_y(y) > 0, \quad x, y \in \mathbb{S}^{n-1}. \tag{4.6}$$

Proof. Part (i) is an immediate consequence of (4.3) and Lemma 4.2. Next, by the defining property of zonal harmonics,

$$P_y(x) = \langle P_y, P_x \rangle = \overline{\langle P_x, P_y \rangle} = \overline{P_x(y)},$$

and part (ii) is established.

Since π_k commutes with $\mathrm{O}(n)$,

$$\langle f, P_{\tau x} \rangle = (\pi_k f)(\tau x) = \pi_k(f \circ \tau)(x) = \langle f \circ \tau, P_x \rangle = \langle f, P_x \circ \tau^{-1} \rangle$$

for every $f \in L^2(\mathbb{S}^{n-1})$. This proves (iii) and hence also its special case (iv). Finally,

$$P_{\tau x}(\tau x) = \left(P_x \circ \tau^{-1}\right)(\tau x) = P_x(x), \quad \tau \in \mathrm{O}(n),$$

is another consequence of (iii). It proves (v), because $P_x(x) = \langle P_x, P_x \rangle > 0$. □

Lemma 4.4. *For each* $x \in \mathbb{S}^{n-1}$*, the space* $\mathcal{H}_1^{n,k}$ *contains a unique* f *such that* $f(x) = 1$ *and* $f = f \circ \tau$ *for every* $\tau \in \mathrm{SO}(n)$ *that fixes* x.

Proof. It is easy to dispense with the case where $k < 2$. Let $k \geqslant 2$. The existence of f follows from Lemma 4.3(iv). To prove the uniqueness, assume that $x = (1, 0, \dots, 0)$, without loss of generality, and write points $y \in \mathbb{R}^n$ in the form $y = (y_1, 'y)$, where $'y = (y_2, \dots, y_n)$. The invariance $f = f \circ \tau$ shows then, for each y_1, that $f(y_1, 'y)$ is a radial polynomial in $'y$ and hence is a polynomial in $|'y|^2$ (see Proposition 1.2). Taking into account that $f \in \mathcal{P}_k^n$, we have

$$f(y) = \sum_{j=0}^{[k/2]} c_j y_1^{k-2j} |'y|^{2j}, \quad c_j \in \mathbb{C},$$

where $c_0 = 1$. Therefore,

$$(\Delta f)(y) = \sum_{j=0}^{[k/2]-1} (c_j a_j + c_{j+1} b_j) y_1^{k-2(j+1)} |'y|^{2j}, \tag{4.7}$$

where $a_j = (k-2j)(k-2j-1)$ and $b_j = 2(j+1)(n+2j-1)$. Since f is harmonic, relation (4.7) yields

$$c_{j+1} = -\frac{a_j}{b_j} c_j, \quad j = 0, \ldots, [k/2]-1.$$

But this asserts that all the coefficients $c_1, \ldots, c_{[k/2]}$ are determined by c_0. Thus, Lemma 4.4 is proved. □

Lemma 4.5. *Let* $A\colon \mathcal{H}_1^{n,k} \to \mathcal{H}_1^{n,l}$ *be a linear mapping and suppose that* A *commutes with the group* SO(n). *Then*

$$A = \begin{cases} c\,\mathrm{Id} & \text{if } k = l, \\ 0 & \text{if } k \neq l, \end{cases} \tag{4.8}$$

where $c \in \mathbb{C}$, *and* Id *is the identity operator.*

Proof. Let $P_x \in \mathcal{H}_1^{n,k}$ be as in (4.3), and let Q_x be the corresponding function in $\mathcal{H}_1^{n,l}$. If $\tau \in \mathrm{SO}(n)$ and $\tau x = x$, then

$$(AP_x) \circ \tau = A(P_x \circ \tau) = AP_x$$

by Lemma 4.3(iv), since A commutes with SO(n). Hence, to each $x \in \mathbb{S}^{n-1}$ there corresponds a $c(x) \in \mathbb{C}$ such that $AP_x = c(x)Q_x$ (see Lemma 4.4). Take $\varphi \in \mathrm{SO}(n)$ arbitrarily. Relations (4.5) and (4.6) imply

$$c(\varphi(x)) = \frac{AP_{\varphi(x)}(\varphi(x))}{Q_{\varphi(x)}(\varphi(x))} = \frac{A(P_x \circ \varphi^{-1})(\varphi(x))}{Q_x(x)} = \frac{(AP_x)(x)}{Q_x(x)} = c(x).$$

We obtain from this that $c(x) = c$, the same for all $x \in \mathbb{S}^{n-1}$. Next, in view of (4.3) and (4.4),

$$f(x) = \int_{\mathbb{S}^{n-1}} f(\xi) P_\xi(x)\,\mathrm{d}\omega(\xi), \quad f \in \mathcal{H}_1^{n,k}. \tag{4.9}$$

Apply the operator A to (4.9). Because $AP_\xi = cQ_\xi$, we have

$$(Af)(x) = c\int_{\mathbb{S}^{n-1}} f(\xi) Q_\xi(x)\,\mathrm{d}\omega(\xi) = c(\pi_l f)(x) \tag{4.10}$$

for every $f \in \mathcal{H}_1^{n,k}$. Finally, Lemma 4.2 shows that (4.10) is just another way of writing (4.8). □

Proof of Theorem 4.1. By Lemma 4.2,

$$T_1^n(\tau) = \bigoplus_{k=0}^{\infty} T_1^{n,k}(\tau).$$

Let $\mathcal{H}$ be an invariant subspace of the representation $T_1^{n,k}(\tau)$. Then so is $\mathcal{H}^{\perp}$, the orthogonal complement of $\mathcal{H}$ relative to $\mathcal{H}_1^{n,k}$. Hence, the projection of $\mathcal{H}_1^{n,k}$ onto $\mathcal{H}$ commutes with $\mathrm{SO}(n)$ and would violate Lemma 4.5 (the case $k = l$) unless $\mathcal{H} = \mathcal{H}_1^{n,k}$ or $\mathcal{H} = \{0\}$. Thus, the representations $T_1^{n,k}(\tau)$, $k \in \mathbb{Z}_+$, are irreducible. The pairwise nonequivalence of $T_1^{n,k}(\tau)$, $k \in \mathbb{Z}_+$, follows from Lemma 4.5 (the case $k \neq l$). □

Corollary 4.3. *The space $\mathcal{H}_1^{n,k}$ is the linear span of functions of the form*

$$(a_1x_1 + \cdots + a_nx_n)^k,$$

where $a_1, \ldots, a_n$ are complex constants such that $a_1^2 + \cdots + a_n^2 = 0$.

Proof. Let $\mathcal{H}$ be the linear span of the system $\{f \circ \tau, \tau \in \mathrm{SO}(n)\}$, where $f(x) = (x_1 + \mathrm{i}x_2)^k$. It is clear that $\mathcal{H}$ is an invariant subspace of $T_1^{n,k}(\tau)$, $\tau \in \mathrm{SO}(n)$. Hence, $\mathcal{H} = \mathcal{H}_1^{n,k}$, since the representation $T_1^{n,k}(\tau)$, $\tau \in \mathrm{SO}(n)$, is irreducible. Now after some simple calculations we obtain the required result. □

We now present an application of Theorem 4.1.

A space H of functions on $\mathbb{S}^{n-1}$ is said to be K*-invariant*, or simply K*-space*, if $f \circ \tau \in H$ whenever $f \in H$ and $\tau \in K$. For example, every $\mathcal{H}_1^{n,k}$ is a K-space, since Δ commutes with K.

Theorem 4.1 allows one to obtain an explicit description of K-invariant subspaces of $\mathfrak{B}^n$, where $\mathfrak{B}^n = C(\mathbb{S}^{n-1})$ or $L^p(\mathbb{S}^{n-1})$, $p \geqslant 1$. To state the corresponding result we need some notation.

For a nonempty set $\Omega \subset \mathbb{Z}_+$, denote by $\mathcal{H}_\Omega$ the algebraic sum of all $\mathcal{H}_1^{n,k}$ with $k \in \Omega$. If $\Omega = \varnothing$, put $\mathcal{H}_\Omega = \{0\}$. The symbol $\mathfrak{B}^n_\Omega$ will stand for the $\mathfrak{B}^n$-closure of $\mathcal{H}_\Omega$.

Trivially, every $\mathfrak{B}^n_\Omega$ is a closed K-space in $\mathfrak{B}^n$. On the other hand, the following result is true.

Theorem 4.2. *If H is a K-invariant subspace of $\mathfrak{B}^n$, then $H = \mathfrak{B}^n_\Omega$ for some set $\Omega \subset \mathbb{Z}_+$.*

To prove the theorem we require one technical lemma.

Lemma 4.6. *If H is a closed K-space in $\mathfrak{B}^n$, then $H \cap C^\infty(\mathbb{S}^{n-1})$ is dense in H.*

Proof. The assertion of Lemma 4.6 can be deduced by means of the standard approximation trick, when a function $f \in H$ is replaced by convolutions of the form

$$\int_K \psi(\tau)\big(f \circ \tau^{-1}\big)\,\mathrm{d}\tau, \quad \psi \in \mathcal{D}(K).$$

We shall not stop to reproduce the details here (see, for example, Rudin [184], the proof of Lemma 12.3.4). □

Proof of Theorem 4.2. Let $\{Y_j^k\}$, $j \in \{1, \dots, d(n,k)\}$, be a fixed orthonormal basis in $\mathcal{H}_1^{n,k}$. To any function $f \in L^1(\mathbb{S}^{n-1})$ there corresponds the Fourier series

$$f \sim \sum_{k=0}^{\infty} \sum_{j=1}^{d(n,k)} f_{k,j} Y_j^k, \tag{4.11}$$

where

$$f_{k,j} = \int_{\mathbb{S}^{n-1}} f(\xi) \overline{Y_j^k(\xi)} \, \mathrm{d}\omega(\xi).$$

For $\eta \in \mathbb{S}^{n-1}$, define

$$(\pi_k f)(\eta) = \sum_{j=1}^{d(n,k)} f_{k,j} Y_j^k(\eta) = \int_{\mathbb{S}^{n-1}} f(\xi) \overline{P_\eta(\xi)} \, \mathrm{d}\omega(\xi).$$

Now set

$$\Omega_H = \left\{k \in \mathbb{Z}_+ : \pi_k H \neq \{0\}\right\}$$

and prove that

$$H = \mathfrak{B}^n_{\Omega_H}. \tag{4.12}$$

Pick $k \in \Omega_H$. Then there exists a function $f \in H$ for which $\pi_k f \neq 0$. In particular, $f_{k,j_0} \neq 0$ for some $1 \leqslant j_0 \leqslant d(n,k)$. By (1.80),

$$f_{k,j_0} Y_{j_0}^k = d(n,k) \int_K \left(f \circ \tau^{-1}\right) \overline{t_{j_0,j_0}^k(\tau)} \, \mathrm{d}\tau, \tag{4.13}$$

where $\{t_{i,j}^k(\tau)\}$, $i, j \in \{1, \dots, d(n,k)\}$, is the matrix of the representation $T_1^n(\tau)$, $\tau \in K$. The integrand in (4.13) is a continuous H-valued function on K. Consequently, $f_{k,j_0} Y_{j_0}^k \in H$. Hence, in view of the irreducibility of $T_1^{n,k}(\tau)$, $\mathcal{H}_1^{n,k} \subset H$. Since $k \in \Omega_H$ above was arbitrary, we obtain $\mathfrak{B}^n_{\Omega_H} \subset H$. Finally, taking Lemma 4.2 into account and using Lemma 4.6, we arrive at (4.12). This finishes the proof. □

4.2 The Group U(n)

Let $T_2^n(\tau)$, $n \geqslant 2$, be the quasi-regular representation of the group U(n) on $L^2(\mathbb{S}^{2n-1})$. Denote by $\mathcal{H}_2^{n,p,q}$, $p, q \in \mathbb{Z}_+$, the vector space of all harmonic homogeneous polynomials on $\mathbb{C}^n$ that have total degree p in the variables $z_1, \dots, z_n$ and total degree q in the variables $\overline{z}_1, \dots, \overline{z}_n$. As before, we identify $\mathcal{H}_2^{n,p,q}$ with the space of restrictions of its elements to $\mathbb{S}^{2n-1}$. It is clear that $\mathcal{H}_2^{n,p,q}$ is an invariant subspace of the representation $T_2^n(\tau)$. Set

$$T_2^{n,p,q}(\tau) = T_2^n(\tau)\big|_{\mathcal{H}_2^{n,p,q}}.$$

The aim of this section is to prove the following result.

Theorem 4.3. *The quasi-regular representation $T_2^n(\tau)$ is the orthogonal direct sum of the pairwise nonequivalent irreducible unitary representations*

$$T_2^{n,p,q}(\tau), \quad p, q \in \mathbb{Z}_+.$$

The proof of Theorem 4.3 requires some preparation.

Identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via correspondence (1.13), we see that

$$\mathcal{H}_2^{n,p,q} \subset \mathcal{H}_1^{2n,k}$$

whenever $p + q = k$. Actually, more is true:

Lemma 4.7. *The space $\mathcal{H}_1^{2n,k}$ is the sum of the pairwise orthogonal spaces $\mathcal{H}_2^{n,p,q}$, where $p + q = k$.*

Proof. Suppose that $(p,q) \neq (r,s)$, $p + q = r + s = k$. For $f \in \mathcal{H}_2^{n,p,q}$ and $g \in \mathcal{H}_2^{n,r,s}$, we have

$$\begin{aligned} \int_{\mathbb{S}^{2n-1}} f(\zeta)\overline{g(\zeta)}\,\mathrm{d}\omega(\zeta) &= \frac{1}{2\pi}\int_{\mathbb{S}^{2n-1}}\int_{-\pi}^{\pi} f\left(\mathrm{e}^{\mathrm{i}\theta}\zeta\right)\overline{g(\mathrm{e}^{\mathrm{i}\theta}\zeta)}\,\mathrm{d}\theta\,\mathrm{d}\omega(\zeta) \\ &= \frac{1}{2\pi}\int_{\mathbb{S}^{2n-1}}\int_{-\pi}^{\pi} f(\zeta)\overline{g(\zeta)}\mathrm{e}^{\mathrm{i}(p-q+s-r)\theta}\,\mathrm{d}\theta\,\mathrm{d}\omega(\zeta) \\ &= 0, \end{aligned}$$

because $p - q + s - r \neq 0$. Hence, the spaces $\mathcal{H}_2^{n,p,q}$ and $\mathcal{H}_2^{n,r,s}$ are orthogonal.

Next, let $a_1, \ldots, a_{2n}$ be complex constants such that $a_1^2 + \cdots + a_{2n}^2 = 0$. If $x = (x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n}$, put

$$\begin{aligned} h(x) &= (a_1x_1 + \cdots + a_{2n}x_{2n})^k, \\ z_k &= x_k + \mathrm{i}x_{n+k}, \quad 1 \leqslant k \leqslant n, \qquad z = (z_1, \ldots, z_n). \end{aligned}$$

We obtain

$$h(x) = \frac{1}{2^k}\sum_{l=0}^{k}\binom{k}{l}h_l(z), \quad \text{where } \binom{k}{l} = \frac{k!}{l!(k-l)!},$$

$$h_l(z) = \left(\sum_{m=1}^{n}(a_m - \mathrm{i}a_{n+m})z_m\right)^l\left(\sum_{m=1}^{n}(a_m + \mathrm{i}a_{n+m})\overline{z}_m\right)^{k-l}.$$

Since

$$\sum_{m=1}^{n}(a_m - \mathrm{i}a_{n+m})(a_m + \mathrm{i}a_{n+m}) = \sum_{j=1}^{2n}a_j^2 = 0,$$

the polynomial h_l belongs to $\mathcal{H}_2^{n,l,k-l}$. Using Corollary 4.3, we complete the proof of Lemma 4.7. □

From this result and Lemma 4.2 we obtain

Corollary 4.4. *The space* $L^2(\mathbb{S}^{2n-1})$ *is the direct sum of the pairwise orthogonal spaces* $\mathcal{H}_2^{n,p,q}$, $p, q \in \mathbb{Z}_+$.

Now we present analogues of Lemmas 4.3 and 4.4 for the spaces $\mathcal{H}_2^{n,p,q}$.

Lemma 4.8. *Fix* (p, q). *To every* $z \in \mathbb{S}^{2n-1}$ *there corresponds a unique* $P_z \in \mathcal{H}_2^{n,p,q}$ *that satisfies*

$$(\pi_{p,q} f)(z) = \int_{\mathbb{S}^{2n-1}} f(\zeta)\overline{P_z(\zeta)}\,\mathrm{d}\omega(\zeta), \quad f \in L^2\big(\mathbb{S}^{2n-1}\big), \tag{4.14}$$

where $\pi_{p,q}$ *is the orthogonal projection of* $L^2(\mathbb{S}^{2n-1})$ *onto* $\mathcal{H}_2^{n,p,q}$. *These functions* P_z *have the following properties.*

(i) $P_z(w) = \overline{P_w(z)}$, $z, w \in \mathbb{S}^{2n-1}$.
(ii) $P_{\tau z} = P_z \circ \tau^{-1}$, $\tau \in \mathrm{U}(n)$.
(iii) $P_z = P_z \circ \tau$ *for all* $\tau \in \mathrm{U}(n)$ *that fix* z.
(iv) $P_z(z) = P_w(w) > 0$, $z, w \in \mathbb{S}^{2n-1}$.

The proof of Lemma 4.8 is quite similar to that of Lemma 4.3.

Remark 4.1. Relation (4.14) allows us to extend the domain of $\pi_{p,q}$ to $L^1(\mathbb{S}^{2n-1})$.

Lemma 4.9. *For each* $z \in \mathbb{S}^{2n-1}$, *the space* $\mathcal{H}_2^{n,p,q}$ *contains a unique* f *such that* $f(z) = 1$ *and* $f = f \circ \tau$ *for every* $\tau \in \mathrm{U}(n)$ *that fixes* z.

Proof. It suffices to prove the uniqueness of f (see Lemma 4.8(iii)). We assume without loss of generality that $z = (1, 0, \dots, 0)$. Let $w = (w_1, \dots, w_n) \in \mathbb{C}^n$, $'w = (w_2, \dots, w_n)$. In the same way as in the proof of Lemma 4.4, we see that

$$f(w) = \sum_{j=0}^{\min\{p,q\}} c_j w_1^{p-j}\overline{w}_1^{q-j}\left|'w\right|^{2j}, \quad c_j \in \mathbb{C},$$

where $c_0 = 1$. In particular,

$$f(w) = \begin{cases} w_1^p & \text{if } q = 0, \\ \overline{w}_1^q & \text{if } p = 0. \end{cases}$$

For $\min\{p, q\} \geqslant 1$, we have

$$(\Delta f)(w) = 4 \sum_{j=0}^{\min\{p,q\}-1} b_j w_1^{p-j-1}\overline{w}_1^{q-j-1}\left|'w\right|^{2j}, \tag{4.15}$$

where

$$b_j = (p-j)(q-j)c_j + (j+1)(n+j-1)c_{j+1}, \quad 0 \leqslant j \leqslant \min\{p,q\} - 1.$$

Since f is harmonic, equality (4.15) gives

$$c_{j+1} = \frac{(p-j)(j-q)}{(j+1)(n+j-1)} c_j, \quad j = 0, \dots, \min\{p, q\} - 1.$$

It follows that the function f is uniquely determined. □

Proof of Theorem 4.3. Corollary 4.4 implies that

$$T_2^n(\tau) = \bigoplus_{p,q=0}^{\infty} T_2^{n,p,q}(\tau).$$

Next, let $A\colon \mathcal{H}_2^{n,p,q} \to \mathcal{H}_2^{n,r,s}$ be a linear operator and assume that A commutes with the group $\mathrm{U}(n)$. Using Lemmas 4.8 and 4.9 and repeating the arguments in the proof of Lemma 4.5, we obtain

$$A = \begin{cases} c\,\mathrm{Id} & \text{if } (p,q) = (r,s), \\ 0 & \text{if } (p,q) \neq (r,s). \end{cases} \tag{4.16}$$

Thus, the representations $T_2^{n,p,q}(\tau)$, $p, q \in \mathbb{Z}_+$, are irreducible and pairwise non-equivalent. □

As a consequence, we have the following description of $\mathcal{H}_2^{n,p,q}$.

Corollary 4.5. *The space $\mathcal{H}_2^{n,p,q}$ is the linear span of functions of the form*

$$(\alpha_1 z_1 + \dots + \alpha_n z_n)^p (\beta_1 \overline{z}_1 + \dots + \beta_n \overline{z}_n)^q,$$

where $\alpha_1, \dots, \alpha_n$, $\beta_1, \dots, \beta_n$ are complex constants such that $\alpha_1\beta_1 + \dots + \alpha_n\beta_n = 0$.

Proof. It is obvious that $z_1^p \overline{z}_2^q \in \mathcal{H}_2^{n,p,q}$. Using the irreducibility of the representation $T_2^{n,p,q}(\tau)$, we arrive at the required assertion (see the proof of Corollary 4.3). □

The following is the analog of Theorem 4.2.

Theorem 4.4. *If H is a $\mathrm{U}(n)$-invariant subspace of $\mathfrak{B}^{2n}$, then H is the $\mathfrak{B}^{2n}$-closure of the algebraic sum of all $\mathcal{H}_2^{n,p,q}$ such that $\pi_{p,q} H \neq \{0\}$.*

The proof of this result is similar to that of Theorem 4.2 (see Theorem 4.3).

4.3 The Group $\mathrm{O}_{\mathbb{C}}(n)$

A suitable realization of the irreducible components of a quasi-regular representation $T_3^n(\tau)$, $n \geqslant 2$, of the group $\mathrm{O}_{\mathbb{C}}(n)$ on $L^2(\mathbb{S}^{2n-1})$ is given here. It will serve us as a pattern for the more complicated cases of the groups $\mathrm{Sp}(n)$, $\mathrm{O}_{\mathbb{Q}}(n)$, and $\mathrm{O}_{\mathbb{C}a}(2)$.

As before, we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via correspondence (1.13). If $k \in \mathbb{Z}_+$ and $m \in \{0, \dots, [k/2]\}$, put

$$\mathcal{H}_3^{n,k,m} = \{f \in \mathcal{H}_1^{2n,k} : \mathcal{L}f = \lambda_{k,m} f\},$$

where

$$\mathcal{L} = \sum_{i,j=1}^{n} (\delta_{i,j} - z_i \overline{z}_j) \frac{\partial^2}{\partial z_i \partial \overline{z}_j}, \quad \lambda_{k,m} = m(m-k). \tag{4.17}$$

Using Corollary 4.5, we find

$$\mathcal{L}f = -pqf$$

for all $f \in \mathcal{H}_2^{n,p,q}$. Therefore,

$$\mathcal{H}_2^{n,m,k-m} + \mathcal{H}_2^{n,k-m,m} \subset \mathcal{H}_3^{n,k,m}. \tag{4.18}$$

Because

$$(\mathcal{L}f)(z) = \frac{1}{4}\left(1 - |z|^2\right)^{-1}(Lf)(z), \quad f \in C^2\left(B_{\mathbb{C}}^n\right), \tag{4.19}$$

where L is the Laplace–Beltrami operator on the complex hyperbolic space $\mathbb{H}_{\mathbb{C}}^n$, Proposition 2.9 shows that $\mathcal{H}_3^{n,k,m}$ is an invariant subspace of $T_3^n(\tau)$. By $T_3^{n,k,m}(\tau)$ denote the restriction of $T_3^n(\tau)$ to $\mathcal{H}_3^{n,k,m}$.

Theorem 4.5. *The quasi-regular representation $T_3^n(\tau)$ is the orthogonal direct sum of the pairwise nonequivalent irreducible unitary representations*

$$T_3^{n,k,m}(\tau), \quad k \in \mathbb{Z}_+, m \in \{0, \dots, [k/2]\}.$$

The proof of this theorem is based on the results of Sect. 4.2. We shall need two auxiliary statements.

Lemma 4.10. *Suppose that $(k_1, m_1) \neq (k_2, m_2)$. Then the spaces $\mathcal{H}_3^{n,k_1,m_1}$ and $\mathcal{H}_3^{n,k_2,m_2}$ are orthogonal.*

Proof. In view of orthogonality of the spaces $\mathcal{H}_1^{2n,k}$, $k \in \mathbb{Z}_+$, we may assume that $k_1 = k_2$ and $m_1 \neq m_2$. Let $f \in \mathcal{H}_3^{n,k_1,m_1}$, $g \in \mathcal{H}_3^{n,k_2,m_2}$. We fix $\varepsilon \in (0,1)$ and consider a function $\varphi \in C^\infty(B_{\mathbb{C}}^n)$ with the following properties:

(a) φ has the form $\varphi(z) = \varphi_0(|z|)$;
(b) $\varphi(z) \geqslant 0$ for $z \in B_{\mathbb{C}}^n$;
(c) $\operatorname{supp} \varphi = \{z \in B_{\mathbb{C}}^n : |z| \leqslant \varepsilon\}$.

Since the operator L is symmetric, we have

$$4\lambda_{k_1,m_1} \int_{B_{\mathbb{C}}^n} f(z)\overline{g(z)}\varphi_0(|z|)\left(1 - |z|^2\right) d\mu(z)$$
$$= \int_{B_{\mathbb{C}}^n} (Lf)(z)\overline{g(z)}\varphi_0(|z|)\, d\mu(z) = \int_{B_{\mathbb{C}}^n} f(z)\overline{L\left(\varphi_0(|z|)g(z)\right)}\, d\mu(z), \tag{4.20}$$

where $\mathrm{d}\mu$ is the Riemannian measure on $\mathbb{H}^n_{\mathbb{C}}$. By (4.17) and (4.19), we find

$$L\big(\varphi_0(|z|)g(z)\big) = \bigg(\big(1-|z|^2\big)^2\varphi_0''(|z|) + \frac{1-|z|^2}{|z|}\big(2n+2k_2-1 - (2k_2+1)|z|^2\big)\varphi_0'(|z|) + 4\lambda_{k_2,m_2}\big(1-|z|^2\big)\varphi_0(|z|)\bigg)g(z). \tag{4.21}$$

Relations (4.20) and (4.21) imply

$$\begin{aligned}
&4(\lambda_{k_1,m_1}-\lambda_{k_2,m_2})\int_{B^n_{\mathbb{C}}} f(z)\overline{g(z)}\varphi_0(|z|)\big(1-|z|^2\big)\,\mathrm{d}\mu(z)\\
&= \int_{B^n_{\mathbb{C}}} f(z)\overline{g(z)}\bigg(\big(1-|z|^2\big)^2\varphi_0''(|z|) + \frac{1-|z|^2}{|z|}\big(2n+2k_2-1\\
&\quad - (2k_2+1)|z|^2\big)\varphi_0'(|z|)\bigg)\,\mathrm{d}\mu(z)\\
&= \int_0^{\varepsilon} \frac{\mathrm{d}}{\mathrm{d}\varrho}\bigg(\frac{\varrho^{2n-1+2k_2}}{(1-\varrho^2)^{n-1}}\varphi_0'(\varrho)\bigg)\,\mathrm{d}\varrho\int_{\mathbb{S}^{2n-1}} f(\sigma)\overline{g(\sigma)}\,\mathrm{d}\omega(\sigma)\\
&= 0.
\end{aligned}$$

Bearing in mind that $\lambda_{k_1,m_1}\neq\lambda_{k_2,m_2}$, we obtain

$$\int_{B^n_{\mathbb{C}}} f(z)\overline{g(z)}\varphi_0(|z|)\big(1-|z|^2\big)\,\mathrm{d}\mu(z) = 0.$$

Equivalently,

$$\int_0^{\varepsilon}\frac{\varrho^{2n-1+2k_2}}{(1-\varrho^2)^n}\varphi_0(\varrho)\,\mathrm{d}\varrho\int_{\mathbb{S}^{2n-1}} f(\sigma)\overline{g(\sigma)}\,\mathrm{d}\omega(\sigma) = 0.$$

Thus,

$$\int_{\mathbb{S}^{2n-1}} f(\sigma)\overline{g(\sigma)}\,\mathrm{d}\omega(\sigma) = 0,$$

and Lemma 4.10 is proved. □

Lemma 4.11. *The space $\mathcal{H}_3^{n,k,m}$ is the sum of the spaces $\mathcal{H}_2^{n,m,k-m}$ and $\mathcal{H}_2^{n,k-m,m}$.*

Proof. By virtue of (4.18), it suffices to prove that

$$\mathcal{H}_3^{n,k,m}\subset\mathcal{H}_2^{n,m,k-m}+\mathcal{H}_2^{n,k-m,m}. \tag{4.22}$$

Let $f\in\mathcal{H}_3^{n,k,m}$. From Lemma 4.7 we have

$$f=\sum_{j=0}^{[k/2]} f_j,$$

where $f_j \in \mathcal{H}_2^{n,j,k-j} + \mathcal{H}_2^{n,k-j,j}$. Using (4.18) and Lemma 4.10, we get $f = f_m$, whence (4.22) follows. □

Combining Lemma 4.11 with Lemmas 4.2 and 4.7, we deduce

Corollary 4.6. *The following decomposition holds:*

$$L^2(\mathbb{S}^{2n-1}) = \bigoplus_{k=0}^{\infty} \bigoplus_{m=0}^{[k/2]} \mathcal{H}_3^{n,k,m}.$$

Proof of Theorem 4.5. Owing to Corollary 4.6,

$$T_3^n(\tau) = \bigoplus_{k=0}^{\infty} \bigoplus_{m=0}^{[k/2]} T_3^{n,k,m}(\tau).$$

Next, let $A\colon \mathcal{H}_3^{n,k_1,m_1} \to \mathcal{H}_3^{n,k_2,m_2}$ be a linear map and suppose that A commutes with $\mathrm{O}_{\mathbb{C}}(n)$. Take $f \in \mathcal{H}_2^{n,m_1,k_1-m_1}$. Lemma 4.11 implies that

$$Af = f_1 + f_2,$$

where $f_1 \in \mathcal{H}_2^{n,m_2,k_2-m_2}$, $f_2 \in \mathcal{H}_2^{n,k_2-m_2,m_2}$. Hence,

$$\begin{aligned} e^{i\theta(2m_1-k_1)}(Af)(z) &= A\big(f\big(e^{i\theta}z\big)\big) \\ &= (Af)\big(e^{i\theta}z\big) \\ &= e^{i\theta(2m_2-k_2)} f_1(z) + e^{i\theta(k_2-2m_2)} f_2(z) \end{aligned} \tag{4.23}$$

for any $\theta \in \mathbb{R}^1$. If $f_2 \not\equiv 0$ and $Af \not\equiv 0$, then (4.23) gives $k_2 = 2m_2$, $k_1 = 2m_1$. Using (4.16), we now see that A sends $\mathcal{H}_2^{n,m_1,k_1-m_1}$ to $\mathcal{H}_2^{n,m_2,k_2-m_2}$ and

$$A|_{\mathcal{H}_2^{n,m_1,k_1-m_1}} = \begin{cases} c_1 \,\mathrm{Id} & \text{if } (k_1,m_1) = (k_2,m_2), \\ 0 & \text{if } (k_1,m_1) \neq (k_2,m_2). \end{cases} \tag{4.24}$$

Similarly,

$$A|_{\mathcal{H}_2^{n,k_1-m_1,m_1}} = \begin{cases} c_2 \,\mathrm{Id} & \text{if } (k_1,m_1) = (k_2,m_2), \\ 0 & \text{if } (k_1,m_1) \neq (k_2,m_2). \end{cases} \tag{4.25}$$

Since A commutes with the mapping $\tau\colon (z_1,\dots,z_n) \to (\overline{z}_1,\dots,\overline{z}_n)$, we conclude from (4.24) and (4.25) that $c_1 = c_2$. This finishes the proof. □

Note, in conclusion, that using Theorems 4.4 and 4.5, it is easy to obtain a description of $\mathrm{O}_{\mathbb{C}}(n)$-invariant subspaces of $\mathfrak{B}^{2n}$. We leave for the reader to state and prove the corresponding result.

4.4 The Group Sp(n)

Having completed our study of the spaces $\mathcal{H}_2^{n,p,q}$, we shall now show how they can be used in the harmonic analysis on $\mathbb{S}^{4n-1} = \mathrm{Sp}(n)/\mathrm{Sp}(n-1)$.

Throughout this section we assume that

$$n \geqslant 2, \quad p, q \in \mathbb{Z}_+, \quad r = \min\{p, q\}, \quad \text{and} \quad s = p + q.$$

Let L be the Laplace–Beltrami operator on the quaternionic hyperbolic space $\mathbb{H}^n_{\mathbb{Q}}$ (see Proposition 2.14). Introduce the differential operator $\mathcal{L}\colon C^2(B^{2n}_{\mathbb{C}}) \to C(B^{2n}_{\mathbb{C}})$ as follows:

$$(\mathcal{L}f)(z) = \frac{(Lf)(z)}{4(1-|z|^2)}, \quad f \in C^2\big(B^{2n}_{\mathbb{C}}\big).$$

For $m \in \{0, \dots, r\}$, we set

$$\mathcal{H}_4^{n,p,q,m} = \big\{f \in \mathcal{H}_2^{2n,p,q} : \mathcal{L}f = \lambda_{p,q,m} f\big\},$$

where

$$\lambda_{p,q,m} = (m-1)(m-s).$$

Since $\mathrm{Sp}(n) \subset I(\mathbb{H}^n_{\mathbb{Q}})$ (see Proposition 2.12), the space $\mathcal{H}_4^{n,p,q,m}$ is an invariant subspace of the quasi-regular representation $T_4^n(\tau)$ of the group $\mathrm{Sp}(n)$ on $L^2(\mathbb{S}^{4n-1})$. We denote by $T_4^{n,p,q,m}(\tau)$ the restriction of $T_4^n(\tau)$ to $\mathcal{H}_4^{n,p,q,m}$.

The objective of this section is to establish the following result.

Theorem 4.6. *The quasi-regular representation $T_4^n(\tau)$ is the orthogonal direct sum of the irreducible unitary representations*

$$T_4^{n,p,q,m}(\tau), \quad p, q \in \mathbb{Z}_+, m \in \{0, \dots, r\}.$$

The representations $T_4^{n,p_i,q_i,m_i}(\tau)$, $i = 1, 2$, are equivalent if and only if

$$p_1 + q_1 = p_2 + q_2, \quad m_1 = m_2.$$

Before giving the proof of our theorem, let us consider some properties of the spaces $\mathcal{H}_4^{n,p,q,m}$.

Put

$$\mathcal{L}_1 = \sum_{i,j=1}^{2n} (\delta_{i,j} - z_i \overline{z}_j) \frac{\partial^2}{\partial z_i \partial \overline{z}_j} + \sum_{i=1}^{2n} \left(z_i \frac{\partial}{\partial z_i} + \overline{z}_i \frac{\partial}{\partial \overline{z}_i} \right),$$

$$\mathcal{L}_2 = \mathcal{L} - \mathcal{L}_1.$$

Using Corollary 4.5, we obtain

$$\mathcal{L}_1 f = (s - pq) f$$

for any polynomial $f \in \mathcal{H}_2^{2n,p,q}$. Hence,

$$\mathcal{H}_4^{n,p,q,m} = \big\{ f \in \mathcal{H}_2^{2n,p,q} : \mathcal{L}_2 f = (\lambda_{p,q,m} - s + pq) f \big\}.$$

Lemma 4.12. *The space $\mathcal{H}_4^{n,p,q,m}$ contains the polynomial*

$$Q_{p,q,m}(z) = z_{n+1}^{p-m} \overline{z}_1^{q-m} (\overline{z}_1 z_{n+2} - \overline{z}_2 z_{n+1})^m.$$

Proof. It is clear that $Q_{p,q,m} \in \mathcal{H}_2^{2n,p,q}$. Next,

$$Q_{p,q,m}(z) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} z_{n+1}^{p-m+k} z_{n+2}^{m-k} \overline{z}_1^{q-k} \overline{z}_2^{k}.$$

One finds

$$\begin{aligned}\mathcal{L}_2\big(z_{n+1}^{p-m+k} z_{n+2}^{m-k} \overline{z}_1^{q-k} \overline{z}_2^{k}\big) &= \big((p-m+k)(q-k) + (m-k)k\big) z_{n+1}^{p-m+k} z_{n+2}^{m-k} \overline{z}_1^{q-k} \overline{z}_2^{k} \\ &\quad + (m-k)(q-k) z_{n+1}^{p-m+k+1} z_{n+2}^{m-k-1} \overline{z}_1^{q-k-1} \overline{z}_2^{k+1} \\ &\quad + k(p-m+k) z_{n+1}^{p-m+k-1} z_{n+2}^{m-k+1} \overline{z}_1^{q-k+1} \overline{z}_2^{k-1}.\end{aligned}$$

Summation over the set of all $k \in \{0, \dots, m\}$ yields

$$\mathcal{L}_2(Q_{p,q,m}) = (\lambda_{p,q,m} - s + pq) Q_{p,q,m},$$

whence $Q_{p,q,m} \in \mathcal{H}_4^{n,p,q,m}$. □

Lemma 4.13. *Suppose that $f \in C^2(B_{\mathbb{C}}^{2n})$ has the form $f(z) = \varphi(|z|) g(z)$, where $g \in \mathcal{H}_4^{n,p,q,m}$. Then*

$$\begin{aligned}(Lf)(z) = \bigg(&\big(1-|z|^2\big)^2 \varphi''(|z|) + \frac{1-|z|^2}{|z|}\big(4n + 2s - 1 - (2s-1)|z|^2\big)\varphi'(|z|) \\ &+ 4\lambda_{p,q,m}\big(1-|z|^2\big)\varphi(|z|)\bigg) g(z). \qquad (4.26)\end{aligned}$$

Proof. This is a direct consequence of Proposition 2.14. □

Lemma 4.14. *The spaces $\mathcal{H}_4^{n,p_1,q_1,m_1}$ and $\mathcal{H}_4^{n,p_2,q_2,m_2}$ are orthogonal, provided that $(p_1, q_1, m_1) \neq (p_2, q_2, m_2)$.*

Proof. Since the spaces $\mathcal{H}_2^{2n,p_1,q_1}$ and $\mathcal{H}_2^{2n,p_2,q_2}$, $(p_1, q_1) \neq (p_2, q_2)$, are orthogonal, it can be assumed that $p_1 = p_2$, $q_1 = q_2$, and $m_1 \neq m_2$. Then we have

$$\lambda_{p_1,q_1,m_1} \neq \lambda_{p_2,q_2,m_2},$$

and the desired assertion is implied by the symmetry of the operator L and formula (4.26) (see the proof of Lemma 4.10). □

We set

$$l_1(z) = \sum_{i=1}^{2n} \alpha_i z_i, \qquad l_2(z) = \sum_{i=1}^{2n} \beta_i \overline{z}_i,$$
$$l_3(z) = \sum_{i=1}^{n} (\alpha_i \overline{z}_{n+i} - \alpha_{n+i}\overline{z}_i), \qquad l_4(z) = \sum_{i=1}^{n} (\beta_{n+i} z_i - \beta_i z_{n+i}),$$

where $\alpha_i, \beta_i \in \mathbb{C}$ and $\alpha_1\beta_1 + \cdots + \alpha_{2n}\beta_{2n} = 0$. It can easily be seen that

$$l_1^{p-m} l_2^{q-m} l_3^m l_4^m \in \mathcal{H}_2^{2n,p,q} \tag{4.27}$$

for any $m \in \{0, \dots, r\}$.

Lemma 4.15. *The relation*

$$\begin{aligned} &\Lambda_{p,q,m} \cdots \Lambda_{p,q,0}\big(l_1^p l_2^q\big) \\ &\quad = (-1)^m pq \cdots (p-m)(q-m) \sum_{k=0}^{m+1} (-1)^{k+1} \binom{m+1}{k} l_1^{p-k} l_2^{q-k} l_3^k l_4^k \end{aligned} \tag{4.28}$$

holds, where

$$\Lambda_{p,q,j} = \mathcal{L} - \lambda_{p,q,j}\,\mathrm{Id}, \quad j \in \{0, \dots, r\}.$$

Proof. Formula (2.41) implies that

$$\begin{aligned} \mathcal{L}\big(l_1^{p-j} l_2^{q-j} l_3^j l_4^j\big) = \big(&s - pq + j(s-2j)\big) l_1^{p-j} l_2^{q-j} l_3^j l_4^j \\ &+ (p-j)(q-j) l_1^{p-j-1} l_2^{q-j-1} l_3^{j+1} l_4^{j+1} \\ &+ j^2 l_1^{p-j+1} l_2^{q-j+1} l_3^{j-1} l_4^{j-1}. \end{aligned}$$

Induction on m now leads to the desired assertion. □

Corollary 4.7. *Let $f \in \mathcal{H}_2^{2n,p,q}$. Then*

$$\Lambda_{p,q,m} \cdots \Lambda_{p,q,0}(f) \in \mathcal{H}_2^{2n,p,q}, \quad m \in \{0, \dots, r\}.$$

In addition,

$$\Lambda_{p,q,r} \cdots \Lambda_{p,q,0}(f) = 0.$$

Proof. The assertion follows from (4.27), (4.28), and the description of the space $\mathcal{H}_2^{2n,p,q}$ (see Sect. 4.2). □

Lemma 4.16. *The space $\mathcal{H}_2^{2n,p,q}$ is the direct sum of the spaces $\mathcal{H}_4^{n,p,q,m}$, $m = 0, \dots, r$.*

Proof. Let $f \in \mathcal{H}_2^{2n,p,q}$. We shall show that

$$\Lambda_{p,q,r-l} \cdots \Lambda_{p,q,0}(f) \in \bigoplus_{k=0}^{l-1} \mathcal{H}_4^{n,p,q,r-k} \tag{4.29}$$

for $l \in \{1, \dots, r+1\}$, where the left-hand side coincides with f for $l = r+1$. If $l = 1$, then (4.29) follows from Corollary 4.7. We suppose that the assertion is true for some $l \in \{1, \dots, r\}$ and then prove it for $l+1$. We have

$$\Lambda_{p,q,r-l} \cdots \Lambda_{p,q,0}(f) = \sum_{k=0}^{l-1} f_{r-k}, \tag{4.30}$$

where $f_{r-k} \in \mathcal{H}_4^{n,p,q,r-k}$. Since

$$f_{r-k} = \frac{\Lambda_{p,q,r-l}(f_{r-k})}{\lambda_{p,q,r-k} - \lambda_{p,q,r-l}}, \quad 0 \leqslant k \leqslant l-1,$$

equality (4.30) gives

$$\Lambda_{p,q,r-l}\left(\Lambda_{p,q,r-l-1} \cdots \Lambda_{p,q,0}(f) - \sum_{k=0}^{l-1} \frac{f_{r-k}}{\lambda_{p,q,r-k} - \lambda_{p,q,r-l}}\right) = 0.$$

Consequently,

$$\Lambda_{p,q,r-l-1} \cdots \Lambda_{p,q,0}(f) \in \bigoplus_{k=0}^{l} \mathcal{H}_4^{n,p,q,r-k}.$$

Thus, relation (4.29) is proved. Putting $l = r+1$ in (4.29), we obtain the statement of the lemma. □

Corollary 4.8. *The space $L^2(\mathbb{S}^{4n-1})$ is the orthogonal direct sum of the spaces*

$$\mathcal{H}_4^{n,p,q,m}, \quad p, q \in \mathbb{Z}_+, m \in \{0, \dots, r\}.$$

Proof. According to Corollary 4.4, we have

$$L^2(\mathbb{S}^{4n-1}) = \bigoplus_{p,q=0}^{\infty} \mathcal{H}_2^{2n,p,q}.$$

Using Lemma 4.16, we arrive at the required assertion. □

Proof of Theorem 4.6. Corollary 4.8 shows that

$$T_4^n(\tau) = \bigoplus_{p,q=0}^{\infty} \bigoplus_{m=0}^{r} T_4^{n,p,q,m}(\tau).$$

Our further purpose is to prove the irreducibility of $T_4^{n,p,q,m}(\tau)$. We require some auxiliary constructions.

Let $\tau(t)$, $t \in \mathbb{R}^1$, be a *one-parameter subgroup* of the group Sp(n), i.e.,

$$\tau(t_1)\tau(t_2) = \tau(t_1 + t_2), \quad t_1, t_2 \in \mathbb{R}^1.$$

The *infinitesimal operator* A of the representation $T_4^{n,p,q,m}(\tau)$ corresponding to $\tau(t)$ has the form

$$(Af)(z) = \frac{\mathrm{d}}{\mathrm{d}t}\big(f\big(\tau^{-1}(t)z\big)\big)\Big|_{t=0}, \qquad f \in \mathcal{H}_4^{n,p,q,m}. \tag{4.31}$$

We define

$$\tau_{k,1}(t) = \begin{pmatrix} \omega_1(t) & \omega_2(t) \\ \omega_2(t) & \omega_1(t) \end{pmatrix}, \qquad \tau_{k,2}(t) = \begin{pmatrix} \omega_1(t) & \mathrm{i}\omega_2(t) \\ \overline{\mathrm{i}\omega_2(t)} & \omega_1(t) \end{pmatrix},$$
$$\tau_{k,3}(t) = \begin{pmatrix} \overline{\omega_3(t)} & 0 \\ 0 & \omega_3(t) \end{pmatrix},$$

where $\omega_1(t)$ and $\omega_3(t)$ are the $n \times n$ matrices obtained from the identity matrix by replacing the kth, $1 \leqslant k \leqslant n$, entry on the principal diagonal by the respective functions $\cos(t/2)$ and $\mathrm{e}^{\mathrm{i}(t/2)}$, and

$$\omega_2(t) = \omega_1(t) - \omega_3(t).$$

Denote by $A_{k,l}$ the infinitesimal operator of $T_4^{n,p,q,m}(\tau)$ corresponding to the one-parameter subgroup $\tau_{k,l}$ ($k \in \{1,\dots,n\}$, $l \in \{1,2,3\}$). Using (4.31), we find

$$A_{k,1} = \frac{\mathrm{i}}{2}\left(z_{n+k}\frac{\partial}{\partial z_k} + z_k\frac{\partial}{\partial z_{n+k}} - \overline{z}_{n+k}\frac{\partial}{\partial \overline{z}_k} - \overline{z}_k\frac{\partial}{\partial \overline{z}_{n+k}}\right), \tag{4.32}$$

$$A_{k,2} = \frac{1}{2}\left(-z_{n+k}\frac{\partial}{\partial z_k} + z_k\frac{\partial}{\partial z_{n+k}} - \overline{z}_{n+k}\frac{\partial}{\partial \overline{z}_k} + \overline{z}_k\frac{\partial}{\partial \overline{z}_{n+k}}\right), \tag{4.33}$$

$$A_{k,3} = \frac{\mathrm{i}}{2}\left(z_k\frac{\partial}{\partial z_k} - z_{n+k}\frac{\partial}{\partial z_{n+k}} - \overline{z}_k\frac{\partial}{\partial \overline{z}_k} + \overline{z}_{n+k}\frac{\partial}{\partial \overline{z}_{n+k}}\right). \tag{4.34}$$

Note that every subspace invariant with respect to the representation $T_4^{n,p,q,m}(\tau)$ is invariant with respect to the operators (4.32)–(4.34) as well.

We write

$$D = -\mathrm{i}A_{1,1} - A_{1,2} = z_{n+1}\frac{\partial}{\partial z_1} - \overline{z}_1\frac{\partial}{\partial \overline{z}_{n+1}}, \tag{4.35}$$

$$E = \mathrm{i}A_{1,1} - A_{1,2} = \overline{z}_{n+1}\frac{\partial}{\partial \overline{z}_1} - z_1\frac{\partial}{\partial z_{n+1}}. \tag{4.36}$$

Clearly,

$$\overline{D^l(f)} = E^l\big(\overline{f}\big), \quad l \in \mathbb{Z}_+. \tag{4.37}$$

In addition one may easily check that

$$D^l\big(z_1^\alpha z_{n+1}^\beta \overline{z}_1^\gamma \overline{z}_{n+1}^\delta\big) = \sum_{k=0}^{l}\binom{l}{k}(-1)^k(-\alpha)_k(-\delta)_{l-k} z_1^{\alpha-k} z_{n+1}^{k+\beta} \overline{z}_1^{\gamma+l-k} \overline{z}_{n+1}^{\delta-l+k}, \tag{4.38}$$

$$E^l\left(z_1^\alpha z_{n+1}^\beta \overline{z}_1^\gamma \overline{z}_{n+1}^\delta\right) = \sum_{k=0}^{l} \binom{l}{k} (-1)^k (-\gamma)_k (-\beta)_{l-k} z_1^{\alpha+l-k} z_{n+1}^{\beta-l+k} \overline{z}_1^{\gamma-k} \overline{z}_{n+1}^{k+\delta}, \tag{4.39}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{Z}_+$ and $(a)_k$ is the Pochhammer symbol, i.e.,

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

In particular,

$$D^{\alpha+\delta}\left(z_1^\alpha z_{n+1}^\beta \overline{z}_1^\gamma \overline{z}_{n+1}^\delta\right) = (-1)^\delta (\alpha+\delta)! z_{n+1}^{\alpha+\beta} \overline{z}_1^{\gamma+\delta}, \tag{4.40}$$

$$E^{\beta+\gamma}\left(z_1^\alpha z_{n+1}^\beta \overline{z}_1^\gamma \overline{z}_{n+1}^\delta\right) = (-1)^\beta (\beta+\gamma)! z_1^{\alpha+\beta} \overline{z}_{n+1}^{\gamma+\delta}. \tag{4.41}$$

We shall also need the following relations:

$$\begin{aligned} \int_{\mathbb{S}^{4n-1}} f(\xi)\left(D^l g\right)(\xi)\, \mathrm{d}\omega(\xi) &= (-1)^l \int_{\mathbb{S}^{4n-1}} \left(D^l f\right)(\xi) g(\xi)\, \mathrm{d}\omega(\xi), \\ \int_{\mathbb{S}^{4n-1}} f(\xi)\left(E^l g\right)(\xi)\, \mathrm{d}\omega(\xi) &= (-1)^l \int_{\mathbb{S}^{4n-1}} \left(E^l f\right)(\xi) g(\xi)\, \mathrm{d}\omega(\xi). \end{aligned} \tag{4.42}$$

To prove (4.42) observe that

$$\int_{\mathbb{S}^{4n-1}} f(\xi) g\left(\tau_{1,k}^{-1}(t)\xi\right) \mathrm{d}\omega(\xi) = \int_{\mathbb{S}^{4n-1}} f\left(\tau_{1,k}(t)\xi\right) g(\xi)\, \mathrm{d}\omega(\xi)$$

for $k = 1, 2$. Therefore,

$$\int_{\mathbb{S}^{4n-1}} f(\xi)\left(A_{1,k} g\right)(\xi)\, \mathrm{d}\omega(\xi) = -\int_{\mathbb{S}^{4n-1}} \left(A_{1,k} f\right)(\xi) g(\xi)\, \mathrm{d}\omega(\xi),$$

whence, in view of (4.35) and (4.36), the desired formulae follow.

Next, let $f \in \mathcal{H}_2^{2n,p,q}$ and suppose that $f \circ \tau = f$ for any $\tau \in \mathrm{Sp}(n-1)$, the isotropy subgroup of the point $e = (1, 0, \ldots, 0) \in \mathbb{S}^{4n-1}$ in the group Sp(n). In the same way as in the proof of Lemma 4.4, we derive that the polynomial f has the form

$$f(z) = \sum_{l=0}^{r} f_l(z) U_l(z), \tag{4.43}$$

where

$$\begin{aligned} f_l(z) &= \sum_{j=0}^{p-l} \sum_{k=0}^{q-l} c_{j,k}^{(l)} z_1^j z_{n+1}^{p-l-j} \overline{z}_1^k \overline{z}_{n+1}^{q-l-k}, \quad c_{j,k}^{(l)} \in \mathbb{C}, \\ U_l(z) &= \left(\sum_{k=2}^{n} |z_k|^2 + |z_{n+k}|^2\right)^l. \end{aligned} \tag{4.44}$$

Then the harmonicity of f implies that

$$\Delta(f_l) = -4(l+1)(2n+l-2)f_{l+1}, \quad 0 \leqslant l \leqslant r-1, \tag{4.45}$$

$$\Delta(f_r) = 0. \tag{4.46}$$

We rewrite (4.44) as

$$f_l(z) = \sum_{i=l-p}^{q-l} \sum_{j \in M_{i,l}} c_{j,i+j}^{(l)} z_1^j z_{n+1}^{p-l-j} \overline{z}_1^{i+j} \overline{z}_{n+1}^{q-l-i-j}, \tag{4.47}$$

where

$$M_{i,l} = \{j \in \mathbb{Z}_+ : 0 \leqslant j \leqslant p-l,\ 0 \leqslant i+j \leqslant q-l\}.$$

By (4.46) and (4.47) we infer that

$$c_{j,i+j}^{(r)} = (-1)^j \frac{(p-r)!}{(p-r-j)!} \frac{(q-r-i)!}{(q-r-i-j)!} \frac{i!}{j!(i+j)!} c_{0,i}^{(r)} \quad \text{for } i > 0 \tag{4.48}$$

and

$$c_{j,i+j}^{(r)} = (-1)^{i+j} \frac{(p-r+i)!}{(p-r-j)!} \frac{(q-r)!}{(q-r-i-j)!} \frac{(-i)!}{j!(i+j)!} c_{-i,0}^{(r)} \quad \text{for } i \leqslant 0. \tag{4.49}$$

The following lemma is the main step in the proof of the irreducibility of the representation $T_4^{n,p,q,m}(\tau)$.

Lemma 4.17. *Every invariant subspace $\mathcal{H} \neq \{0\}$ of the representation $T_4^{n,p,q,m}(\tau)$ contains the polynomial*

$$H_{p,q,m}(z) = \sum_{l=0}^{m} U_l(z) V_l(z),$$

where

$$V_l(z) = z_1^{p-m} \overline{z}_{n+1}^{q-m} \left(|z_1|^2 + |z_{n+1}|^2\right)^{m-l} \prod_{k=l}^{m-1} \frac{(k+1)(2n+k-2)}{(m-k)(k+m-s-1)}$$

$$\text{if } 0 \leqslant l \leqslant m-1$$

and

$$V_l(z) = z_1^{p-m} \overline{z}_{n+1}^{q-m} \quad \text{if } l = m.$$

Proof. Let $g \in \mathcal{H}$, $g \not\equiv 0$. It can be assumed without loss of generality that $g(e) \neq 0$, where $e = (1, 0, \dots, 0) \in \mathbb{S}^{4n-1}$. Averaging g over the group $\mathrm{Sp}(n-1)$, we see that $\mathcal{H}$ contains a polynomial f such that

$$f \circ \tau = f, \quad \tau \in \mathrm{Sp}(n-1), \tag{4.50}$$

$$f(e) \neq 0. \tag{4.51}$$

Condition (4.50) means that f has the form (4.43). Denote by $\varkappa$ the largest l for which the coefficient of U_l in (4.43) is not identically zero (see (4.51)). Pick $j_0 \in \{0, \dots, p-\varkappa\}$ and $k_0 \in \{0, \dots, q-\varkappa\}$ such that $c^{(\varkappa)}_{j_0,k_0} \neq 0$. Put

$$\mathcal{Q} = \big\{(j_1, k_1)\colon\ j_1 \in \{0, \dots, p\},\ k_1 \in \{0, \dots, q\},\ k_1 - j_1 \neq k_0 - j_0\big\}.$$

Fix $(j_1, k_1) \in \mathcal{Q}$. The polynomial

$$-\mathrm{i}(A_{1,3} f)(z) - \frac{(2(j_1 - k_1) - p + q)}{2} f(z)$$
$$= \sum_{l=0}^{\varkappa} \left(\sum_{j=0}^{p-l} \sum_{k=0}^{q-l} c^{(l)}_{j,k} (j - k - j_1 + k_1) z_1^j z_{n+1}^{p-l-j} \overline{z}_1^k \overline{z}_{n+1}^{q-l-k} \right) U_l(z)$$

belongs to $\mathcal{H}$, because $A_{1,3}$ is an infinitesimal operator of the representation $T_4^{n,p,q,m}(\tau)$ (see (4.34)). For the same reason, $\mathcal{H}$ contains the polynomial

$$u(z) = \sum_{l=0}^{\varkappa} \left(\sum_{j=0}^{p-l} \sum_{k=0}^{q-l} b^{(l)}_{j,k} z_1^j z_{n+1}^{p-l-j} \overline{z}_1^k \overline{z}_{n+1}^{q-l-k} \right) U_l(z),$$

where

$$b^{(l)}_{j,k} = c^{(l)}_{j,k} \prod_{(j_1,k_1) \in \mathcal{Q}} (j - k - j_1 + k_1).$$

It is clear that

$$b^{(\varkappa)}_{j_0,k_0} \neq 0.$$

In addition,

$$b^{(l)}_{j,k} = 0$$

if $0 \leqslant j \leqslant p - l$, $0 \leqslant k \leqslant q - l$, $k - j \neq k_0 - j_0$, and $0 \leqslant l \leqslant \varkappa$. Therefore,

$$u(z) = \sum_{l=0}^{\varkappa} \left(\sum_{\substack{j=0 \\ k-j=k_0-j_0}}^{p-l} \sum_{k=0}^{q-l} b^{(l)}_{j,k} z_1^j z_{n+1}^{p-l-j} \overline{z}_1^k \overline{z}_{n+1}^{q-l-k} \right) U_l(z),$$

and we have

$$\sum_{\substack{j=0 \\ k-j=k_0-j_0}}^{p-\varkappa} \sum_{k=0}^{q-\varkappa} b^{(\varkappa)}_{j,k} z_1^j z_{n+1}^{p-\varkappa-j} \overline{z}_1^k \overline{z}_{n+1}^{q-\varkappa-k} \not\equiv 0. \tag{4.52}$$

By virtue of (4.39) and (4.41),

$$\big(E^{p-\varkappa+k_0-j_0} u\big)(z) = c z_1^{p-\varkappa} \overline{z}_{n+1}^{q-\varkappa} U_\varkappa(z)$$
$$+ \sum_{l=0}^{\varkappa-1} \left(\sum_{j=0}^{p-l} \sum_{k=0}^{q-l} d^{(l)}_{j,k} z_1^j z_{n+1}^{p-l-j} \overline{z}_1^k \overline{z}_{n+1}^{q-l-k} \right) U_l(z),$$

where

$$c = \sum_{\substack{j=0\\k-j=k_0-j_0}}^{p-\varkappa} \sum_{k=0}^{q-\varkappa} b_{j,k}^{(\varkappa)} (-1)^{p-\varkappa-j} (p-\varkappa+k_0-j_0)!.$$

Relations (4.48), (4.49), and (4.52) imply that $c \neq 0$. Repeating the above argument with $E^{p-\varkappa+k_0-j_0}u$ instead of f, we deduce that $\mathcal{H}$ contains the polynomial

$$v(z) = \sum_{l=0}^{\varkappa} W_l(z) U_l(z), \tag{4.53}$$

where

$$W_l(z) = \sum_{j=p-\varkappa}^{p-l} a_{l,j} z_1^j z_{n+1}^{p-l-j} \overline{z}_1^{j+\varkappa-p} \overline{z}_{n+1}^{q-l-j-\varkappa+p}, \quad a_{l,j} \in \mathbb{C},\ a_{\varkappa,p-\varkappa} = 1. \tag{4.54}$$

Let us compute $\mathcal{L}v$. For $\alpha, \beta, \gamma, \delta \in \mathbb{Z}_+$, one finds

$$\begin{aligned}
\mathcal{L}_1\big(z_1^\alpha z_{n+1}^\beta \overline{z}_1^\gamma \overline{z}_{n+1}^\delta U_l(z)\big) &= l(2n+l-3) z_1^\alpha z_{n+1}^\beta \overline{z}_1^\gamma \overline{z}_{n+1}^\delta U_{l-1}(z) \\
&\quad + \big(\alpha\gamma z_1^{\alpha-1} z_{n+1}^\beta \overline{z}_1^{\gamma-1} \overline{z}_{n+1}^\delta + \beta\delta z_1^\alpha z_{n+1}^{\beta-1} \overline{z}_1^\gamma \overline{z}_{n+1}^{\delta-1} \\
&\quad + \big(\alpha+\beta+\gamma+\delta+2l \\
&\quad - (\alpha+\beta+l)(\gamma+\delta+l)\big) z_1^\alpha z_{n+1}^\beta \overline{z}_1^\gamma \overline{z}_{n+1}^\delta\big) U_l(z), \\
\mathcal{L}_2\big(z_1^\alpha z_{n+1}^\beta \overline{z}_1^\gamma \overline{z}_{n+1}^\delta U_l(z)\big) &= \big((\beta\gamma+\alpha\delta-l) z_1^\alpha z_{n+1}^\beta \overline{z}_1^\gamma \overline{z}_{n+1}^\delta \\
&\quad - \alpha\gamma z_1^{\alpha-1} z_{n+1}^{\beta+1} \overline{z}_1^{\gamma-1} \overline{z}_{n+1}^{\delta+1} \\
&\quad - \beta\delta z_1^{\alpha+1} z_{n+1}^{\beta-1} \overline{z}_1^{\gamma+1} \overline{z}_{n+1}^{\delta-1}\big) U_l(z), \\
\mathcal{L}\big(z_1^\alpha z_{n+1}^\beta \overline{z}_1^\gamma \overline{z}_{n+1}^\delta U_l(z)\big) &= (1-l)(\alpha+\beta+\gamma+\delta+l) z_1^\alpha z_{n+1}^\beta \overline{z}_1^\gamma \overline{z}_{n+1}^\delta U_l(z) \\
&\quad + l(2n+l-3) z_1^\alpha z_{n+1}^\beta \overline{z}_1^\gamma \overline{z}_{n+1}^\delta U_{l-1}(z) \\
&\quad - \frac{(|z_1|^2+|z_{n+1}|^2-1)}{4} \Delta\big(z_1^\alpha z_{n+1}^\beta \overline{z}_1^\gamma \overline{z}_{n+1}^\delta\big) U_l(z).
\end{aligned}$$

Consequently,

$$\begin{aligned}
(\mathcal{L}v)(z) &= \sum_{l=0}^{\varkappa} (1-l)(s-l) W_l(z) U_l(z) + \sum_{l=0}^{\varkappa-1} (l+1)(2n+l-2) W_{l+1}(z) U_l(z) \\
&\quad - \frac{(|z_1|^2+|z_{n+1}|^2-1)}{4} \sum_{l=0}^{\varkappa} (\Delta W_l)(z) U_l(z).
\end{aligned}$$

Using (4.45) and (4.46) with $f_l = W_l$ and $r = \varkappa$, we obtain

$$(\mathcal{L}v)(z) = \sum_{l=0}^{\varkappa}(1-l)(s-l)W_l(z)U_l(z) + \big(|z_1|^2 + |z_{n+1}|^2\big)\sum_{l=0}^{\varkappa-1}(l+1)(2n+l-2)W_{l+1}(z)U_l(z).$$

Now the condition $\mathcal{L}v = \lambda_{p,q,m}v$ gives

$$\varkappa = m \tag{4.55}$$

and

$$W_l(z) = \frac{(l+1)(2n+l-2)}{(m-l)(l+m-s-1)}\big(|z_1|^2 + |z_{n+1}|^2\big)W_{l+1}(z), \quad 0 \leqslant l \leqslant m-1. \tag{4.56}$$

From (4.53)–(4.56) we conclude that $v = H_{p,q,m}$. Thus, $H_{p,q,m} \in \mathcal{H}$, which proves the lemma. □

We proceed to the proof of the irreducibility of the representation $T_4^{n,p,q,m}(\tau)$.

Let $\mathcal{H}$ be an invariant subspace of $T_4^{n,p,q,m}(\tau)$. Then so is $\mathcal{H}^\perp$, the orthogonal complement of $\mathcal{H}$ relative to $\mathcal{H}_4^{n,p,q,m}$, since the representation $T_4^{n,p,q,m}(\tau)$ is unitary. From Lemma 4.17 it follows that $\mathcal{H} = \{0\}$ or $\mathcal{H}^\perp = \{0\}$. Hence, $\mathcal{H} = \{0\}$ or $\mathcal{H} = \mathcal{H}_4^{n,p,q,m}$, as required. This concludes the proof of the first statement of Theorem 4.6.

The main question that remains to be discussed is to find conditions of equivalence for the representations $T_4^{n,p_i,q_i,m_i}(\tau)$, $i = 1, 2$. To do this we must study in more detail the structure of the spaces

$$\mathcal{Z}_{p,q,m}^n = \big\{f \in \mathcal{H}_4^{n,p,q,m} : f \circ \tau = f \ \forall \tau \in \mathrm{Sp}(n-1)\big\}.$$

We begin with the following assertion.

Lemma 4.18. *Let $f \in \mathcal{Z}_{p,q,m}^n$. Then*

$$f(z) = \sum_{l=0}^{m} f_l(z)U_l(z), \tag{4.57}$$

where

$$f_m(z) = \sum_{k=0}^{s-2m} \alpha_k D^k\big(z_1^{p-m}\overline{z}_{n+1}^{q-m}\big), \quad \alpha_k \in \mathbb{C}, \tag{4.58}$$

$$f_l(z) = f_m(z)\big(|z_1|^2 + |z_{n+1}|^2\big)^{m-l}\prod_{k=l}^{m-1}\frac{(k+1)(2n+k-2)}{(m-k)(k+m-s-1)}, \quad 0 \leqslant l \leqslant m-1. \tag{4.59}$$

Proof. It suffices to consider the case where $f \not\equiv 0$. Repeating the arguments from the proof of Lemma 4.17 (see (4.53), (4.55), (4.56)), we obtain (4.57), where f_l has the form (4.59). Let us prove the expansion (4.58). In accordance with (4.47),

$$f_m(z) = \sum_{i=m-p}^{q-m} \sum_{j \in M_{i,m}} c_{j,i+j}^{(m)} z_1^j z_{n+1}^{p-m-j} \overline{z}_1^{i+j} \overline{z}_{n+1}^{q-m-i-j}. \tag{4.60}$$

By (4.38), (4.48), and (4.49) we infer that

$$\sum_{j \in M_{i,m}} c_{j,i+j}^{(m)} z_1^j z_{n+1}^{p-m-j} \overline{z}_1^{i+j} \overline{z}_{n+1}^{q-m-i-j}$$
$$= \frac{(-i)!}{(p-m)!} c_{-i,0}^{(m)} D^{i+p-m}\left(z_1^{p-m} \overline{z}_{n+1}^{q-m}\right), \quad i \leqslant 0,$$
$$\sum_{j \in M_{i,m}} c_{j,i+j}^{(m)} z_1^j z_{n+1}^{p-m-j} \overline{z}_1^{i+j} \overline{z}_{n+1}^{q-m-i-j}$$
$$= \frac{(-1)^i i!(q-m-i)!}{(q-m)!(p-m+i)!} c_{0,i}^{(m)} D^{i+p-m}\left(z_1^{p-m} \overline{z}_{n+1}^{q-m}\right), \quad i > 0.$$

These relations together with (4.60) imply the assertion of Lemma 4.18. □

An immediate consequence of Lemma 4.18 is the following:

Corollary 4.9. *Let $f, g \in \mathcal{Z}_{p,q,m}^n$, and let*

$$f(z) = \sum_{l=0}^{m} f_l(z) U_l(z), \qquad g(z) = \sum_{l=0}^{m} g_l(z) U_l(z).$$

In this case, if $f_m \equiv g_m$ then $f \equiv g$.

Lemma 4.19. *The system of functions*

$$\left\{D^j(H_{p,q,m})\right\}_{j=0}^{s-2m} \tag{4.61}$$

forms an orthogonal basis in $\mathcal{Z}_{p,q,m}^n$.

Proof. It follows from (4.40) and Corollary 4.9 that the functions $D^j(H_{p,q,m})$, $0 \leqslant j \leqslant s-2m$, are nonzero and that

$$D^{s-2m+1}(H_{p,q,m}) \equiv 0.$$

To prove the orthogonality of the system (4.61), we use the relation

$$\int_{\mathbb{S}^{4n-1}} f(\xi)\,\mathrm{d}\omega(\xi) = \frac{1}{2\pi} \int_{\mathbb{S}^{4n-1}} \mathrm{d}\omega(\xi) \int_{-\pi}^{\pi} f\left(\mathrm{e}^{\mathrm{i}t}\xi_1, {}'\xi\right) \mathrm{d}t,$$

where $'\xi = (\xi_2, \dots, \xi_{2n})$. One finds (see (4.38))

$$U_l\big(\mathrm{e}^{\mathrm{i}t}\xi_1, '\xi\big) = U_l(\xi),$$
$$D^j(V_l)\big(\mathrm{e}^{\mathrm{i}t}\xi_1, '\xi\big) = \mathrm{e}^{\mathrm{i}t(p-m-j)} D^j(V_l)(\xi),$$

whence

$$\int_{\mathbb{S}^{4n-1}} D^{j_1}(V_l)(\xi)\overline{D^{j_2}(V_k)(\xi)}U_l(\xi)U_k(\xi)\,\mathrm{d}\omega(\xi) = 0, \tag{4.62}$$

where $j_1 \neq j_2$ and $l, k \in \{0, \dots, m\}$. Since

$$D^j(H_{p,q,m}) = \sum_{l=0}^{m} D^j(V_l)U_l,$$

from (4.62) we obtain

$$\int_{\mathbb{S}^{4n-1}} D^{j_1}(H_{p,q,m})(\xi)\overline{D^{j_2}(H_{p,q,m})(\xi)}\,\mathrm{d}\omega(\xi) = 0, \quad j_1 \neq j_2,$$

that is, (4.61) is an orthogonal system. We now show that an arbitrary polynomial $f \in \mathcal{Z}^n_{p,q,m}$ is a linear combination of functions belonging to (4.61). Owing to Lemma 4.18, f has the decomposition (4.57). Consider the function

$$g = \sum_{k=0}^{s-2m} \alpha_k D^k(H_{p,q,m}).$$

We rewrite it as

$$g = \sum_{l=0}^{m} g_l U_l,$$

where

$$g_l = \sum_{k=0}^{s-2m} \alpha_k D^k(V_l).$$

Because $f_m \equiv g_m$, Corollary 4.9 implies that $f \equiv g$. Thus, the lemma is proved. □

Let us fix a point $z \in \mathbb{S}^{4n-1}$ and consider the linear functional on $\mathcal{H}_4^{n,p,q,m}$ which associates with each function $f \in \mathcal{H}_4^{n,p,q,m}$ the number $f(z)$. By the Riesz representation theorem, there is a unique function $P_z \in \mathcal{H}_4^{n,p,q,m}$ such that

$$f(z) = \int_{\mathbb{S}^{4n-1}} f(\xi)\overline{P_z(\xi)}\,\mathrm{d}\omega(\xi), \quad f \in \mathcal{H}_4^{n,p,q,m}. \tag{4.63}$$

Lemma 4.20. *The functions P_z possess the following properties.*

(i) *For any function $f \in L^2(\mathbb{S}^{4n-1})$, we have*

$$(\pi_{p,q,m} f)(z) = \int_{\mathbb{S}^{4n-1}} f(\xi)\overline{P_z(\xi)}\, d\omega(\xi),$$

where $\pi_{p,q,m}$ is the orthogonal projection of $L^2(\mathbb{S}^{4n-1})$ onto $\mathcal{H}_4^{n,p,q,m}$.

(ii) $P_z(w) = \overline{P_w(z)}$, $z, w \in \mathbb{S}^{4n-1}$.

(iii) $P_{\tau z} = P_z \circ \tau^{-1}$, $\tau \in \mathrm{Sp}(n)$.

(iv) $P_z = P_z \circ \tau$ *for any $\tau \in \mathrm{Sp}(n)$ such that $\tau z = z$. In particular,*

$$P_e \in \mathcal{Z}^n_{p,q,m}.$$

(v) $P_z(z) = P_w(w) > 0$, $z, w \in \mathbb{S}^{4n-1}$.

Proof. The argument is similar to that of Lemma 4.3. □

Lemma 4.21. *The relation*

$$P_e(z) = \kappa_{p,q,m} D^{q-m}(H_{p,q,m})(z) \tag{4.64}$$

holds, where

$$\kappa_{p,q,m} = D^{q-m}(H_{p,q,m})(e)\left(\int_{\mathbb{S}^{4n-1}} \left|D^{q-m}(H_{p,q,m})(\xi)\right|^2 d\omega(\xi)\right)^{-1}.$$

Proof. We write P_e in the form

$$P_e = \sum_{k=0}^{s-2m} \alpha_k D^k(H_{p,q,m}), \quad \alpha_k \in \mathbb{C} \tag{4.65}$$

(see Lemmas 4.19 and 4.20 (iv)). It follows from (4.63), (4.65), and (4.62) that

$$\begin{aligned}
D^j(H_{p,q,m})(e) &= \int_{\mathbb{S}^{4n-1}} D^j(H_{p,q,m})(\xi)\overline{P_e(\xi)}\, d\omega(\xi) \\
&= \sum_{i,l=0}^{m} \sum_{k=0}^{s-2m} \overline{\alpha}_k \int_{\mathbb{S}^{4n-1}} D^j(V_l)(\xi)\overline{D^k(V_i)(\xi)} U_l(\xi) U_i(\xi)\, d\omega(\xi) \\
&= \overline{\alpha}_j \int_{\mathbb{S}^{4n-1}} \sum_{i,l=0}^{m} D^j(V_l)(\xi)\overline{D^j(V_i)(\xi)} U_l(\xi) U_i(\xi)\, d\omega(\xi) \\
&= \overline{\alpha}_j \int_{\mathbb{S}^{4n-1}} \left|D^j(H_{p,q,m})(\xi)\right|^2 d\omega(\xi),
\end{aligned}$$

whence

$$\alpha_j = \overline{D^j(H_{p,q,m})(e)}\left(\int_{\mathbb{S}^{4n-1}} \left|D^j(H_{p,q,m})(\xi)\right|^2 d\omega(\xi)\right)^{-1}. \tag{4.66}$$

Using (4.38), we find that

$$D^j(H_{p,q,m})(e) = D^j(V_0)(e) = \begin{cases} (-1)^{q-m}(q-m)! \prod_{k=0}^{m-1} \dfrac{(k+1)(2n+k-2)}{(m-k)(k+m-s-1)} \\ \qquad \text{if } j = q-m, \\ 0 \quad \text{if } j \neq q-m, \end{cases} \tag{4.67}$$

where the product is set to be equal to one for $m = 0$. Relations (4.66), (4.67), and (4.65) give (4.64). □

Now we can go to the description of the intertwining operators for the representations $T_4^{n,p_i,q_i,m_i}(\tau)$, $i = 1, 2$.

Let $\mathfrak{T}$ be a nonzero linear operator from $\mathcal{H}_4^{n,p_1,q_1,m_1}$ to $\mathcal{H}_4^{n,p_2,q_2,m_2}$ commuting with the group Sp(n). Using the irreducibility of $T_4^{n,p_i,q_i,m_i}(\tau)$ and the Schur lemma (see Adams [1], Lemma 3.22, and Vilenkin [220], Chap. 1, Sect. 3.2), we see that $\mathfrak{T}$ is an isomorphism of the spaces $\mathcal{H}_4^{n,p_i,q_i,m_i}$, $i = 1, 2$. We shall study the properties of that isomorphism.

Lemma 4.22. *The relation*

$$\mathfrak{T}(H_{p_1,q_1,m_1}) = c_1(\mathfrak{T}) H_{p_2,q_2,m_2} \tag{4.68}$$

holds, where $c_1(\mathfrak{T})$ is a nonzero constant.

Proof. Since the mapping $\mathfrak{T}$ sends $\mathcal{Z}^n_{p_1,q_1,m_1}$ to $\mathcal{Z}^n_{p_2,q_2,m_2}$, Lemma 4.19 implies that

$$\mathfrak{T}(H_{p_1,q_1,m_1}) = \sum_{j=0}^{p_2+q_2-2m_2} \alpha_j D^j(H_{p_2,q_2,m_2}),$$

where

$$\alpha_j = \int_{\mathbb{S}^{4n-1}} \mathfrak{T}(H_{p_1,q_1,m_1})(\xi) \overline{D^j(H_{p_2,q_2,m_2})(\xi)}\, \mathrm{d}\omega(\xi) \times \left(\int_{\mathbb{S}^{4n-1}} \left| D^j(H_{p_2,q_2,m_2})(\xi) \right|^2 \mathrm{d}\omega(\xi) \right)^{-1}.$$

In accordance with (4.37) and (4.42) we have

$$\int_{\mathbb{S}^{4n-1}} \mathfrak{T}(H_{p_1,q_1,m_1})(\xi) \overline{D^j(H_{p_2,q_2,m_2})(\xi)}\, \mathrm{d}\omega(\xi) = (-1)^j \int_{\mathbb{S}^{4n-1}} (E^j \mathfrak{T})(H_{p_1,q_1,m_1})(\xi) \overline{H_{p_2,q_2,m_2}(\xi)}\, \mathrm{d}\omega(\xi).$$

The definition of an infinitesimal operator shows that $\mathfrak{T}$ commutes with the mapping E. Furthermore, by Corollary 4.9 we conclude that $E(H_{p_1,q_1,m_1}) \equiv 0$.

Hence,

$$E^j\mathfrak{T}(H_{p_1,q_1,m_1}) = \mathfrak{T}E^j(H_{p_1,q_1,m_1}) \equiv 0 \quad \text{for } j \geqslant 1.$$

Thus, $\alpha_j = 0$ for $j \geqslant 1$, whence (4.68) follows.

□

Corollary 4.10. *Let* $f \in \mathcal{H}_4^{n,p_1,q_1,m_1}$. *Then*

$$(\mathfrak{T}f)(z) = \kappa_{p_1,q_1,m_1} c_1(\mathfrak{T}) \int_{\mathbb{S}^{4n-1}} f(\xi) D^{q_1-m_1}(H_{p_2,q_2,m_2})\big(\tau_\xi^{-1} z\big)\, \mathrm{d}\omega(\xi), \tag{4.69}$$

where τ_ξ *is arbitrary element of* $\mathrm{Sp}(n)$ *such that* $\tau_\xi e = \xi$.

Proof. By Lemmas 4.20 and 4.21,

$$(\mathfrak{T}f)(z) = \kappa_{p_1,q_1,m_1} \int_{\mathbb{S}^{4n-1}} f(\xi)\mathfrak{T}D^{q_1-m_1}(H_{p_1,q_1,m_1})\big(\tau_\xi^{-1} z\big)\, \mathrm{d}\omega(\xi).$$

Bearing in mind that $\mathfrak{T}D^{q_1-m_1} = D^{q_1-m_1}\mathfrak{T}$ and using (4.68), we obtain (4.69). □

Corollary 4.11. *For* $j \neq q_2 - m_2$, *we have*

$$\int_{\mathbb{S}^{4n-1}} D^j(H_{p_1,q_1,m_1})(\xi) D^{q_1-m_1}(H_{p_2,q_2,m_2})\big(\tau_\xi^{-1} e\big)\, \mathrm{d}\omega(\xi) = 0. \tag{4.70}$$

Proof. We set $f = D^j(H_{p_1,q_1,m_1})$ and $z = e$ in (4.69). In view of the equality

$$\mathfrak{T}D^j(H_{p_1,q_1,m_1}) = D^j\mathfrak{T}(H_{p_1,q_1,m_1}) = c_1(\mathfrak{T})D^j(H_{p_2,q_2,m_2}), \tag{4.71}$$

we get

$$\int_{\mathbb{S}^{4n-1}} D^j(H_{p_1,q_1,m_1})(\xi) D^{q_1-m_1}(H_{p_2,q_2,m_2})\big(\tau_\xi^{-1} e\big)\, \mathrm{d}\omega(\xi) \\ = (\kappa_{p_1,q_1,m_1})^{-1} D^j(H_{p_2,q_2,m_2})(e). \tag{4.72}$$

This and (4.67) imply (4.70). □

Lemma 4.23. *Let* $f \in \mathcal{H}_4^{n,p_1,q_1,m_1}$. *Then*

$$(\mathfrak{T}f)(z) = c_2(\mathfrak{T})|z|^{p_2+q_2} \int_{\mathbb{S}^{4n-1}} f(\tau_{z/|z|}\xi)\overline{D^{q_2-m_2}(H_{p_1,q_1,m_1})(\xi)}\, \mathrm{d}\omega(\xi),$$

where

$$c_2(\mathfrak{T}) = c_1(\mathfrak{T})D^{q_2-m_2}(H_{p_2,q_2,m_2})(e)\left(\int_{\mathbb{S}^{4n-1}} \left|D^{q_2-m_2}(H_{p_1,q_1,m_1})(\xi)\right|^2 \mathrm{d}\omega(\xi)\right)^{-1}.$$

Proof. We first assume that $f \in \mathcal{Z}^n_{p_1,q_1,m_1}$. By Lemma 4.19,

$$f = \sum_{j=0}^{p_1+q_1-2m_1} \alpha_j D^j(H_{p_1,q_1,m_1}), \tag{4.73}$$

where

$$\alpha_j = \int_{\mathbb{S}^{4n-1}} f(\xi)\overline{D^j(H_{p_1,q_1,m_1})(\xi)}\,\mathrm{d}\omega(\xi) \times \left(\int_{\mathbb{S}^{4n-1}} \left|D^j(H_{p_1,q_1,m_1})(\xi)\right|^2 \mathrm{d}\omega(\xi)\right)^{-1}. \tag{4.74}$$

Relations (4.73) and (4.69) give

$$(\mathfrak{T}f)(e) = \kappa_{p_1,q_1,m_1} c_1(\mathfrak{T}) \sum_{j=0}^{p_1+q_1-2m_1} \alpha_j \times \int_{\mathbb{S}^{4n-1}} D^j(H_{p_1,q_1,m_1})(\xi) D^{q_1-m_1}(H_{p_2,q_2,m_2})\big(\tau_\xi^{-1}e\big)\,\mathrm{d}\omega(\xi). \tag{4.75}$$

Using (4.75), (4.70), (4.71), and (4.67), we infer that $q_2 - m_2 \leqslant p_1 + q_1 - 2m_1$. In addition,

$$(\mathfrak{T}f)(e) = \kappa_{p_1,q_1,m_1} c_1(\mathfrak{T}) \alpha_{q_2-m_2} \times \int_{\mathbb{S}^{4n-1}} D^{q_2-m_2}(H_{p_1,q_1,m_1})(\xi) D^{q_1-m_1}(H_{p_2,q_2,m_2})\big(\tau_\xi^{-1}e\big)\,\mathrm{d}\omega(\xi).$$

Now from (4.74) and (4.72) we obtain

$$(\mathfrak{T}f)(e) = c_2(\mathfrak{T}) \int_{\mathbb{S}^{4n-1}} f(\xi)\overline{D^{q_2-m_2}(H_{p_1,q_1,m_1})(\xi)}\,\mathrm{d}\omega(\xi). \tag{4.76}$$

Averaging over the group Sp($n-1$) shows that (4.76) is preserved for functions $f \in \mathcal{H}_4^{n,p_1,q_1,m_1}$. Therefore,

$$(\mathfrak{T}f)(\sigma) = c_2(\mathfrak{T}) \int_{\mathbb{S}^{4n-1}} f(\tau_\sigma \xi)\overline{D^{q_2-m_2}(H_{p_1,q_1,m_1})(\xi)}\,\mathrm{d}\omega(\xi) \tag{4.77}$$

for any $\sigma \in \mathbb{S}^{4n-1}$, whence the assertion of Lemma 4.23 follows. □

Define the operators

$$\mathfrak{D} = \sum_{k=1}^{n}\left(\overline{z}_k \frac{\partial}{\partial z_{n+k}} - \overline{z}_{n+k}\frac{\partial}{\partial z_k}\right),$$

$$\mathfrak{E} = \sum_{k=1}^{n}\left(z_k \frac{\partial}{\partial \overline{z}_{n+k}} - z_{n+k}\frac{\partial}{\partial \overline{z}_k}\right).$$

Lemma 4.24. *The operator $\mathfrak{T}$ has the form*

$$(\mathfrak{T}f)(z) = c|z|^{p_2+q_2-p_1-q_1}(\mathfrak{F}f)(z), \quad c \in \mathbb{C}\backslash\{0\}, \tag{4.78}$$

where

$$\mathfrak{F} = \begin{cases} \mathfrak{D}^{q_2-m_2-q_1+m_1} & \text{for } q_2-m_2 \geqslant q_1-m_1, \\ \mathfrak{E}^{q_1-m_1-q_2+m_2} & \text{for } q_1-m_1 > q_2-m_2. \end{cases}$$

Proof. Suppose that $q_2 - m_2 \geqslant q_1 - m_1$. Let $\sigma \in \mathbb{S}^{4n-1}$. Because of (4.77), (4.37), and (4.42), we have

$$\begin{aligned}(\mathfrak{T}f)(\sigma) &= c_2(\mathfrak{T}) \int_{\mathbb{S}^{4n-1}} f(\tau_\sigma \xi) E^{\alpha+q_1-m_1}\big(\overline{H_{p_1,q_1,m_1}}\big)(\xi)\,\mathrm{d}\omega(\xi) \\ &= (-1)^\alpha c_2(\mathfrak{T}) \int_{\mathbb{S}^{4n-1}} E^\alpha(f\circ\tau_\sigma)(\xi)\overline{D^{q_1-m_1}(H_{p_1,q_1,m_1})(\xi)}\,\mathrm{d}\omega(\xi),\end{aligned}$$

where $\alpha = q_2 - m_2 - q_1 + m_1$. This, together with (4.64), (4.63), and (4.36), implies that

$$\begin{aligned}(\mathfrak{T}f)(\sigma) &= (-1)^\alpha (\kappa_{p_1,q_1,m_1})^{-1} c_2(\mathfrak{T}) E^\alpha (f\circ\tau_\sigma)(e) \\ &= (\kappa_{p_1,q_1,m_1})^{-1} c_2(\mathfrak{T}) \frac{\partial^\alpha}{\partial z_{n+1}^\alpha}(f\circ\tau_\sigma)(e).\end{aligned}$$

Put $w_j(z) = -\overline{z}_{n+j}$ and $w_{n+j}(z) = \overline{z}_j$, $j = \{1,\dots,n\}$. Keeping in mind that $\tau_\sigma \in \mathrm{Sp}(n)$ and $\tau_\sigma e = \sigma$, we derive

$$(\mathfrak{T}f)(\sigma) = (\kappa_{p_1,q_1,m_1})^{-1} c_2(\mathfrak{T}) \sum_{j_1,\dots,j_\alpha=1}^{2n} \frac{\partial^\alpha f}{\partial z_{j_1}\dots\partial z_{j_\alpha}}(\sigma) w_{j_1}(\sigma)\cdots w_{j_\alpha}(\sigma).$$

Hence,

$$\begin{aligned}(\mathfrak{T}f)(z) &= (\kappa_{p_1,q_1,m_1})^{-1} c_2(\mathfrak{T}) |z|^{p_2+q_2-p_1-q_1} \\ &\quad\times \sum_{j_1,\dots,j_\alpha=1}^{2n} \frac{\partial^\alpha f}{\partial z_{j_1}\cdots\partial z_{j_\alpha}}(z) w_{j_1}(z)\cdots w_{j_\alpha}(z) \\ &= c|z|^{p_2+q_2-p_1-q_1}\big(\mathfrak{D}^{q_2-m_2-q_1+m_1} f\big)(z).\end{aligned}$$

Now let $q_1 - m_1 > q_2 - m_2$. We see from Lemma 4.19 and (4.71) that

$$p_1 + q_1 - 2m_1 = p_2 + q_2 - 2m_2. \tag{4.79}$$

Therefore, $p_2 - m_2 > p_1 - m_1$. Consider the linear operator $\mathfrak{U}\colon \mathcal{H}_4^{n,q_1,p_1,m_1} \to \mathcal{H}_4^{n,q_2,p_2,m_2}$ defined as follows:

$$(\mathfrak{U}f)(z) = \overline{\mathfrak{T}\big(\overline{f}\big)}, \quad f \in \mathcal{H}_4^{n,q_1,p_1,m_1}.$$

It is clear that $\mathfrak{U}$ commutes with the group $\mathrm{Sp}(n)$ and $\mathfrak{U} \neq 0$. By the above,

$$\begin{aligned}(\mathfrak{U}f)(z) &= c|z|^{p_2+q_2-p_1-q_1}\big(\mathfrak{D}^{p_2-m_2-p_1+m_1} f\big)(z) \\ &= c|z|^{p_2+q_2-p_1-q_1}\big(\mathfrak{D}^{q_1-m_1-q_2+m_2} f\big)(z).\end{aligned}$$

Consequently,

$$(\mathfrak{T} f)(z) = c|z|^{p_2+q_2-p_1-q_1}\left(\mathfrak{E}^{q_1-m_1-q_2+m_2} f\right)(z).$$

Thus, the lemma is proved. □

Remark 4.2. A direct check shows that the operators $\mathfrak{D}$ and $\mathfrak{E}$ commute with the group Sp(n). In addition,

$$\mathfrak{D}(Q_{p,q,m}) = (p-m)Q_{p-1,q+1,m},$$
$$\mathfrak{E}(Q_{p,q,m}) = (m-q)Q_{p+1,q-1,m},$$

where $Q_{p,q,m}$ is the polynomial defined in Lemma 4.12. Using the irreducibility of the representations $T_4^{n,p,q,m}(\tau)$, we see that $\mathfrak{D}$ maps $\mathcal{H}_4^{n,p,q,m}$ onto the space

$$X = \begin{cases} \mathcal{H}_4^{n,p-1,q+1,m} & \text{if } p \geqslant m+1, \\ 0 & \text{if } p = m. \end{cases}$$

Analogously, $\mathfrak{E}$ maps $\mathcal{H}_4^{n,p,q,m}$ onto the space

$$Y = \begin{cases} \mathcal{H}_4^{n,p+1,q-1,m} & \text{if } q \geqslant m+1, \\ 0 & \text{if } q = m. \end{cases}$$

Now we are in a position to obtain the following description of intertwining operators for the representations $T_4^{n,p_i,q_i,m_i}(\tau)$, $i = 1, 2$.

Proposition 4.1. *Let* $\mathfrak{T}\colon \mathcal{H}_4^{n,p_1,q_1,m_1} \to \mathcal{H}_4^{n,p_2,q_2,m_2}$ *be a linear operator and assume that* $\mathfrak{T}$ *commutes with the group* Sp(n)*. Then*

$$\mathfrak{T} = \begin{cases} c\mathfrak{D}^{p_1-p_2} & \text{if } p_1+q_1 = p_2+q_2,\, m_1 = m_2,\, p_1 \geqslant p_2, \\ c\mathfrak{E}^{p_2-p_1} & \text{if } p_1+q_1 = p_2+q_2,\, m_1 = m_2,\, p_1 < p_2, \\ 0 & \text{if } p_1+q_1 \neq p_2+q_2 \text{ or } m_1 \neq m_2, \end{cases} \tag{4.80}$$

for some $c \in \mathbb{C}$*. Conversely, for every* $c \in \mathbb{C}$*, the operator* (4.80) *maps* $\mathcal{H}_4^{n,p_1,q_1,m_1}$ *into* $\mathcal{H}_4^{n,p_2,q_2,m_2}$ *and commutes with the group* Sp(n)*.*

Proof. Suppose that $\mathfrak{T}\colon \mathcal{H}_4^{n,p_1,q_1,m_1} \to \mathcal{H}_4^{n,p_2,q_2,m_2}$ is a nonzero linear mapping that commutes with Sp(n). Since harmonic polynomials of different degrees are orthogonal on the unit sphere, we obtain from (4.78) that $p_1+q_1 = p_2+q_2$. Then, taking into account relation (4.79), we find that $m_1 = m_2$. Now Proposition 4.1 follows from Lemma 4.24 and Remark 4.2. □

Proposition 4.1 and the Schur lemma (see Adams [1], Lemma 3.22) imply that the representations $T_4^{n,p_i,q_i,m_i}(\tau)$, $i = 1, 2$, are equivalent if and only if $p_1+q_1 = p_2+q_2$ and $m_1 = m_2$. This concludes the proof of the second statement of Theorem 4.6. Thus, Theorem 4.6 is completely proved. □

We now proceed to description of $\mathrm{Sp}(n)$-invariant subspaces of $\mathfrak{B}^{4n}$, $n \geqslant 2$ (see the definition of $\mathfrak{B}^{4n}$ in Sect. 4.1). Note that we cannot use the method suggested in Sect. 4.1 since among the irreducible components of $T_4^n(\tau)$ there exist equivalent representations.

For $f \in \mathfrak{B}^{4n}$, denote by $\{f\}_{\mathrm{Sp}(n)}$ the linear span of the system $\{f \circ \tau, \tau \in \mathrm{Sp}(n)\}$.

Let $k \in \mathbb{Z}_+$, $m \in \{0, \dots, [k/2]\}$, $l \in \{1, \dots, k-2m+1\}$, and let $\mathfrak{Y}$ be the set of all $\mathrm{Sp}(n)$-spaces of the form

$$\{\alpha_1 H_{p_1,q_1,m} + \dots + \alpha_l H_{p_l,q_l,m}\}_{\mathrm{Sp}(n)}, \tag{4.81}$$

where $\alpha_i \in \mathbb{C}$, $p_i, q_i \geqslant m$, and $p_i + q_i = k$, $1 \leqslant i \leqslant l$.

Evidently, the $\mathfrak{B}^{4n}$-closure of the algebraic sum of an arbitrary subset in $\mathfrak{Y}$ is a closed $\mathrm{Sp}(n)$-space in $\mathfrak{B}^{4n}$. At the same time the following result holds.

Theorem 4.7. *Every* $\mathrm{Sp}(n)$*-invariant subspace in* $\mathfrak{B}^{4n}$ *is the* $\mathfrak{B}^{4n}$*-closure of the algebraic sum of some subset in* $\mathfrak{Y}$.

The proof of Theorem 4.7 requires some preparation.

Let $\{Y_j^{p,q,m}\}$, $j \in \{1, \dots, d^{p,q,m}\}$, be a fixed orthonormal basis in $\mathcal{H}_4^{n,p,q,m}$. Denote by $\{t_{i,j}^{p,q,m}(\tau)\}$, $i, j \in \{1, \dots, d^{p,q,m}\}$, the matrix of the representation $T_4^{n,p,q,m}(\tau)$ in the basis $\{Y_j^{p,q,m}\}$, that is,

$$T_4^{n,p,q,m}(\tau) Y_j^{p,q,m} = \sum_{i=1}^{d^{p,q,m}} t_{i,j}^{p,q,m}(\tau) Y_i^{p,q,m}. \tag{4.82}$$

Next, for $f \in L^1(\mathbb{S}^{4n-1})$, we put

$$f_{p,q,m,j} = \int_{\mathbb{S}^{4n-1}} f(\xi) \overline{Y_j^{p,q,m}(\xi)} \, \mathrm{d}\omega(\xi).$$

Owing to Corollary 4.8, every $f \in L^2(\mathbb{S}^{4n-1})$ has a unique expansion

$$f = \sum_{p,q=0}^{\infty} \sum_{m=0}^{r} \sum_{i=1}^{d^{p,q,m}} f_{p,q,m,j} Y_j^{p,q,m} \tag{4.83}$$

that converges unconditionally to f in the L^2-norm topology.

Lemma 4.25. *For* $f \in L^1(\mathbb{S}^{4n-1})$*, we have*

$$d^{p,q,m} \int_{\mathrm{Sp}(n)} (f \circ \tau^{-1}) \overline{t_{i,j}^{p,q,m}(\tau)} \, \mathrm{d}\tau = \sum_{\substack{\alpha+\beta=s \\ \alpha,\beta \geqslant m}} f_{\alpha,\beta,m,j} Y_i^{\alpha,\beta,m}, \tag{4.84}$$

where $s = p + q$*, and* $\mathrm{d}\tau$ *is the normalized Haar measure on the group* $\mathrm{Sp}(n)$.

Proof. We can assume, without loss of generality, that $f \in L^2(\mathbb{S}^{4n-1})$. Then (4.82), (4.83), and the continuity of the representation operator imply

$$f \circ \tau^{-1} = \sum_{\alpha,\beta=0}^{\infty} \sum_{m_1=0}^{\min\{\alpha,\beta\}} \sum_{j_1=1}^{d^{\alpha,\beta,m_1}} f_{\alpha,\beta,m_1,j_1} \sum_{i_1=1}^{d^{\alpha,\beta,m_1}} t_{i_1,j_1}^{\alpha,\beta,m_1}(\tau) Y_{i_1}^{\alpha,\beta,m_1}. \tag{4.85}$$

Using (4.85), the orthogonality relations for $t_{i_1,j_1}^{\alpha,\beta,m_1}(\tau)$, and Theorem 4.6, we arrive at the assertion of Lemma 4.25. □

Corollary 4.12. *Let H be an* Sp(n)*-invariant subspace of $L^2(\mathbb{S}^{4n-1})$ and suppose that $\pi_{p,q,m}H \neq \{0\}$. Then*

$$H \cap \Bigg(\bigoplus_{\substack{\alpha+\beta=s \\ \alpha,\beta \geqslant m}} \mathcal{H}_4^{n,\alpha,\beta,m} \Bigg) \neq \{0\}.$$

Proof. Take $f \in H$ such that $\pi_{p,q,m} f \neq 0$. Since

$$\pi_{p,q,m} f = \sum_{j=1}^{d^{p,q,m}} f_{p,q,m,j} Y_j^{p,q,m},$$

$f_{p,q,m,j_0} \neq 0$ for some $j_0 \in \{1, \ldots, d^{p,q,m}\}$. Put

$$g = \sum_{\substack{\alpha+\beta=s \\ \alpha,\beta \geqslant m}} f_{\alpha,\beta,m,j_0} Y_{j_0}^{\alpha,\beta,m}.$$

Then, by definition,

$$g \in \bigoplus_{\substack{\alpha+\beta=s \\ \alpha,\beta \geqslant m}} \mathcal{H}_4^{n,\alpha,\beta,m}, \quad g \neq 0.$$

On the other hand, formula (4.84) with $i = j = j_0$ shows that $g \in H$. Thereby Corollary 4.12 is established. □

Lemma 4.26. *Every* Sp(n)*-invariant subspace H of $L^2(\mathbb{S}^{4n-1})$ is the direct sum of the pairwise orthogonal spaces* $\mathrm{H}_{k,m}$, $k \in \mathbb{Z}_+$, $m \in \{0, \ldots, [k/2]\}$, *where*

$$\mathrm{H}_{k,m} = H \cap \Bigg(\bigoplus_{\substack{\alpha+\beta=k \\ \alpha,\beta \geqslant m}} \mathcal{H}_4^{n,\alpha,\beta,m} \Bigg).$$

Proof. We set

$$H_1 = \bigoplus_{k=0}^{\infty} \bigoplus_{m=0}^{[k/2]} \mathrm{H}_{k,m}. \tag{4.86}$$

One has $H = H_1 \oplus H_1^{\perp}$, where H_1 and $H_1^{\perp}$ are $\mathrm{Sp}(n)$-invariant subspaces of H. We claim that $H_1^{\perp} = \{0\}$. Suppose the contrary. Pick numbers $p_0, q_0 \in \mathbb{Z}_+$ and $m_0 \in \{0, \dots, \min\{p_0, q_0\}\}$ for which $\pi_{p_0,q_0,m_0} H_1^{\perp} \neq \{0\}$. On account of Corollary 4.12, the space

$$H_1^{\perp} \cap \Bigg(\bigoplus_{\substack{\alpha+\beta=p_0+q_0 \\ \alpha,\beta \geqslant m_0}} \mathcal{H}_4^{n,\alpha,\beta,m_0} \Bigg)$$

contains a nonzero function f. Hence, by (4.86), $f \in \mathrm{H}_{p_0+q_0,m_0} \subset H_1$ and $f \in H_1^{\perp}$. Consequently, $f = 0$, which is a contradiction. Thus, $H_1^{\perp} = \{0\}$, and the lemma is proved. □

Lemma 4.27. *Let $l \in \{1, \dots, k-2m+1\}$, and let H be a nontrivial $\mathrm{Sp}(n)$-invariant subspace of the direct sum*

$$\bigoplus_{i=1}^{l} \mathcal{H}_4^{n,p_i,q_i,m}, \tag{4.87}$$

where $p_i, q_i \geqslant m$, $p_i + q_i = k$, and (p_i, q_i) are pairwise distinct. Assume that

$$H \cap \Bigg(\bigoplus_{\substack{j=1 \\ j\neq i}}^{l} \mathcal{H}_4^{n,p_j,q_j,m} \Bigg) = \{0\}, \quad 1 \leqslant i \leqslant l. \tag{4.88}$$

Then H is of the form (4.81) for some $\alpha_1, \dots, \alpha_l \in \mathbb{C} \setminus \{0\}$.

Proof. Because $H \neq \{0\}$, there is a function $f \in H$ such that $f(e) \neq 0$. Represent f as

$$f = h_1 + \dots + h_l \tag{4.89}$$

with $h_i \in \mathcal{H}_4^{n,p_i,q_i,m}$, $1 \leqslant i \leqslant l$. Using the averaging over the group $\mathrm{Sp}(n-1)$, we may assume that $h_i \in \mathcal{Z}^n_{p_i,q_i,m}$. In view of condition (4.88), the functions h_i, $1 \leqslant i \leqslant l$, are nonzero. By Lemma 4.19,

$$h_i = \sum_{j=\eta_i}^{k-2m} c_{i,j} D^j (H_{p_i,q_i,m}),$$

where $c_{i,\eta_i} \in \mathbb{C} \setminus \{0\}$. Put $\eta = k - 2m - \min\{\eta_1, \dots, \eta_l\}$. Applying the operator D^{η} to (4.89) and taking into account (4.88), we get $\eta_1 = \dots = \eta_l$. Then

$$E^{k-2m} D^{\eta} f = \sum_{i=1}^{l} \alpha_i H_{p_i,q_i,m} \in H, \tag{4.90}$$

where

$$\alpha_1, \dots, \alpha_l \in \mathbb{C} \setminus \{0\} \tag{4.91}$$

(see (4.38), (4.39) and Corollary 4.9). We now prove that $H = \{E^{k-2m} D^{\eta} f\}_{\mathrm{Sp}(n)}$.

Fix $h \in H$. Suppose that

$$\int_{\mathbb{S}^{4n-1}} \big(E^{k-2m} D^{\eta} f\big)(\tau\xi)\overline{h(\xi)}\,\mathrm{d}\omega(\xi) = 0$$

for all $\tau \in \mathrm{Sp}(n)$. If $h \not\equiv 0$, reasoning as above, we obtain

$$\int_{\mathbb{S}^{4n-1}} \overline{\big(E^{k-2m} D^{\eta} f\big)(\tau\xi)} \sum_{i=1}^{l} \beta_i H_{p_i,q_i,m}(\xi)\,\mathrm{d}\omega(\xi) = 0, \tag{4.92}$$

where

$$\beta_i \in \mathbb{C}\setminus\{0\} \quad \text{and} \quad \sum_{i=1}^{l} \beta_i H_{p_i,q_i,m} \in H. \tag{4.93}$$

It follows from (4.90), (4.91), (4.93), and (4.88) that the vectors $(\alpha_1,\dots,\alpha_l)$ and $(\beta_1,\dots,\beta_l)$ are proportional. Therefore, relations (4.90) and (4.92) give

$$\alpha_1 H_{p_1,q_1,m} + \dots + \alpha_l H_{p_l,q_l,m} \equiv 0,$$

contrary to (4.91). Thus, h vanishes identically. Hence, by the Hahn–Banach theorem, $H = \{E^{k-2m} D^{\eta} f\}_{\mathrm{Sp}(n)}$, as we wished to prove. □

Lemma 4.28. *Every* $\mathrm{Sp}(n)$*-invariant subspace of* $L^2(\mathbb{S}^{4n-1})$ *is the orthogonal direct sum of some subset in* $\mathfrak{Y}$.

Proof. Denote by $\mathfrak{H}_{k,m,l}$ the space (4.87) in Lemma 4.27. By virtue of Lemma 4.26, it suffices to verify that every nontrivial $\mathrm{Sp}(n)$-invariant subspace in $\mathfrak{H}_{k,m,l}$ is the orthogonal direct sum of spaces of the form

$$\{\alpha_{i_1} H_{p_{i_1},q_{i_1},m} + \dots + \alpha_{i_j} H_{p_{i_j},q_{i_j},m}\}_{\mathrm{Sp}(n)}, \tag{4.94}$$

where $1 \leqslant j \leqslant l$, $1 \leqslant i_1 < \dots < i_j \leqslant l$, and $\alpha_{i_1},\dots,\alpha_{i_j} \in \mathbb{C}\setminus\{0\}$. In case $l = 1$ this assertion follows from the irreducibility of the representation $T_4^{n,p_1,q_1,m}(\tau)$. Assume that it is true for certain $l \in \{1,\dots,k-2m\}$, and let us prove it for $l+1$. Consider an arbitrary nontrivial $\mathrm{Sp}(n)$-invariant subspace $\mathfrak{H}$ of $\mathfrak{H}_{k,m,l+1}$. Using the induction hypothesis, it is not hard to establish the decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, where $\mathfrak{H}_1$ is the orthogonal direct sum of spaces of the form (4.94), and $\mathfrak{H}_2$ is an $\mathrm{Sp}(n)$-invariant subspace in $\mathfrak{H}_{k,m,l+1}$ such that

$$\mathfrak{H}_2 \cap \Bigg(\bigoplus_{\substack{j=1\\ j\neq i}}^{l+1} \mathcal{H}_4^{n,p_j,q_j,m}\Bigg) = \{0\}, \quad 1 \leqslant i \leqslant l+1.$$

Now invoking Lemma 4.27 we obtain the desired assertion. Hence the lemma. □

Lemma 4.28 implies the statement of Theorem 4.7 for the space $L^2(\mathbb{S}^{4n-1})$. The following result will makes it easy to pass from $L^2(\mathbb{S}^{4n-1})$ to $\mathfrak{B}^{4n}$.

Lemma 4.29. *If $H \subset C(\mathbb{S}^{4n-1})$, H is an* $\mathrm{Sp}(n)$*-space, and some $g \in C(\mathbb{S}^{4n-1})$ is not in the uniform closure of H, then g is not in the L^2-closure of H.*

Proof. In view of the Hahn–Banach theorem and the Riesz representation theorem, there is a complex Borel measure μ on $\mathbb{S}^{4n-1}$ with the following condition:

$$\int_{\mathbb{S}^{4n-1}} f(\xi)\,\mathrm{d}\mu(\xi) = 0 \quad \text{for all } f \in H, \tag{4.95}$$

but

$$\int_{\mathbb{S}^{4n-1}} g(\xi)\,\mathrm{d}\mu(\xi) = 1.$$

Choose a neighborhood $\mathcal{U}$ of the identity in $\mathrm{Sp}(n)$ such that

$$\mathrm{Re} \int_{\mathbb{S}^{4n-1}} g(\tau\xi)\,\mathrm{d}\mu(\xi) \geqslant \frac{1}{2} \quad \text{for each } \tau \in \mathcal{U}. \tag{4.96}$$

Let $\psi\colon \mathrm{Sp}(n) \to [0, \infty)$ be continuous, with support in $\mathcal{U}$ and

$$\int_{\mathrm{Sp}(n)} \psi(\tau)\,\mathrm{d}\tau = 1. \tag{4.97}$$

For $h \in L^2(\mathbb{S}^{4n-1})$, we put

$$F(h) = \int_{\mathrm{Sp}(n)} \psi(\tau)\left(\int_{\mathbb{S}^{4n-1}} h(\tau\xi)\,\mathrm{d}\mu(\xi)\right)\mathrm{d}\tau. \tag{4.98}$$

Then

$$F = 0 \quad \text{on } H \quad \text{and} \quad \mathrm{Re}\ F(g) \geqslant 1/2 \tag{4.99}$$

(see (4.95)–(4.97)). Next, changing the order of integration in (4.98) and using the Schwarz inequality, we obtain

$$|F(h)| \leqslant \frac{1}{\sqrt{\omega_{4n-1}}}\|\mu\|\|\psi\|_2\|h\|_2,$$

where $\|\mu\|$ is the complete variation of μ on $\mathbb{S}^{4n-1}$. Thus, F is a bounded linear functional on $L^2(\mathbb{S}^{4n-1})$. Combining this with (4.99), we infer that g is not in the L^2-closure of H, as required. □

We are now in a position to prove Theorem 4.7 in the general case.

Proof of Theorem 4.7. Let H be an $\mathrm{Sp}(n)$-invariant subspace of $\mathfrak{B}^{4n}$. Define $\mathfrak{H}$ to be the L^2-closure of $H \cap C$, where $C = C(\mathbb{S}^{4n-1})$. According to Lemma 4.28, $\mathfrak{H}$ is the L^2-closure of $\Sigma(\mathfrak{Y}_0)$ for some $\mathfrak{Y}_0 \subset \mathfrak{Y}$, where $\Sigma(\mathfrak{Y}_0)$ is the algebraic sum of $\mathfrak{Y}_0$. Hence, by Lemma 4.29,

$$\mathfrak{H} \cap C = \text{uniform closure of } \Sigma(\mathfrak{Y}_0). \tag{4.100}$$

On the other hand, as $H \cap C$ is uniformly closed, another application of Lemma 4.29 gives

$$\mathfrak{H} \cap C = H \cap C. \tag{4.101}$$

Since $H \cap C$ is $\mathfrak{B}^{4n}$-dense in H (see the proof of Lemma 4.6), (4.100) and (4.101) imply that H is the $\mathfrak{B}^{4n}$-closure of $\Sigma(\mathfrak{Y}_0)$. This is the assertion of Theorem 4.7. □

4.5 The Group $O_{\mathbb{Q}}(n)$

The technique developed in Sect. 4.4 allows us to obtain an analog of Theorem 4.6 for the quasi-regular representation $T_5^n(\tau)$, $n \geqslant 2$, of the group $O_{\mathbb{Q}}(n)$ on $L^2(\mathbb{S}^{4n-1})$.

As above, let $\mathcal{H}_1^{4n,k}$ be the space of homogeneous harmonic polynomials of degree k on $\mathbb{R}^{4n} = \mathbb{C}^{2n}$. For $m \in \{0, \dots, [k/2]\}$, we set

$$\mathcal{H}_5^{n,k,m} = \left\{ f \in \mathcal{H}_1^{4n,k} \colon (Lf)(z) = 4(m-1)(m-k)\left(1 - |z|^2\right) f(z) \right\},$$

where L is the Laplace–Beltrami operator on the quaternionic hyperbolic space $\mathbb{H}_{\mathbb{Q}}^n$. Since $O_{\mathbb{Q}}(n)$ is a subgroup of the group $I(\mathbb{H}_{\mathbb{Q}}^n)$ (see Proposition 2.12), the space $\mathcal{H}_5^{n,k,m}$ is an invariant subspace of the representation $T_5^n(\tau)$. We denote by $T_5^{n,k,m}(\tau)$ the restriction of $T_5^n(\tau)$ to $\mathcal{H}_5^{n,k,m}$.

The main result of this section is stated in the following theorem.

Theorem 4.8. *The quasi-regular representation $T_5^n(\tau)$ is the orthogonal direct sum of the pairwise nonequivalent irreducible unitary representations*

$$T_5^{n,k,m}(\tau), \qquad k \in \mathbb{Z}_+, m \in \{0, \dots, [k/2]\}.$$

To prove the theorem a couple of lemmas will be needed.

Lemma 4.30. *Let $(k_1, m_1) \neq (k_2, m_2)$. Then the spaces $\mathcal{H}_5^{n,k_1,m_1}$ and $\mathcal{H}_5^{n,k_2,m_2}$ are orthogonal.*

Proof. The argument is quite similar to that of Lemma 4.14. □

Lemma 4.31. *The space $\mathcal{H}_5^{n,k,m}$ is the sum of the spaces $\mathcal{H}_4^{n,p,q,m}$, where $p+q = k$, $p \geqslant m$, $q \geqslant m$.*

Proof. It is obvious that

$$\sum_{\substack{p+q=k \\ p \geqslant m, q \geqslant m}} \mathcal{H}_4^{n,p,q,m} \subset \mathcal{H}_5^{n,k,m}. \tag{4.102}$$

Next, let $f \in \mathcal{H}_5^{n,k,m}$. In accordance with Lemmas 4.7 and 4.16, we have

$$f = \sum_{j=0}^{[k/2]} f_j,$$

where

$$f_j \in \sum_{\substack{p+q=k \\ p\geqslant j, q\geqslant j}} \mathcal{H}_4^{n,p,q,j}.$$

Taking the relation $f - f_m \in \mathcal{H}_5^{n,k,m}$ into account and using Lemma 4.30, we conclude that $f = f_m$. Thus,

$$\mathcal{H}_5^{n,k,m} \subset \sum_{\substack{p+q=k \\ p\geqslant m, q\geqslant m}} \mathcal{H}_4^{n,p,q,m}. \tag{4.103}$$

Together, (4.102) and (4.103) imply the assertion of Lemma 4.31. □

Corollary 4.13. *The space $L^2(\mathbb{S}^{4n-1})$ is the orthogonal direct sum of the spaces*

$$\mathcal{H}_5^{n,k,m}, \quad k \in \mathbb{Z}_+, m \in \{0, \ldots, [k/2]\}.$$

Proof. From Lemmas 4.7 and 4.16 we have

$$\mathcal{H}_1^{4n,k} = \bigoplus_{p+q=k} \mathcal{H}_2^{2n,p,q} = \bigoplus_{p+q=k} \bigoplus_{m=0}^{\min\{p,q\}} \mathcal{H}_4^{n,p,q,m} = \bigoplus_{m=0}^{[k/2]} \bigoplus_{\substack{p+q=k \\ p\geqslant m, q\geqslant m}} \mathcal{H}_4^{n,p,q,m}.$$

Now using Lemmas 4.2 and 4.31, we obtain

$$L^2\big(\mathbb{S}^{4n-1}\big) = \bigoplus_{k=0}^{\infty} \bigoplus_{m=0}^{[k/2]} \mathcal{H}_5^{n,k,m}, \tag{4.104}$$

as required. □

Proof of Theorem 4.8. By virtue of (4.104),

$$T_5^n(\tau) = \bigoplus_{k=0}^{\infty} \bigoplus_{m=0}^{[k/2]} T_5^{n,k,m}(\tau).$$

We now prove that the representations $T_5^{n,k,m}(\tau)$ are irreducible. First, let us calculate some infinitesimal operators of the representation $T_5^{n,k,m}(\tau)$. For $t \in \mathbb{R}^1$ and $z = (z_1, \ldots, z_{2n}) \in \mathbb{C}^{2n}$, we set $\tau_1(t)z = (v_1, \ldots, v_{2n})$ and $\tau_2(t)z = (w_1, \ldots, w_{2n})$, where

$$\begin{aligned} v_1 &= (\cos t)\mathrm{e}^{\mathrm{i}t} z_1 - (\sin t)\mathrm{e}^{\mathrm{i}t} \overline{z}_{n+1}, \\ v_{n+1} &= (\sin t)\mathrm{e}^{-\mathrm{i}t} \overline{z}_1 + (\cos t)\mathrm{e}^{-\mathrm{i}t} z_{n+1}, \end{aligned}$$

$$\begin{aligned} v_j &= z_j \cos t - \overline{z}_{n+j} \sin t, \\ v_{n+j} &= \overline{z}_j \sin t + z_{n+j} \cos t, \quad 2 \leqslant j \leqslant n, \end{aligned}$$

and

$$\begin{aligned} w_1 &= (\cos t)\mathrm{e}^{\mathrm{i}t} z_1 - \mathrm{i}(\sin t)\mathrm{e}^{\mathrm{i}t}\overline{z}_{n+1}, \\ w_{n+1} &= \mathrm{i}(\sin t)\mathrm{e}^{-\mathrm{i}t}\overline{z}_1 + (\cos t)\mathrm{e}^{-\mathrm{i}t} z_{n+1}, \\ w_j &= z_j \cos t - \mathrm{i}\overline{z}_{n+j} \sin t, \\ w_{n+j} &= \mathrm{i}\overline{z}_j \sin t + z_{n+j} \cos t, \quad 2 \leqslant j \leqslant n. \end{aligned}$$

It is easy to see that $\tau_1(t)$ and $\tau_2(t)$ are one-parameter subgroups of the group $O_{\mathbb{Q}}(n)$. Denote by A_i the infinitesimal operator of the representation $T_5^{n,k,m}(\tau)$ corresponding to $\tau_i(t)$, $i = 1, 2$. We find

$$\begin{aligned} A_1 &= \mathrm{i}\left(-z_1 \frac{\partial}{\partial z_1} + z_{n+1} \frac{\partial}{\partial z_{n+1}} + \overline{z}_1 \frac{\partial}{\partial \overline{z}_1} - \overline{z}_{n+1} \frac{\partial}{\partial \overline{z}_{n+1}}\right) \\ &\quad + \sum_{k=1}^{n} \left(\overline{z}_{n+k} \frac{\partial}{\partial z_k} - \overline{z}_k \frac{\partial}{\partial z_{n+k}} + z_{n+k} \frac{\partial}{\partial \overline{z}_k} - z_k \frac{\partial}{\partial \overline{z}_{n+k}}\right), \\ A_2 &= \mathrm{i}\left(-z_1 \frac{\partial}{\partial z_1} + z_{n+1} \frac{\partial}{\partial z_{n+1}} + \overline{z}_1 \frac{\partial}{\partial \overline{z}_1} - \overline{z}_{n+1} \frac{\partial}{\partial \overline{z}_{n+1}}\right) \\ &\quad + \mathrm{i} \sum_{k=1}^{n} \left(\overline{z}_{n+k} \frac{\partial}{\partial z_k} - \overline{z}_k \frac{\partial}{\partial z_{n+k}} - z_{n+k} \frac{\partial}{\partial \overline{z}_k} + z_k \frac{\partial}{\partial \overline{z}_{n+k}}\right). \end{aligned}$$

Hence,

$$\begin{aligned} -\frac{A_1 + 2A_{1,3} - (A_2 + 2A_{1,3})\mathrm{i}}{2} &= \mathfrak{D}, \\ -\frac{A_1 + 2A_{1,3} + (A_2 + 2A_{1,3})\mathrm{i}}{2} &= \mathfrak{E}, \end{aligned}$$

where $A_{1,3}$, $\mathfrak{D}$ and $\mathfrak{E}$ are the operators defined in Sect. 4.4. Since

$$\mathrm{Sp}(n) \subset O_{\mathbb{Q}}(n), \tag{4.105}$$

it follows that every subspace invariant with respect to the representation $T_5^{n,k,m}(\tau)$ is invariant with respect to the operators $\mathfrak{D}$ and $\mathfrak{E}$ as well.

Next, let $f \in \mathcal{H}_5^{n,k,m}$, $f \not\equiv 0$. We write f in the form

$$f = \sum_{\mu=0}^{\nu} f_\mu, \quad \nu \in \{0, \ldots, k - 2m\},$$

where $f_\mu \in \mathcal{H}_4^{n,m+\mu,k-m-\mu,m}$ and $f_\nu \not\equiv 0$ (see Lemma 4.31). Then Remark 4.2 and the Schur lemma show that

$$\mathfrak{E}^j \mathfrak{D}^\nu f = \mathfrak{E}^j \mathfrak{D}^\nu f_\nu \in \mathcal{H}_4^{n,m+j,k-m-j,m} \backslash \{0\}$$

for every $j \in \{0, \ldots, k-2m\}$. In particular,

$$\{f\}_{\mathrm{O}_{\mathbb{Q}}(n)} \cap \mathcal{H}_4^{n,m+j,k-m-j,m} \neq \{0\}, \tag{4.106}$$

where $\{f\}_{\mathrm{O}_{\mathbb{Q}}(n)}$ is the linear span of the system $\{f \circ \tau, \tau \in \mathrm{O}_{\mathbb{Q}}(n)\}$. Using (4.105), (4.106), and Theorem 4.6, we conclude that

$$\mathcal{H}_4^{n,m+j,k-m-j,m} \subset \{f\}_{\mathrm{O}_{\mathbb{Q}}(n)}, \quad 0 \leqslant j \leqslant k-2m.$$

Thus, $\{f\}_{\mathrm{O}_{\mathbb{Q}}(n)} = \mathcal{H}_5^{n,k,m}$, whence the irreducibility of the representation $T_5^{n,k,m}(\tau)$ follows.

It remains to prove that the representations $T_5^{n,k_i,m_i}(\tau)$, $i = 1, 2$, are equivalent if and only if $k_1 = k_2$ and $m_1 = m_2$. Let $A\colon \mathcal{H}_5^{n,k_1,m_1} \to \mathcal{H}_5^{n,k_2,m_2}$ be a nonzero linear operator and suppose that A commutes with the group $\mathrm{O}_{\mathbb{Q}}(n)$. Because $A\mathfrak{D} = \mathfrak{D}A$, we derive from Remark 4.2 and Lemma 4.31 that A sends $\mathcal{H}_4^{n,m_1,k_1-m_1,m_1}$ to $\mathcal{H}_4^{n,m_2,k_2-m_2,m_2}$. Applying Theorem 4.6, we arrive at the desired assertion. □

Remark 4.3. Theorems 4.7 and 4.8 imply the description of $\mathrm{O}_{\mathbb{Q}}(n)$-invariant subspaces of $\mathfrak{B}^{4n}$ analogous to that given in Sect. 4.1 for the group $\mathrm{O}(n)$. In this case we must replace the spaces $\mathcal{H}_1^{n,k}$ by $\mathcal{H}_5^{n,k,m}$.

4.6 The Group $\mathrm{O}_{\mathbb{C}a}(2)$

In this section we shall obtain an explicit description of the decomposition of the space $\mathcal{H}_1^{16,k}$ into irreducible subspaces under the action of the group $\mathrm{O}_{\mathbb{C}a}(2)$.

Let L be the Laplace–Beltrami operator on the Cayley hyperbolic plane $\mathbb{H}^2_{\mathbb{C}a}$ (see Proposition 2.21). For $m \in \{0, \ldots, [k/2]\}$, we set

$$\mathcal{H}_6^{k,m} = \left\{f \in \mathcal{H}_1^{16,k} : (Lf)(x) = 4\lambda_{k,m}\big(1-|x|^2\big)f(x)\right\},$$

where

$$\lambda_{k,m} = (m-3)(m-k).$$

Because $\mathrm{O}_{\mathbb{C}a}(2) \subset I(\mathbb{H}^2_{\mathbb{C}a})$ (see Sect. 2.4), the space $\mathcal{H}_6^{k,m}$ is an invariant subspace of the quasi-regular representation $T_6(\tau)$ of the group $\mathrm{O}_{\mathbb{C}a}(2)$ on $L^2(\mathbb{S}^{15})$. We denote by $T_6^{k,m}(\tau)$ the restriction of $T_6(\tau)$ to $\mathcal{H}_6^{k,m}$. Our goal is now to prove the following result.

Theorem 4.9. *The quasi-regular representation $T_6(\tau)$ is the orthogonal direct sum of the pairwise nonequivalent irreducible unitary representations*

$$T_6^{k,m}(\tau), \quad k \in \mathbb{Z}_+, m \in \{0, \ldots, [k/2]\}.$$

It will be convenient to break up the proof of Theorem 4.9 into a number of lemmas. We shall begin with some relations concerning the Beltrami parameters on $\mathbb{H}^2_{\mathbb{C}a}$. Recall that the first Beltrami parameter ∇_1 on $\mathbb{H}^2_{\mathbb{C}a}$ is defined by

$$(\nabla_1 f)(x) = \sum_{i,j=1}^{16} g^{ij}(x)\frac{\partial f}{\partial x_i}(x)\frac{\partial f}{\partial x_j}(x), \quad f \in C^1\big(B^{16}_{\mathbb{R}}\big).$$

Here and in the rest of this section, we are using the notation established in Sect. 2.4. The second Beltrami parameter Δ_2 on $\mathbb{H}^2_{\mathbb{C}a}$ coincides with the operator L. Note that the operators ∇_1 and L can be naturally extended to $C^1(\mathbb{R}^{16})$ and $C^2(\mathbb{R}^{16})$, respectively. In addition,

$$\nabla_1 f = \frac{1}{2}L\big(f^2\big) - f \cdot Lf.$$

Lemma 4.32. *Let $a = (a_1, \dots, a_{16}) \in \mathbb{R}^{16}$ be fixed. Then*

(i) $\nabla_1\big(\Phi_{\mathbb{C}a}(x,a)\big) = 4\big(1-|x|^2\big)\Phi_{\mathbb{C}a}(x,a)\big(|a|^2 - \Phi_{\mathbb{C}a}(x,a)\big)$;

(ii) $\nabla_1\big(\Psi_{\mathbb{C}a}(x,a)\big) = 4\big(1-|x|^2\big)\Psi_{\mathbb{C}a}(x,a)\big(|a|^2 - \Phi_{\mathbb{C}a}(x,a)\big)$.

Proof. Relation (2.51) gives

$$\frac{\partial}{\partial x_i}\big(\Phi_{\mathbb{C}a}(x,a)\big) = -2\big(1-|a|^2\big)x_i + 2\big(1-|a|^2\big)^2\sum_{j=1}^{16} x_j g_{ij}(a),$$

$$\frac{\partial}{\partial x_i}\big(\Psi_{\mathbb{C}a}(x,a)\big) = -2a_i - 2\big(1-|a|^2\big)x_i + 2\big(1-|a|^2\big)^2\sum_{j=1}^{16} x_j g_{ij}(a).$$

Hence, by (1.17) and (2.58),

$$\sum_{i,j=1}^{16} g^{ij}(y)\frac{\partial}{\partial y_i}\big(\Phi_{\mathbb{C}a}(y,a)\big)\frac{\partial}{\partial y_j}\big(\Phi_{\mathbb{C}a}(y,a)\big)\Big|_{y=|x|e_1}$$
$$= 4\big(1-|x|^2\big)\Phi_{\mathbb{C}a}(|x|e_1,a)\big(|a|^2 - \Phi_{\mathbb{C}a}(|x|e_1,a)\big),$$
$$\sum_{i,j=1}^{16} g^{ij}(y)\frac{\partial}{\partial y_i}\big(\Psi_{\mathbb{C}a}(y,a)\big)\frac{\partial}{\partial y_j}\big(\Psi_{\mathbb{C}a}(y,a)\big)\Big|_{y=|x|e_1}$$
$$= 4\big(1-|x|^2\big)\Psi_{\mathbb{C}a}(|x|e_1,a)\big(|a|^2 - \Phi_{\mathbb{C}a}(|x|e_1,a)\big).$$

Now keeping in mind the invariance of ∇_1 under the group $O_{\mathbb{C}a}(2)$ and using Proposition 1.1, we arrive at the desired assertion. □

From this result and from (2.54) and (2.55) we obtain

Corollary 4.14. *The following equalities are valid:*

(i) $$\sum_{i,j=1}^{16} g^{ij}(x)a_i \frac{\partial}{\partial x_j}\big(\Phi_{\mathbb{C}a}(x,a)\big) = 2\big(1-|x|^2\big)\big(|a|^2 - \Phi_{\mathbb{C}a}(x,a)\big)\langle x,a\rangle_{\mathbb{R}}.$$

(ii) $$\sum_{i,j=1}^{16} g^{ij}(x)a_i \frac{\partial}{\partial x_j}\big(\Psi_{\mathbb{C}a}(x,a)\big) = 2\big(1-|x|^2\big)\big(|a|^2 - \Phi_{\mathbb{C}a}(x,a)\big)\big(\langle x,a\rangle_{\mathbb{R}} - 1\big).$$

For $f \in C^2(\mathbb{R}^{16})$, we put

$$(\mathcal{L}f)(x) = \frac{1}{4}\sum_{i,j=1}^{16} b_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + 3\sum_{i=1}^{16} x_i \frac{\partial f}{\partial x_i}(x)$$

(see Sect. 2.4). Propositions 2.17 and 2.21 show that

$$4\big(1-|x|^2\big)(\mathcal{L}f)(x) = (Lf)(x).$$

Lemma 4.33. *For $a \in \mathbb{R}^{16}$, we have*

(i) $$\mathcal{L}\big(\Phi_{\mathbb{C}a}(x,a)\big) = 4|a|^2 + 2\Phi_{\mathbb{C}a}(x,a).$$

(ii) $$\mathcal{L}\big(\Psi_{\mathbb{C}a}(x,a)\big) = 4|a|^2 + 2\Phi_{\mathbb{C}a}(x,a) - 6\langle x,a\rangle_{\mathbb{R}}.$$

Proof. By virtue of (2.50),

$$\frac{\partial^2}{\partial x_i \partial x_j}\big(\Phi_{\mathbb{C}a}(x,a)\big) = \frac{\partial^2}{\partial x_i \partial x_j}\big(\Psi_{\mathbb{C}a}(x,a)\big) = -2\delta_{i,j}\big(1-|a|^2\big)+2\big(1-|a|^2\big)^2 g_{ij}(a).$$

Then, taking into account relations (2.55) and (2.60), we find that

$$\begin{aligned}\sum_{i,j=1}^{16} b_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}\big(\Phi_{\mathbb{C}a}(x,a)\big) &= \sum_{i,j=1}^{16} b_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}\big(\Psi_{\mathbb{C}a}(x,a)\big)\\ &= 16\big(|a|^2 - \Phi_{\mathbb{C}a}(x,a)\big).\end{aligned}$$

This, together with Euler's formula for homogeneous functions, brings us to the assertion of Lemma 4.33. □

Lemma 4.34. *The relation*

$$\begin{aligned}(\mathcal{L} - \lambda_{k,m}\,\mathrm{Id})&\big(\langle x,a\rangle_{\mathbb{R}}^{k-2m}\big(\Phi_{\mathbb{C}a}(x,a)\big)^m\big)\\ &= m(k-m+3)|a|^2\langle x,a\rangle_{\mathbb{R}}^{k-2m}\big(\Phi_{\mathbb{C}a}(x,a)\big)^{m-1}\\ &\quad + \frac{(k-2m)(k-2m-1)}{4}\langle x,a\rangle_{\mathbb{R}}^{k-2m-2}\big(\Phi_{\mathbb{C}a}(x,a)\big)^m\big(|a|^2 - \Phi_{\mathbb{C}a}(x,a)\big)\end{aligned} \tag{4.107}$$

holds. Here, as above, $a \in \mathbb{R}^{16}$, $k \in \mathbb{Z}_+$, and $m \in \{0,\dots,[k/2]\}$.

Proof. Invoking Corollary 4.14 and Lemmas 4.32 and 4.33, we obtain (4.107) at the point $|x|e_1$ by a direct calculation. Since the group $O_{\mathbb{C}a}(2)$ acts transitively on $\mathbb{S}^{15}$ and $\mathcal{L}$ is invariant under $O_{\mathbb{C}a}(2)$, equality (4.107) is valid for any $x \in \mathbb{R}^{16}$. □

Denote by $\mathbb{C}^{16}_{\text{isot}}$ the set of all *isotropic* vectors in $\mathbb{C}^{16}$. We recall that a vector $b = (b_1, \dots, b_{16}) \in \mathbb{C}^{16}$ belongs to $\mathbb{C}^{16}_{\text{isot}}$ if and only if

$$b_1^2 + \cdots + b_{16}^2 = 0.$$

An immediate consequence of Lemma 4.34 is the following:

Corollary 4.15. *Let $b \in \mathbb{C}^{16}_{\text{isot}}$ and $\nu \in \{0, \dots, [k/2] - m\}$. Then*

$$\begin{aligned}&\left(\prod_{j=m}^{m+\nu} (\mathcal{L} - \lambda_{k,j}\,\mathrm{Id})\right)\left(\langle x, b\rangle_{\mathbb{R}}^{k-2m}\big(\Phi_{\mathbb{C}a}(x,b)\big)^m\right)\\ &\quad= \left(-\frac{1}{4}\right)^{\nu+1}\left(\prod_{j=m}^{m+\nu}(k-2j)(k-2j-1)\right)\langle x, b\rangle_{\mathbb{R}}^{k-2(m+\nu+1)}\big(\Phi_{\mathbb{C}a}(x,b)\big)^{m+\nu+1}.\end{aligned} \tag{4.108}$$

Remark 4.4. In the same way as in the proof of Lemma 4.34, we see that

$$\begin{aligned}\Delta\left(\langle x, a\rangle_{\mathbb{R}}^{k-2m}\big(\Phi_{\mathbb{C}a}(x,a)\big)^m\right) &= |a|^2\langle x, a\rangle_{\mathbb{R}}^{k-2m-2}\big(\Phi_{\mathbb{C}a}(x,a)\big)^{m-1}\\ &\quad\times\big(4m(k-m+3)\langle x, a\rangle_{\mathbb{R}}^2\\ &\quad+(k-2m)(k-2m-1)\Phi_{\mathbb{C}a}(x,a)\big).\end{aligned}$$

In particular,

$$\Delta\left(\langle x, b\rangle_{\mathbb{R}}^{k-2m}\big(\Phi_{\mathbb{C}a}(x,b)\big)^m\right) = 0 \quad \text{for } b \in \mathbb{C}^{16}_{\text{isot}}. \tag{4.109}$$

Hence, it follows from Corollary 4.15 that $\mathcal{H}_6^{k,m}$ contains the polynomial

$$\begin{aligned}P_{k,m,b}(x) = &\sum_{j=0}^{[k/2]-m}\left(-\frac{1}{4}\right)^j (j+1)(j+2)\binom{k-2m-j+2}{j+2}\\ &\times\langle x, b\rangle_{\mathbb{R}}^{k-2m-2j}\big(\Phi_{\mathbb{C}a}(x,b)\big)^{m+j}.\end{aligned}$$

In addition (see the proof of Lemma 4.16),

$$\begin{aligned}&\langle x, b\rangle_{\mathbb{R}}^{k-2m}\big(\Phi_{\mathbb{C}a}(x,b)\big)^m - \frac{1}{(k-2m+1)(k-2m+2)}P_{k,m,b}(x)\\ &\quad\in \sum_{m+1\leqslant j\leqslant [k/2]}\mathcal{H}_6^{k,j},\end{aligned}$$

and if $m+1 \leqslant [k/2]$, then

$$\langle x, b\rangle_{\mathbb{R}}^{k-2m}\big(\Phi_{\mathbb{C}a}(x,b)\big)^m - \frac{1}{(k-2m+1)(k-2m+2)} P_{k,m,b}(x) - \frac{k-2m-2}{4(k-2m+2)} P_{k,m+1,b}(x) \in \sum_{m+2\leqslant j\leqslant [k/2]} \mathcal{H}_6^{k,j}.$$

(The sums are set to be equal to zero when the set of indices of summation is empty.)

Corollary 4.16. *For $f \in \mathcal{H}_1^{16,k}$, we have*

$$\left(\prod_{j=0}^{m} (\mathcal{L} - \lambda_{k,j}\,\mathrm{Id})\right)(f) \in \mathcal{H}_1^{16,k}.$$

Moreover,

$$\left(\prod_{j=0}^{[k/2]} (\mathcal{L} - \lambda_{k,j}\,\mathrm{Id})\right)(f) = 0.$$

Proof. The statement is clear from (4.108), (4.109), and the description of the space $\mathcal{H}_1^{16,k}$ (see Sect. 4.1). □

Lemma 4.35. *Let $f \in C^2(B_{\mathbb{R}}^{16})$ and suppose that f has the form*

$$f(x) = \varphi(|x|)P(x), \quad \text{where } P \in \mathcal{H}_6^{k,m}.$$

Then

$$(Lf)(x) = \bigg(\big(1-|x|^2\big)^2\varphi''(|x|) + \frac{1-|x|^2}{|x|}\big(15+2k+(5-2k)|x|^2\big)\varphi'(|x|) + 4\lambda_{k,m}\big(1-|x|^2\big)\varphi(|x|)\bigg)P(x). \tag{4.110}$$

Proof. We use the formula

$$L(f_1 f_2)(x) = f_1(x)(Lf_2)(x) + f_2(x)(Lf_1)(x) + 2\sum_{i,j=1}^{16} g^{ij}(x)\frac{\partial f_1}{\partial x_i}(x)\frac{\partial f_2}{\partial x_j}(x)$$

with $f_1(x) = \varphi(|x|)$ and $f_2(x) = P(x)$. Relations (2.52) and (2.55) and Euler's theorem for homogeneous functions yield

$$\sum_{i,j=1}^{16} g^{ij}(x)\frac{\partial}{\partial x_i}\big(\varphi(|x|)\big)\frac{\partial}{\partial x_j}\big(P(x)\big) = k\frac{(1-|x|^2)^2}{|x|}\varphi'(|x|)P(x). \tag{4.111}$$

Furthermore, the definition of $\mathcal{H}_6^{k,m}$ shows that

$$(LP)(x) = 4\lambda_{k,m}\big(1-|x|^2\big)P(x). \tag{4.112}$$

Combining (4.111), (4.112), and (2.72), we get (4.110). □

Lemma 4.36. *The space $\mathcal{H}_1^{16,k}$ is the sum of the pairwise orthogonal spaces $\mathcal{H}_6^{k,m}$, where $m \in \{0, \ldots, [k/2]\}$.*

Proof. The orthogonality of the spaces $\mathcal{H}_6^{k,m}$ is implied by the symmetry of the operator L (see (4.110) and the proof of Lemma 4.10). The decomposition

$$\mathcal{H}_1^{16,k} = \sum_{m=0}^{[k/2]} \mathcal{H}_6^{k,m}$$

follows from Corollary 4.16 in the same manner as in Lemma 4.16. □

Corollary 4.17. *The space $L^2(\mathbb{S}^{15})$ is the orthogonal direct sum of the spaces*

$$\mathcal{H}_6^{k,m}, \quad k \in \mathbb{Z}_+, m \in \{0, \ldots, [k/2]\}.$$

Proof. Thanks to Lemma 4.2, we have

$$L^2(\mathbb{S}^{15}) = \bigoplus_{k=0}^{\infty} \mathcal{H}_1^{16,k}.$$

Applying Lemma 4.36, we complete the proof. □

Now we require some properties of polynomials invariant with respect to the isotropy subgroup of the point e_1 in the group $O_{\mathbb{C}a}(2)$. We first prove the following assertion.

Lemma 4.37. *Suppose that $\tau \in O_{\mathbb{C}a}(2)$, $\tau e_1 = e_1$, and let $\|t_{i,j}\|_{i,j=1}^{16}$ be the matrix of τ in the standard basis $e_1, \ldots, e_{16} \in \mathbb{R}^{16}$, that is,*

$$\tau e_j = \sum_{i=1}^{16} t_{i,j} e_i, \quad 1 \leqslant j \leqslant 16.$$

Then $t_{2i,2j-1} = t_{2j-1,2i} = 0$ and $t_{2i-1,1} = t_{1,2i-1} = \delta_{1,2i-1}$ for $i, j \in \{1, 2, \ldots, 8\}$.

Proof. Since $\tau \in O(16)$ and $\tau e_1 = e_1$, we get $t_{1,1} = 1$ and

$$t_{1,j} = t_{j,1} = 0$$

for $j \in \{2, 3, \ldots, 16\}$. Using now the relation

$$\Phi_{\mathbb{C}a}(\tau x, e_1) = \Phi_{\mathbb{C}a}(x, e_1), \quad x \in \mathbb{R}^{16},$$

we see that $t_{i,j} = 0$ if $i \in \{3, 5, \ldots, 15\}$ and $j \in \{2, 4, \ldots, 16\}$. Analogously, the equality

$$|\tau x|^2 - \Phi_{\mathbb{C}a}(\tau x, e_1) = |x|^2 - \Phi_{\mathbb{C}a}(x, e_1), \quad x \in \mathbb{R}^{16},$$

yields $t_{i,j} = 0$ when $i \in \{2, 4, \dots, 16\}$ and $j \in \{3, 5, \dots, 15\}$. Thus, the lemma is proved. □

Next, with our notation in Sect. 2.4, we have the following:

Lemma 4.38. *Let f be a homogeneous polynomial on $\mathbb{R}^{16}$ of degree k and assume that $f \circ \tau = f$ for every $\tau \in \mathrm{O}_{\mathbb{C}a}(2)$ that fixes e_1. Then*

$$f(x) = \sum_{\alpha+2\beta+2\gamma=k} c_{\alpha,\beta,\gamma} x_1^{\alpha} \big(p_9(x) - x_1^2\big)^{\beta} \big(p_{10}(x)\big)^{\gamma}, \tag{4.113}$$

for some $c_{\alpha,\beta,\gamma} \in \mathbb{C}$. Conversely, for all $c_{\alpha,\beta,\gamma} \in \mathbb{C}$, the polynomial (4.113) belongs to $\mathcal{P}_k^{16}$ and is invariant with respect to the isotropy subgroup of the point e_1 in the group $\mathrm{O}_{\mathbb{C}a}(2)$.

Proof. We write $f(x)$ as

$$f(x) = \sum_{\alpha+\beta\leqslant k} x_1^{\alpha} x_2^{\beta} f_{\alpha,\beta}(x_3, x_4, \dots, x_{16}),$$

where $f_{\alpha,\beta}$ is a homogeneous polynomial of degree $k - \alpha - \beta$. Since $f \circ A_{\varphi} = f$ for any $\varphi \in G_2$ (see Example 1.2 and (1.15)), we deduce from Corollary 1.2 that

$$f(x) = \sum_{\alpha+2\beta+2\gamma+2\delta+\varepsilon=k} d_{\alpha,\beta,\gamma,\delta,\varepsilon} x_1^{\alpha} \big(p_9(x) - x_1^2\big)^{\beta} \big(p_{10}(x)\big)^{\gamma} x_2^{\delta} \big(p_1(x)\big)^{\varepsilon} \tag{4.114}$$

with some constants $d_{\alpha,\beta,\gamma,\delta,\varepsilon} \in \mathbb{C}$. Next, let

$$\begin{aligned}
&X_1 = x_1, && X_2 = x_2 \cos t - x_6 \sin t, && X_3 = x_3 \cos t - x_7 \sin t, && X_4 = x_4,\\
&X_5 = x_5, && X_6 = x_6 \cos t + x_2 \sin t, && X_7 = x_7 \cos t + x_3 \sin t, && X_8 = x_8,\\
&X_9 = x_9, && X_{10} = x_{10} \cos t - x_{14} \sin t, && X_{11} = x_{11} \cos t - x_{15} \sin t, && X_{12} = x_{12},\\
&X_{13} = x_{13}, && X_{14} = x_{14} \cos t + x_{10} \sin t, && X_{15} = x_{15} \cos t + x_{11} \sin t, && X_{16} = x_{16},
\end{aligned}$$

$$X = (X_1, \dots, X_{16}).$$

Then

$$\begin{aligned}
&p_1(X) = p_1(x) \cos t - p_3(x) \sin t, && p_2(X) = p_2(x),\\
&p_3(X) = p_3(x) \cos t + p_1(x) \sin t, && p_4(X) = p_4(x),\\
&p_5(X) = p_5(x) \cos t - p_7(x) \sin t, && p_6(X) = p_6(x),\\
&p_7(X) = p_7(x) \cos t + p_5(x) \sin t, && p_8(X) = p_8(x),\\
&p_9(X) = p_9(x), && p_{10}(X) = p_{10}(x),
\end{aligned}$$

and $X = e_1$ for $x = e_1$. These relations imply that the transformation $\tau : x \to X$ belongs to $\mathrm{O}_{\mathbb{C}a}(2)$ and $\tau e_1 = e_1$. Therefore, by the hypothesis and (4.114), for $Y = (Y_1, \dots, Y_{16}) \in \mathbb{C}^{16}$,

$$\sum_{\alpha+2\beta+2\gamma+2\delta+\varepsilon=k} d_{\alpha,\beta,\gamma,\delta,\varepsilon} Y_1^\alpha \big(p_9(Y)-Y_1^2\big)^\beta \big(p_{10}(Y)\big)^\gamma Y_2^\delta \big(p_1(Y)\big)^\varepsilon$$
$$= \sum_{\alpha+2\beta+2\gamma+2\delta+\varepsilon=k} d_{\alpha,\beta,\gamma,\delta,\varepsilon} Y_1^\alpha \big(p_9(Y)-Y_1^2\big)^\beta \big(p_{10}(Y)\big)^\gamma$$
$$\times (Y_2\cos t - Y_6 \sin t)^\delta \big(p_1(Y)\cos t - p_3(Y)\sin t\big)^\varepsilon. \tag{4.115}$$

Take $Z_1,\dots,Z_7 \in \mathbb{C}$ arbitrarily. Putting

$$Y_1 = Z_1,\quad Y_2 = Z_4,\quad Y_3 = Z_1Z_4 - Z_5,\quad Y_4 = 1,\quad Y_5 = 0,$$
$$Y_6 = Z_6,\quad Y_7 = Z_1Z_6 - Z_7,\quad Y_8 = Y_9 = 0,$$
$$Y_{10} = \sqrt{Z_3 - Z_4^2 - Z_6^2 - 1},$$
$$Y_{11} = \sqrt{Z_2 - (Z_1Z_4 - Z_5)^2 - (Z_1Z_6 - Z_7)^2},$$
$$Y_{12} = Y_{13} = Y_{14} = Y_{15} = Y_{16} = 0,$$

and $t = -\pi/2$ in (4.115), we obtain

$$\sum_{\alpha+2\beta+2\gamma+2\delta+\varepsilon=k} d_{\alpha,\beta,\gamma,\delta,\varepsilon} Z_1^\alpha Z_2^\beta Z_3^\gamma Z_4^\delta Z_5^\varepsilon$$
$$= \sum_{\alpha+2\beta+2\gamma+2\delta+\varepsilon=k} d_{\alpha,\beta,\gamma,\delta,\varepsilon} Z_1^\alpha Z_2^\beta Z_3^\gamma Z_6^\delta Z_7^\varepsilon.$$

Hence,

$$d_{\alpha,\beta,\gamma,\delta,\varepsilon} = 0 \quad \text{if } \delta \neq 0 \text{ or } \varepsilon \neq 0. \tag{4.116}$$

Together, (4.114) and (4.116) give (4.113). The converse assertion follows from Lemma 4.37. □

As before, let $k \in \mathbb{Z}_+$, $m \in \{0,\dots,[k/2]\}$. For $\alpha \in \{0,\dots,m\}$, we set

$$Q_{\alpha,k,m}(x) = c_{\alpha,k,m}\big(p_9(x)\big)^{m-\alpha} \sum_{\beta=0}^{[k/2]-m} d_{\beta,k,m} x_1^{k-2m-2\beta}\big(p_9(x)-x_1^2\big)^\beta,$$

where

$$c_{\alpha,k,m} = \begin{cases} 1 & \text{if } \alpha = 0,\\ \displaystyle\prod_{\mu=0}^{\alpha-1} \frac{(m-\mu)(\mu+m-k-3)}{\mu^2+5\mu+4} & \text{if } 1 \leqslant \alpha \leqslant m,\end{cases}$$
$$d_{\beta,k,m} = \begin{cases} 1 & \text{if } \beta = 0,\\ \displaystyle\prod_{\mu=0}^{\beta-1} \frac{(2m+2\mu-k)(k-2m-2\mu-1)}{4\mu^2+18\mu+14} & \text{if } 1 \leqslant \beta \leqslant [k/2]-m.\end{cases}$$

Now define

$$P_{k,m}(x) = \sum_{\alpha=0}^{m} \big(p_{10}(x)\big)^{\alpha} Q_{\alpha,k,m}(x).$$

Lemma 4.39. *Let* $f \in \mathcal{H}_6^{k,m}$, $f(e_1) = 1$ *and suppose that* $f \circ \tau = f$ *for all* $\tau \in \mathrm{O}_{\mathbb{C}a}(2)$ *that fix* e_1. *Then* $f = P_{k,m}$.

Proof. By Lemma 4.38,

$$f(x) = \sum_{\alpha=0}^{[k/2]} \big(p_{10}(x)\big)^{\alpha} f_\alpha(x)$$

with

$$f_\alpha(x) = \sum_{\beta=0}^{[k/2]-\alpha} \gamma_{\beta,k,\alpha} x_1^{k-2m-2\beta} \big(p_9(x) - x_1^2\big)^{\beta}, \quad \gamma_{\beta,k,\alpha} \in \mathbb{C}. \tag{4.117}$$

Put $\varkappa = \max\{\alpha \in \{0, \dots, [k/2]\} \colon f_\alpha \not\equiv 0\}$. The harmonicity of f means that

$$\Delta(f_\alpha) = -4\big(\alpha^2 + 5\alpha + 4\big) f_{\alpha+1}, \quad 0 \leqslant \alpha \leqslant \varkappa - 1, \tag{4.118}$$

$$\Delta(f_\varkappa) = 0. \tag{4.119}$$

Let us compute $\mathcal{L}f$. We represent $\mathcal{L}$ in the form $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$, where

$$\mathcal{L}_1 = \frac{1}{4}\sum_{i=1}^{16} b_{ii}(x)\frac{\partial^2}{\partial x_i^2}, \qquad \mathcal{L}_2 = \frac{1}{4}\sum_{\substack{i,j=1\\ i\neq j}}^{16} b_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}, \qquad \mathcal{L}_3 = 3\sum_{i=1}^{16} x_i \frac{\partial}{\partial x_i}.$$

It is easy to make sure that

$$(\mathcal{L}_1 g_\alpha)(x) = \alpha(3+\alpha)\big(1 - p_{10}(x)\big)\big(p_{10}(x)\big)^{\alpha-1} f_\alpha(x) + \frac{1}{4}\big(1 - p_9(x)\big)\big(p_{10}(x)\big)^{\alpha} (\Delta f_\alpha)(x), \tag{4.120}$$

$$(\mathcal{L}_3 g_\alpha)(x) = 3k g_\alpha(x), \tag{4.121}$$

where $g_\alpha(x) = (p_{10}(x))^{\alpha} f_\alpha(x)$. To calculate $\mathcal{L}_2 g_\alpha$ we need the relation

$$\sum_{\substack{j=1\\ j\neq i}}^{16} x_j b_{ij}(x) = \begin{cases} -x_i p_{10}(x) & \text{if } i \in E_1, \\ -x_i p_9(x) & \text{if } i \in E_2, \end{cases}$$

with $E_1 = \{1, 3, 5, \dots, 15\}$ and $E_2 = \{2, 4, 6, \dots, 16\}$ (see (2.52)). For $i \in E_1$, we have

$$\frac{1}{4}\sum_{\substack{j=1\\j\neq i}}^{16} b_{ij}(x)\frac{\partial^2 g_\alpha}{\partial x_i \partial x_j}(x) = \frac{1}{4}\sum_{j\in E_2} b_{ij}(x)\frac{\partial^2 g_\alpha}{\partial x_j \partial x_j}(x)$$
$$= \frac{\alpha}{2}\big(p_{10}(x)\big)^{\alpha-1}\frac{\partial f_\alpha}{\partial x_i}(x)\sum_{\substack{j=1\\j\neq i}}^{16} x_j b_{ij}(x)$$
$$= -\frac{\alpha}{2}\big(p_{10}(x)\big)^{\alpha} x_i \frac{\partial f_\alpha}{\partial x_i}(x),$$

whence

$$\frac{1}{4}\sum_{i\in E_1}\sum_{\substack{j=1\\j\neq i}}^{16} b_{ij}(x)\frac{\partial^2 g_\alpha}{\partial x_i \partial x_j}(x) = \frac{(2\alpha-k)\alpha}{2} g_\alpha(x). \tag{4.122}$$

Let $i \in E_2$. Then

$$\frac{1}{4}\sum_{\substack{j=1\\j\neq i}}^{16} b_{ij}(x)\frac{\partial^2 g_\alpha}{\partial x_i \partial x_j}(x) = \frac{1}{4}\sum_{j\in E_1} b_{ij}(x)\frac{\partial^2 g_\alpha}{\partial x_i \partial x_j}(x)$$
$$= \frac{\alpha}{2}\big(p_{10}(x)\big)^{\alpha-1} x_i \sum_{j=1}^{16} b_{ij}(x)\frac{\partial f_\alpha}{\partial x_j}(x).$$

Therefore,

$$\frac{1}{4}\sum_{i\in E_2}\sum_{\substack{j=1\\j\neq i}}^{16} b_{ij}(x)\frac{\partial^2 g_\alpha}{\partial x_i \partial x_j}(x) = \frac{\alpha}{2}\big(p_{10}(x)\big)^{\alpha-1}\sum_{j\in E_1}\frac{\partial f_\alpha}{\partial x_j}(x)\sum_{\substack{i=1\\i\neq j}}^{16} x_i b_{ji}(x)$$
$$= \frac{(2\alpha-k)\alpha}{2} g_\alpha(x). \tag{4.123}$$

According to (4.122) and (4.123),

$$\mathcal{L}_2(g_\alpha) = \alpha(2\alpha-k)g_\alpha. \tag{4.124}$$

In view of (4.120), (4.121), and (4.124),

$$(\mathcal{L}g_\alpha)(x) = (\alpha-3)(\alpha-k)g_\alpha(x) + \alpha(3+\alpha)\big(p_{10}(x)\big)^{\alpha-1} f_\alpha(x)$$
$$+ \frac{1}{4}\big(1-p_9(x)\big)\big(p_{10}(x)\big)^{\alpha}(\Delta f_\alpha)(x). \tag{4.125}$$

Using (4.125), (4.118), and (4.119), we see that

$$(\mathcal{L}f)(x) = \sum_{\alpha=0}^{\varkappa}(\alpha-3)(\alpha-k)g_\alpha(x) + p_9(x)\sum_{\alpha=0}^{\varkappa-1}\big(\alpha^2+5\alpha+4\big)\big(p_{10}(x)\big)^{\alpha} f_{\alpha+1}(x).$$

Now the condition $\mathcal{L} f = \lambda_{k,m} f$ yields

$$\varkappa = m, \tag{4.126}$$

$$f_\alpha(x) = \big(p_9(x)\big)^{m-\alpha} f_m(x) \prod_{\mu=\alpha}^{m-1} \frac{\mu^2 + 5\mu + 4}{(m-\mu)(\mu + m - k - 3)}, \quad 0 \leqslant \alpha \leqslant m-1. \tag{4.127}$$

Since $f(e_1) = 1$, we conclude from (4.117), (4.119), and (4.126) that

$$\gamma_{0,k,m} = \begin{cases} 1 & \text{if } m = 0, \\ \displaystyle\prod_{\mu=0}^{m-1} \frac{(m-\mu)(\mu + m - k - 3)}{\mu^2 + 5\mu + 4} & \text{if } 1 \leqslant m \leqslant [k/2], \end{cases}$$

$$\gamma_{\beta,k,m} = \gamma_{0,k,m} \prod_{\mu=0}^{\beta-1} \frac{(2m + 2\mu - k)(k - 2m - 2\mu - 1)}{4\mu^2 + 18\mu + 14}, \quad 1 \leqslant \beta \leqslant [k/2] - m.$$

In combination with (4.127) this implies the assertion of Lemma 4.39. □

From Lemma 4.39 and Proposition 1.1 we obtain the following:

Corollary 4.18. *For each $x \in \mathbb{S}^{15}$, the space $\mathcal{H}_6^{k,m}$ contains a unique f such that $f(x) = 1$ and $f \circ \tau = f$ for every $\tau \in \mathrm{O}_{\mathbb{C}a}(2)$ that fixes x.*

Proof of Theorem 4.9. By Corollary 4.17, $T_6(\tau)$ is the orthogonal direct sum of the representations $T_6^{k,m}(\tau)$, $k \in \mathbb{Z}_+$, $m \in \{0, \ldots, [k/2]\}$. Next, suppose that $A\colon \mathcal{H}_6^{k_1,m_1} \to \mathcal{H}_6^{k_2,m_2}$ is a linear map that commutes with the group $\mathrm{O}_{\mathbb{C}a}(2)$. Then

$$A = \begin{cases} c\,\mathrm{Id} & \text{if } (k_1, m_1) = (k_2, m_2), \\ 0 & \text{if } (k_1, m_1) \neq (k_2, m_2); \end{cases}$$

see Corollary 4.18 and the proof of Lemma 4.5. It follows that the representations $T_6^{k,m}(\tau)$, $k \in \mathbb{Z}_+$, $m \in \{0, \ldots, [k/2]\}$, are irreducible and pairwise nonequivalent. Thus, Theorem 4.9 is proved. □

Finally, we state the analog of Theorem 4.2 for the group $\mathrm{O}_{\mathbb{C}a}(2)$.

Let $\pi^{k,m}$ be the orthogonal projection of $L^2(\mathbb{S}^{15})$ onto $\mathcal{H}_6^{k,m}$. As in Sect. 4.2 (see Remark 4.1), the domain of $\pi^{k,m}$ may be extended to $L^1(\mathbb{S}^{15})$. We now have:

Theorem 4.10. *Let H be an $\mathrm{O}_{\mathbb{C}a}(2)$-invariant subspace of $\mathfrak{B}^{16}$. Then H is the $\mathfrak{B}^{16}$-closure of the algebraic sum of all $\mathcal{H}_6^{k,m}$ such that $\pi^{k,m} H \neq \{0\}$.*

The proof can be given along the lines of the proof of Theorem 4.2 by using Theorem 4.9.

Chapter 5
Non-Euclidean Analogues of Plane Waves

Having completed our study of the spherical harmonics, mainly from the point of view of functions on the sphere, we shall now show how they can be used in harmonic analysis of functions on $\mathbb{H}^n_{\mathbb{K}}$ and $\mathbb{H}^2_{\mathbb{C}a}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{Q}$). More precisely, for functions of the form $f(|x|)H(x/|x|)$, where H is a spherical harmonic, we study the following general problem: describe all eigenfunctions of the Laplacian by means of some "Poisson integral" of hyperfunctions.

On Euclidean space $\mathbb{R}^n$, we have the remarkable functions $\mathrm{e}^{\langle x,y\rangle_{\mathbb{R}}}$ with the property that any power of this function yields an eigenfunction of the Laplacian. In this case,

$$\Delta\big(\mathrm{e}^{\mathrm{i}\lambda\langle x,y\rangle_{\mathbb{R}}}\big) = -\lambda^2|y|^2\mathrm{e}^{\mathrm{i}\lambda\langle x,y\rangle_{\mathbb{R}}} \tag{5.1}$$

and

$$\int_{\mathbb{S}^{n-1}} \mathrm{e}^{\mathrm{i}\lambda\langle x,\eta\rangle_{\mathbb{R}}} H_k(\eta)\,\mathrm{d}\omega(\eta) = \mathrm{i}^k(2\pi)^{\frac{n}{2}}\frac{J_{k+\frac{n}{2}-1}(\lambda|x|)}{(\lambda|x|)^{\frac{n}{2}-1}} H_k\big(x/|x|\big) \tag{5.2}$$

for $\lambda \in \mathbb{C}$, $x \in \mathbb{R}^n\backslash\{0\}$, and $H_k \in \mathcal{H}_1^{n,k}$. In Sects. 5.1–5.4 we prove that, on the spaces $\mathbb{H}^n_{\mathbb{K}}$ and $\mathbb{H}^2_{\mathbb{C}a}$, similar statements are valid for the functions

$$x \to \frac{1-|x|^2}{|1-\langle x,y\rangle_{\mathbb{K}}|^2} \quad \big(x, y \in \mathbb{K}^n,\ |x| < 1,\ |y| = 1\big) \tag{5.3}$$

and

$$x \to \frac{1-|x|^2}{\Psi_{\mathbb{C}a}(x,y)} \quad \big(x, y \in \mathbb{R}^{16},\ |x| < 1,\ |y| = 1\big), \tag{5.4}$$

respectively. In this sense, (5.3) and (5.4) can be regarded as non-Euclidean analogues of the plane waves $\mathrm{e}^{\langle x,y\rangle_{\mathbb{R}}}$.

Such functions exist on all symmetric spaces of noncompact type and appear in the kernel of Helgason's Fourier transform. We even find such functions on compact two-point homogeneous spaces if we allow complex-valued functions and local eigenfunctions. These cases will be treated in Part II.

V.V. Volchkov, V.V. Volchkov, *Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group,* Springer Monographs in Mathematics,

5.1 The Case of $\mathbb{H}^n_{\mathbb{R}}$

Let $\lambda \in \mathbb{C}$, $\eta \in \mathbb{S}^{n-1}$. In this section we want to show that the function

$$e_{\nu,\eta}(x) = \left(\frac{\sqrt{1-|x|^2}}{1-\langle x,\eta\rangle_{\mathbb{R}}}\right)^{\nu}, \quad |x|<1,$$

where

$$\nu = \nu(\lambda) = \frac{n-1-\mathrm{i}\lambda}{2},$$

satisfies relations similar to (5.1) and (5.2) on the real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$. We shall use the notation established in Sect. 2.1.

Proposition 5.1. *The relation*

$$(Le_{\nu,\eta})(x) = -\frac{\lambda^2+(n-1)^2}{4}e_{\nu,\eta}(x), \quad x \in \mathbb{H}^n_{\mathbb{R}}, \tag{5.5}$$

holds.

Proof. Let η and x be fixed. Set $\xi = \sigma_x(\eta)$. By Proposition 2.4(ii), (v), $\xi \in \mathbb{S}^{n-1}$ and $\eta = \sigma_x(\xi)$. Therefore, formulae (2.13) and (2.14) imply that

$$e_{\nu,\eta}\big(\sigma_x(y)\big) = \left(\frac{\sqrt{1-|y|^2}}{1-\langle y,\xi\rangle_{\mathbb{R}}}\right)^{\nu}\left(\frac{1-\langle x,\xi\rangle_{\mathbb{R}}}{\sqrt{1-|x|^2}}\right)^{\nu} \tag{5.6}$$

for $y \in B^n_{\mathbb{R}}$. In view of (2.16), (2.13), and (5.6),

$$(Le_{\nu,\eta})(x) = \Delta(e_{\nu,\xi})(0)e_{\nu,\eta}(x) = -\frac{\lambda^2+(n-1)^2}{4}e_{\nu,\eta}(x),$$

as desired. □

Corollary 5.1. *Spherical functions on* $\mathbb{H}^n_{\mathbb{R}}$ *have the form*

$$\varphi_\lambda(x) = \int_{\mathbb{S}^{n-1}} e_{\nu,\eta}(x)\, \mathrm{d}\omega_{\mathrm{norm}}(\eta), \tag{5.7}$$

where $\mathrm{d}\omega_{\mathrm{norm}}$ *is a surface element of* $\mathbb{S}^{n-1}$ *normalized by*

$$\int_{\mathbb{S}^{n-1}} \mathrm{d}\omega_{\mathrm{norm}} = 1.$$

Proof. Denote by $f(x)$ the integral on the right-hand side of (5.7). Since f is radial, $f(0)=1$, and

$$(Lf)(x) = -\frac{\lambda^2+(n-1)^2}{4}f(x), \quad x \in B^n_{\mathbb{R}}, \tag{5.8}$$

we obtain our result. □

Proposition 5.2. *Let $x \in \mathbb{H}^n_{\mathbb{R}}$ and $H_k \in \mathcal{H}_1^{n,k}$. Then*

$$\int_{\mathbb{S}^{n-1}} e_{\nu,\eta}(x) H_k(\eta)\, \mathrm{d}\omega_{\mathrm{norm}}(\eta) = 2^{-k}\big((n/2)_k\big)^{-1}(\nu)_k\big(1-|x|^2\big)^{\frac{\nu}{2}} \times F\left(\frac{\nu+k}{2}, \frac{\nu+k+1}{2}; k+\frac{n}{2}; |x|^2\right) H_k(x). \tag{5.9}$$

The proof of Proposition 5.2 requires some preparation.

Let P be a polynomial in the n variables $x_1, \dots, x_n$. Denote by $P(\partial)$ the differential operator associated with P, i.e., we replace each monomial x^α by ∂^α in P. Put

$$D_k = H_k(\partial).$$

Lemma 5.1. *For $f \in C^k(B^n_{\mathbb{R}})$, we have*

$$D_k\big(|x|^{2N} f(x)\big)(0) = 0, \quad N \in \mathbb{N}, \tag{5.10}$$

$$D_k\big(\big(1-|x|^2\big)^{\lambda} f(x)\big)(0) = (D_k f)(0), \quad \lambda \in \mathbb{C}. \tag{5.11}$$

Proof. By virtue of Corollary 4.3, we can assume, without loss of generality, that

$$H_k(x) = (a_1x_1 + \dots + a_nx_n)^k,$$

where $a_1, \dots, a_n \in \mathbb{C}$ and $a_1^2 + \dots + a_n^2 = 0$. In this situation,

$$D_k\big(|x|^{2N} f(x)\big) = (a_1x_1 + \dots + a_nx_n) f_1(x) + |x|^{2N} f_2(x)$$

for some $f_1, f_2 \in C(B^n_{\mathbb{R}})$, which proves (5.10). Now (5.11) follows from (5.10). □

Lemma 5.2. *The equality*

$$D_k(e_{\nu,\eta})(0) = (\nu)_k H_k(\eta) \tag{5.12}$$

is valid.

Proof. We can write

$$e_{\nu,\eta}(x) = \big(1-|x|^2\big)^{\frac{\nu}{2}} \sum_{j=0}^{\infty} \frac{(\nu)_j}{j!} \langle x, \eta\rangle^j_{\mathbb{R}}.$$

Hence, according to Lemma 5.1,

$$D_k(e_{\nu,\eta})(0) = \sum_{j=0}^{\infty} \frac{(\nu)_j}{j!} D_k\big(\langle x, \eta\rangle^j_{\mathbb{R}}\big)(0). \tag{5.13}$$

Using Corollary 4.3, we obtain

$$D_k\big(\langle x,\eta\rangle_{\mathbb{R}}^j\big)(0)=\begin{cases}0, & j\neq k,\\ k!H_k(\eta), & j=k.\end{cases}\tag{5.14}$$

Together, (5.13) and (5.14) give (5.12). □

Proof of Proposition 5.2. Let $f(x)$ be the integral on the left-hand side of (5.9). By (4.3),

$$\begin{aligned}f(x)&=\int_{\mathbb{S}^{n-1}}e_{\nu,\eta}(x)\bigg(\int_{\mathbb{S}^{n-1}}H_k(\xi)\overline{P_\eta(\xi)}\,\mathrm{d}\omega(\xi)\bigg)\mathrm{d}\omega_{\mathrm{norm}}(\eta)\\ &=\int_{\mathbb{S}^{n-1}}H_k(\xi)\bigg(\int_{\mathbb{S}^{n-1}}e_{\nu,\eta}(x)\overline{P_\eta(\xi)}\,\mathrm{d}\omega_{\mathrm{norm}}(\eta)\bigg)\mathrm{d}\omega(\xi).\end{aligned}\tag{5.15}$$

Pick $\tau\in\mathrm{O}(n)$ for which $\tau x=|x|e_1$, where $e_1=(1,0,\dots,0)\in\mathbb{S}^{n-1}$. We shall regard $\mathrm{O}(n-1)$ as the isotropy subgroup of the point e_1 in the group $\mathrm{O}(n)$. Owing to (4.5),

$$\begin{aligned}\int_{\mathbb{S}^{n-1}}e_{\overline{\nu},\eta}(x)P_\eta(\xi)\,\mathrm{d}\omega_{\mathrm{norm}}(\eta)&=\int_{\mathbb{S}^{n-1}}e_{\overline{\nu},(\varsigma\tau)^{-1}\eta}(x)P_{(\varsigma\tau)^{-1}\eta}(\xi)\,\mathrm{d}\omega_{\mathrm{norm}}(\eta)\\ &=\int_{\mathbb{S}^{n-1}}e_{\overline{\nu},\eta}(|x|e_1)P_\eta(\varsigma\tau\xi)\,\mathrm{d}\omega_{\mathrm{norm}}(\eta)\end{aligned}$$

for every $\varsigma\in\mathrm{O}(n-1)$. Consequently,

$$\int_{\mathbb{S}^{n-1}}e_{\overline{\nu},\eta}(x)P_\eta(\xi)\,\mathrm{d}\omega_{\mathrm{norm}}(\eta)=\int_{\mathbb{S}^{n-1}}e_{\overline{\nu},\eta}(|x|e_1)\psi_\eta(\tau\xi)\,\mathrm{d}\omega_{\mathrm{norm}}(\eta),\tag{5.16}$$

where

$$\psi_\eta(\xi)=\int_{\mathrm{O}(n-1)}P_\eta(\varsigma\xi)\,\mathrm{d}\varsigma.$$

Since ψ_η is invariant with respect to $\mathrm{O}(n-1)$, Lemmas 4.3(iv) and 4.4 imply that

$$\psi_\eta(\xi)=\frac{P_\eta(e_1)P_{e_1}(\xi)}{P_{e_1}(e_1)}.$$

Now (5.15), (5.16), (4.3), and (4.5) yield

$$f(x)=g(|x|)H_k(x/|x|)\tag{5.17}$$

with

$$g(\varrho)=\frac{1}{P_{e_1}(e_1)}\int_{\mathbb{S}^{n-1}}e_{\nu,\eta}(\varrho e_1)\overline{P_\eta(e_1)}\,\mathrm{d}\omega_{\mathrm{norm}}(\eta),\quad 0\leqslant\varrho<1.$$

Next, by Proposition 5.1, f satisfies (5.8). Therefore, from (5.17) and (2.9) we derive the equation

$$\big(1-\varrho^2\big)^2 g''(\varrho)+\frac{1-\varrho^2}{\varrho}\big(n-1-2\varrho^2\big)g'(\varrho)-k(n+k-2)\frac{1-\varrho^2}{\varrho}g(\varrho)$$
$$=-\frac{(\lambda^2+(n-1)^2)}{4}g(\varrho).$$

Because the function g is smooth at the origin, we get

$$g(\varrho)=c\varrho^k\big(1-\varrho^2\big)^{\frac{\nu}{2}}F\left(\frac{\nu+k}{2},\frac{\nu+k+1}{2};k+\frac{n}{2};\varrho^2\right)$$

for some $c\in\mathbb{C}$ (see Erdélyi (ed.) [73], Chap. 2, 2.1 (2)). It remains to calculate the constant c. Set

$$h_k(x)=\sqrt{\frac{(n/2)_k}{k!}}(x_1-\mathrm{i}x_2)^k.$$

The polynomial h_k belongs to $\mathcal{H}_1^{n,k}$, and

$$\int_{\mathbb{S}^{n-1}}|h_k(\eta)|^2\,\mathrm{d}\omega_{\mathrm{norm}}(\eta)=1. \tag{5.18}$$

By the above,

$$\int_{\mathbb{S}^{n-1}}e_{\nu,\eta}(x)h_k(\eta)\,\mathrm{d}\omega_{\mathrm{norm}}(\eta)=g(|x|)h_k(x/|x|).$$

Hence,

$$\int_{\mathbb{S}^{n-1}}\big(\overline{h}_k(\partial)e_{\nu,\eta}\big)(0)h_k(\eta)\,\mathrm{d}\omega_{\mathrm{norm}}(\eta)=\overline{h}_k(\partial)\big(g(|x|)h_k(x/|x|)\big)\big|_{x=0}.$$

Using (5.10)–(5.12) and (5.18), we find

$$c=2^{-k}\big((n/2)_k\big)^{-1}(\nu)_k.$$

Thus, Proposition 5.2 is proved. □

Corollary 5.2. *For $h\in\mathcal{H}_1^{n,k}$ and $H\in\mathcal{H}_1^{n,l}$, we have*

$$\big(H(\partial)h\big)(0)=2^k(n/2)_k\int_{\mathbb{S}^{n-1}}h(\eta)H(\eta)\,\mathrm{d}\omega_{\mathrm{norm}}(\eta). \tag{5.19}$$

Proof. By (5.9)–(5.11),

$$\int_{\mathbb{S}^{n-1}}\big(H(\partial)e_{\nu,\eta}\big)(0)h(\eta)\,\mathrm{d}\omega_{\mathrm{norm}}(\eta)=2^{-k}\big((n/2)_k\big)^{-1}(\nu)_k\big(H(\partial)h\big)(0).$$

Taking (5.12) and Lemma 4.2 into account, we arrive at (5.19). □

To close this section we consider an analogue of (5.5) and (5.9) for the ball $B^n_{\mathbb{R}}$ with the Poincaré metric (2.21).

Proposition 5.3. *The following relations hold:*

$$L\left(\left(\frac{1-|x|^2}{|x-\eta|^2}\right)^{\nu}\right)=-\frac{(\lambda^2+(n-1)^2)}{4}\left(\frac{1-|x|^2}{|x-\eta|^2}\right)^{\nu}, \tag{5.20}$$

$$\int_{\mathbb{S}^{n-1}}\left(\frac{1-|x|^2}{|x-\eta|^2}\right)^{\nu}H_k(\eta)\,\mathrm{d}\omega_{\mathrm{norm}}(\eta)$$
$$=\big((n/2)_k\big)^{-1}(\nu)_k\big(1-|x|^2\big)^{\nu}F\left(\nu+k,\nu+1-\frac{n}{2};k+\frac{n}{2};|x|^2\right)H_k(x). \tag{5.21}$$

Proof. Let

$$f(x)=\int_{\mathbb{S}^{n-1}}\left(\frac{1-|x|^2}{|x-\eta|^2}\right)^{\nu}H_k(\eta)\,\mathrm{d}\omega_{\mathrm{norm}}(\eta).$$

Then

$$f\left(\frac{x}{1+\sqrt{1-|x|^2}}\right)=\int_{\mathbb{S}^{n-1}}e_{\nu,\eta}(x)H_k(\eta)\,\mathrm{d}\omega_{\mathrm{norm}}(\eta).$$

By Proposition 5.2,

$$f\left(\frac{x}{1+\sqrt{1-|x|^2}}\right)$$
$$=2^{-k}\big((n/2)_k\big)^{-1}(\nu)_k\big(1-|x|^2\big)^{\frac{\nu}{2}}F\left(\frac{\nu+k}{2},\frac{\nu+k+1}{2};k+\frac{n}{2};|x|^2\right)H_k(x).$$

Applying the formula

$$F\left(a,a+\frac{1}{2};b;t\right)=2^{2a}\big(1+\sqrt{1-t}\big)^{-2a}F\left(2a,2a-b+1;b;\frac{1-\sqrt{1-t}}{1+\sqrt{1-t}}\right)$$

(see Erdélyi (ed.) [73], Chap. 2, 2.1 (26)), we deduce (5.21). Relation (5.21) and Corollary 2.3 (ii) show that f satisfies (5.8). This implies (5.20), since the linear span of $\{\mathcal{H}_1^{n,k}, k\in\mathbb{Z}_+\}$ is dense in $C(\mathbb{S}^{n-1})$. □

Corollary 5.3. *For the ball $B^n_{\mathbb{R}}$ with metric* (2.21)*, spherical functions are of the form*

$$\varphi_\lambda(x)=\int_{\mathbb{S}^{n-1}}\left(\frac{1-|x|^2}{|x-\eta|^2}\right)^{\nu}\mathrm{d}\omega_{\mathrm{norm}}(\eta).$$

Proof. The statement is clear from (5.20). □

5.2 The Case of $\mathbb{H}^n_{\mathbb{C}}$

Let $B^n_{\mathbb{C}} = \{z \in \mathbb{C}^n : |z| < 1\}$. For $\lambda \in \mathbb{C}$ and $\eta \in \mathbb{S}^{2n-1}$, we set

$$e_{\nu,\eta}(z) = \left(\frac{1-|z|^2}{|1-\langle z,\eta\rangle_{\mathbb{C}}|^2}\right)^{\nu}, \quad z \in B^n_{\mathbb{C}},$$

where

$$\nu = \nu(\lambda) = \frac{n - \mathrm{i}\lambda}{2}.$$

Denote by L the Laplace–Beltrami operator on the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$. Also let $\mathcal{H}_2^{n,p,q}$, $p, q \in \mathbb{Z}_+$, be the space of homogeneous harmonic polynomials on $\mathbb{C}^n$ of bidegree (p,q) (see Sect. 4.2).

The purpose of this section is to prove the following result.

Proposition 5.4. *Suppose that $z \in \mathbb{H}^n_{\mathbb{C}}$ and $H_{p,q} \in \mathcal{H}_2^{n,p,q}$. Then*

$$(Le_{\nu,\eta})(z) = -\big(\lambda^2 + n^2\big)e_{\nu,\eta}(z) \tag{5.22}$$

and

$$\int_{\mathbb{S}^{2n-1}} e_{\nu,\eta}(z)H_{p,q}(\eta)\,\mathrm{d}\omega_{\mathrm{norm}}(\eta)$$
$$= \big((n)_{p+q}\big)^{-1}(\nu)_p(\nu)_q\big(1-|z|^2\big)^{\nu}F\big(\nu+p, \nu+q; n+p+q; |z|^2\big)H_{p,q}(z). \tag{5.23}$$

We require one auxiliary assertion.

Lemma 5.3. *Viewing elements of $\mathcal{H}_2^{n,p,q}$ as polynomials on $\mathbb{R}^{2n}$, we have*

$$\big(H_{p,q}(\partial)e_{\nu,\eta}\big)(0) = 2^{p+q}(\nu)_p(\nu)_q H_{p,q}(\eta).$$

Proof. Because of Corollary 4.5, we can assume that

$$H_{p,q}(z) = (\alpha_1 z_1 + \cdots + \alpha_n z_n)^p(\beta_1\overline{z}_1 + \cdots + \beta_n\overline{z}_n)^q,$$

where the numbers $\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n$ satisfy

$$\alpha_1\beta_1 + \cdots + \alpha_n\beta_n = 0.$$

In this case,

$$H_{p,q}(\partial) = 2^{p+q}\left(\alpha_1\frac{\partial}{\partial\overline{z}_1} + \cdots + \alpha_n\frac{\partial}{\partial\overline{z}_n}\right)^p\left(\beta_1\frac{\partial}{\partial z_1} + \cdots + \beta_n\frac{\partial}{\partial z_n}\right)^q.$$

Then by (5.11)

$$\begin{aligned}\big(H_{p,q}(\partial)e_{\nu,\eta}\big)(0) &= \big(H_{p,q}(\partial)\big(|1-\langle z,\eta\rangle_{\mathbb{C}}|^{-2\nu}\big)\big)(0)\\ &= 2^{p+q}\left(\alpha_1\frac{\partial}{\partial\overline{z}_1}+\cdots+\alpha_n\frac{\partial}{\partial\overline{z}_n}\right)^p\big((1-\langle\eta,z\rangle_{\mathbb{C}})^{-\nu}\big)(0)\\ &\quad\times\left(\beta_1\frac{\partial}{\partial z_1}+\cdots+\beta_n\frac{\partial}{\partial z_n}\right)\big((1-\langle z,\eta\rangle_{\mathbb{C}})^{-\nu}\big)(0)\\ &= 2^{p+q}(\nu)_p(\nu)_q H_{p,q}(\eta),\end{aligned}$$

and Lemma 5.3 is proved. □

Proof of Proposition 5.4. Equality (5.22) follows from (2.31)–(2.33) by a direct calculation. Next, in the same way as in the proof of Proposition 5.2 (see Lemmas 4.8, 4.9 and (4.21)), we conclude that

$$\begin{aligned}&\int_{\mathbb{S}^{2n-1}} e_{\nu,\eta}(z)H_{p,q}(\eta)\,\mathrm{d}\omega_{\mathrm{norm}}(\eta)\\ &\quad= c\big(1-|z|^2\big)^{\nu}F\big(\nu+p,\nu+q;n+p+q;|z|^2\big)H_{p,q}(z),\end{aligned}$$

where c is a complex constant. Now by (5.10), (5.11), and (5.19)

$$\begin{aligned}&\int_{\mathbb{S}^{2n-1}}\big(\overline{H}_{p,q}(\partial)e_{\nu,\eta}\big)(0)H_{p,q}(\eta)\,\mathrm{d}\omega_{\mathrm{norm}}(\eta)\\ &\quad= c2^{p+q}(n)_{p+q}\int_{\mathbb{S}^{2n-1}}|H_{p,q}(\eta)|^2\,\mathrm{d}\omega_{\mathrm{norm}}(\eta).\end{aligned}$$

Applying Lemma 5.3, we derive

$$c=\big((n)_{p+q}\big)^{-1}(\nu)_p(\nu)_q,$$

which proves (5.23). □

Corollary 5.4. *Let* $z\in B^n_{\mathbb{C}}$ *and* $H\in\mathcal{H}_3^{n,k,m}$. *Then*

$$\begin{aligned}&\int_{\mathbb{S}^{2n-1}} e_{\nu,\eta}(z)H(\eta)\,\mathrm{d}\omega_{\mathrm{norm}}(\eta)\\ &\quad=\big((n)_k\big)^{-1}(\nu)_{k-m}(\nu)_m\big(1-|z|^2\big)^{\nu}F\big(\nu+k-m,\nu+m;n+k;|z|^2\big)H(z).\end{aligned} \tag{5.24}$$

Proof. Combine (5.23) with Lemma 4.11. □

Corollary 5.5. *Spherical functions on* $\mathbb{H}^n_{\mathbb{C}}$ *are of the form*

$$\varphi_\lambda(z)=\int_{\mathbb{S}^{2n-1}} e_{\nu,\eta}(z)\,\mathrm{d}\omega_{\mathrm{norm}}(\eta).$$

Proof. This follows from relation (5.22). □

5.3 The Case of $\mathbb{H}^n_{\mathbb{Q}}$

In this section we shall obtain the analogue of Propositions 5.1–5.4 for the quaternionic hyperbolic space $\mathbb{H}^n_{\mathbb{Q}}$.

For $\lambda \in \mathbb{C}$ and $\eta \in \mathbb{S}^{4n-1}$, we put

$$e_{\nu,\eta}(z) = \left(\frac{1-|z|^2}{|1-\langle z,\eta\rangle_{\mathbb{Q}}|^2}\right)^{\nu}, \quad z \in \mathbb{H}^n_{\mathbb{Q}},$$

where

$$\nu = \nu(\lambda) = \frac{2n+1-\mathrm{i}\lambda}{2}.$$

With our notation in Sect. 4.5, we have the following:

Proposition 5.5. *Let $z \in \mathbb{H}^n_{\mathbb{Q}}$ and $H \in \mathcal{H}_4^{n,p,q,m}$. Then*

$$(Le_{\nu,\eta})(z) = -\big(\lambda^2 + (2n+1)^2\big)e_{\nu,\eta}(z) \tag{5.25}$$

and

$$\begin{aligned}\int_{\mathbb{S}^{4n-1}} & e_{\nu,\eta}(z)H(\eta)\,\mathrm{d}\omega_{\mathrm{norm}}(\eta) \\ &= \big((2n)_{p+q}\big)^{-1}(\nu)_{p+q-m}(\nu-1)_m\big(1-|z|^2\big)^{\nu} \\ &\quad \times F\big(\nu+p+q-m, \nu+m-1; 2n+p+q; |z|^2\big)H(z).\end{aligned} \tag{5.26}$$

Proof. By (2.44)–(2.46),

$$(Le_{\nu,\eta})(z) = \Delta(e_{\nu,\eta}\circ\sigma_z)(0) = \Delta(e_{\nu,\sigma_z(\eta)})(0)e_{\nu,\eta}(z). \tag{5.27}$$

A simple calculation shows that

$$\Delta(e_{\nu,\sigma_z(\eta)})(0) = -\big(\lambda^2 + (2n+1)^2\big). \tag{5.28}$$

Combining (5.28) with (5.27), we deduce (5.25). To prove (5.26) a couple of lemmas will be needed.

Lemma 5.4. *The relation*

$$\big(H(\partial)e_{\nu,\eta}\big)(0) = 2^{p+q}(\nu)_{p+q-m}(\nu-1)_m H(\eta) \tag{5.29}$$

holds.

Proof. We define $h(z) = z_1^{p+q-2m}(z_1\overline{z}_{n+2} - z_2\overline{z}_{n+1})^m$. Thanks to Lemmas 4.12 and 4.31,

$$h \in \mathcal{H}_4^{n,p+q-m,m,m} \subset \mathcal{H}_5^{n,p+q,m}.$$

Let us verify that

$$\big(h(\partial)\big(f^{-\nu}\big)\big)(0) = 2^k(\nu)_{k-m}(\nu-1)_m h(\eta), \tag{5.30}$$

where $k = p+q$ and $f(z) = |1-\langle z,\eta\rangle_{\mathbb{Q}}|^2$. One has

$$\begin{aligned}
h(\partial) &= 2^k\left(\frac{\partial^2}{\partial\overline{z}_1\partial z_{n+2}} - \frac{\partial^2}{\partial\overline{z}_2\partial z_{n+1}}\right)^m\left(\frac{\partial}{\partial\overline{z}_1}\right)^{k-2m}\\
&= 2^k\sum_{\alpha=0}^{m}(-1)^{m+\alpha}\binom{m}{\alpha}\left(\frac{\partial}{\partial z_{n+1}}\right)^{m-\alpha}\left(\frac{\partial}{\partial z_{n+2}}\right)^{\alpha}\left(\frac{\partial}{\partial\overline{z}_2}\right)^{m-\alpha}\left(\frac{\partial}{\partial\overline{z}_1}\right)^{\alpha+k-2m}.
\end{aligned}$$

We write

$$f_1 = \frac{\partial f}{\partial\overline{z}_1},\qquad f_2 = \frac{\partial f}{\partial\overline{z}_2},\qquad g_1 = \frac{\partial f}{\partial z_{n+1}},\qquad g_2 = \frac{\partial f}{\partial z_{n+2}}.$$

Since $f(z) = 1+|\langle z,\eta\rangle_{\mathbb{C}}|^2+|[z,\eta]_{\mathbb{C}}|^2-2\operatorname{Re}\langle z,\eta\rangle_{\mathbb{Q}}$, by successive differentiation we find

$$\left(\frac{\partial}{\partial\overline{z}_2}\right)^{m-\alpha}\left(\frac{\partial}{\partial\overline{z}_1}\right)^{\alpha+k-2m}\big(f^{-\nu}\big) = (-1)^{k-m}(\nu)_{k-m}f_1^{\alpha+k-2m}f_2^{m-\alpha}f^{m-k-\nu},$$

$$\begin{aligned}
&\left(\frac{\partial}{\partial z_{n+2}}\right)^{\alpha}\big(f_1^{\alpha+k-2m}f_2^{m-\alpha}f^{m-k-\nu}\big)\\
&\quad= f_2^{m-\alpha}\left(\frac{\partial}{\partial z_{n+2}}\right)^{\alpha}\big(f_1^{\alpha+k-2m}f^{m-k-\nu}\big)\\
&\quad= \sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}f_2^{m-\alpha}\left(\frac{\partial}{\partial z_{n+2}}\right)^{\alpha-\beta}\big(f_1^{\alpha+k-2m}\big)\left(\frac{\partial}{\partial z_{n+2}}\right)^{\beta}\big(f^{m-k-\nu}\big)\\
&\quad= \sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}\frac{(\alpha+k-2m)!}{(\beta+k-2m)!}(\nu+k-m)_\beta(\overline{\eta}_{n+2}\eta_1-\overline{\eta}_{n+1}\eta_2)^{\alpha-\beta}\\
&\qquad\times f_2^{m-\alpha}f_1^{\beta+k-2m}(-g_2)^\beta f^{m-k-\nu-\beta},
\end{aligned}$$

$$\begin{aligned}
&\left(\frac{\partial}{\partial z_{n+1}}\right)^{m-\alpha}\big(f_2^{m-\alpha}f_1^{\beta+k-2m}(-g_2)^\beta f^{m-k-\nu-\beta}\big)\\
&\quad= f_1^{\beta+k-2m}(-g_2)^\beta\left(\frac{\partial}{\partial z_{n+1}}\right)^{m-\alpha}\big(f_2^{m-\alpha}f^{m-k-\nu-\beta}\big)\\
&\quad= \sum_{\gamma=0}^{m-\alpha}\binom{m-\alpha}{\gamma}\frac{(m-\alpha)!}{\gamma!}(\nu+k+\beta-m)_\gamma(\overline{\eta}_{n+1}\eta_2-\eta_1\overline{\eta}_{n+2})^{m-\alpha-\gamma}\\
&\qquad\times f_1^{\beta+k-2m}(-g_2)^\beta(-g_1f_2)^\gamma f^{m-k-\nu-\beta-\gamma}.
\end{aligned}$$

Consequently,

$$\begin{aligned}&\big(h(\partial)\big(f^{-\nu}\big)\big)(0)\\&=2^k(\nu)_{k-m}\eta_1^{k-2m}\sum_{\alpha=0}^{m}\sum_{\beta=0}^{\alpha}\sum_{\gamma=0}^{m-\alpha}\binom{m}{\alpha}\binom{\alpha}{\beta}\binom{m-\alpha}{\gamma}\frac{(\alpha+k-2m)!}{(\beta+k-2m)!}\frac{(m-\alpha)!}{\gamma!}\\&\quad\times(\nu+k-m)_{\beta+\gamma}(\overline{\eta}_{n+1}\eta_2-\eta_1\overline{\eta}_{n+2})^{m-\beta-\gamma}(\eta_1\overline{\eta}_{n+2})^{\beta}(-\eta_2\overline{\eta}_{n+1})^{\gamma}.\end{aligned}$$

Equivalently,

$$\begin{aligned}&\big(h(\partial)\big(f^{-\nu}\big)\big)(0)\\&=2^k(\nu)_{k-m}\eta_1^{k-2m}m!\sum_{\alpha=0}^{m}\sum_{\beta=0}^{\alpha}\sum_{\gamma=0}^{m-\alpha}\binom{\alpha+k-2m}{\beta+k-2m}\binom{m-\alpha}{\gamma}(\nu+k-m)_{\beta+\gamma}\\&\quad\times(\overline{\eta}_{n+1}\eta_2-\eta_1\overline{\eta}_{n+2})^{m-\beta-\gamma}\frac{(\eta_1\overline{\eta}_{n+2})^{\beta}}{\beta!}\frac{(-\eta_2\overline{\eta}_{n+1})^{\gamma}}{\gamma!}\\&=2^k(\nu)_{k-m}\eta_1^{k-2m}m!\sum_{\delta=0}^{m}\sum_{\beta+\gamma=\delta}\left(\sum_{\alpha=\beta}^{m-\gamma}\binom{\alpha+k-2m}{\beta+k-2m}\binom{m-\alpha}{\gamma}\right)(\nu+k-m)_{\delta}\\&\quad\times(\overline{\eta}_{n+1}\eta_2-\eta_1\overline{\eta}_{n+2})^{m-\beta}\frac{(\eta_1\overline{\eta}_{n+2})^{\beta}}{\beta!}\frac{(-\eta_2\overline{\eta}_{n+1})^{\gamma}}{\gamma!}.\end{aligned}$$

Next, we require the identity

$$\sum_{j=0}^{N}\binom{N}{j}(a)_j(b)_{N-j}=(a+b)_N, \tag{5.31}$$

which is proved by induction on N using the relations

$$\binom{N+1}{j}=\binom{N}{j}+\binom{N}{j-1},\qquad (a)_{j+1}=(a)_j(a+j). \tag{5.32}$$

Identity (5.31) gives

$$\begin{aligned}\sum_{\alpha=\beta}^{m-\gamma}\binom{\alpha+k-2m}{\beta+k-2m}\binom{m-\alpha}{\gamma}&=\sum_{j=0}^{m-\beta-\gamma}\binom{\beta+k-2m+j}{\beta+k-2m}\binom{m-\beta-j}{\gamma}\\&=\sum_{j=0}^{m-\beta-\gamma}\frac{(\beta+k-2m+1)_j}{j!}\frac{(\gamma+1)_{m-\beta-\gamma-j}}{(m-\beta-\gamma-j)!}\\&=\frac{(\beta+\gamma+k-2m+2)_{m-\beta-\gamma}}{(m-\beta-\gamma)!}.\end{aligned}$$

Therefore,

$$\big(h(\partial)\big(f^{-\nu}\big)\big)(0) = 2^k(\nu)_{k-m}\eta_1^{k-2m}m!\sum_{\delta=0}^{m}(\nu+k-m)_\delta\frac{(\delta+k-2m+2)_{m-\delta}}{(m-\delta)!}$$

$$\times(\overline{\eta}_{n+1}\eta_2-\eta_1\overline{\eta}_{n+2})^{m-\delta}\sum_{\beta+\gamma=\delta}\frac{(\eta_1\overline{\eta}_{n+2})^\beta}{\beta!}\frac{(-\eta_2\overline{\eta}_{n+1})^\gamma}{\gamma!}$$

$$= 2^k(\nu)_{k-m}\eta_1^{k-2m}m!\sum_{\delta=0}^{m}(-1)^{m+\delta}\frac{(\nu+k-m)_\delta}{\delta!}$$

$$\times\frac{(\delta+k-2m+2)_{m-\delta}}{(m-\delta)!}(\eta_1\overline{\eta}_{n+2}-\eta_2\overline{\eta}_{n+1})^m.$$

Taking into account that $(\delta+k-2m+2)_{m-\delta} = (-1)^{m-\delta}(m-k-1)_{m-\delta}$ and applying (5.31), we obtain (5.30). Hence, by Lemma 5.1 and Theorem 4.8,

$$\big(P(\partial)e_{\nu,\eta}\big)(0) = 2^k(\nu)_{k-m}(\nu-1)_m P(\eta)$$

for every $P\in\mathcal{H}_5^{n,k,m}$. Because $\mathcal{H}_4^{n,p,q,m}\subset\mathcal{H}_5^{n,k,m}$ (see Lemma 4.31), we arrive at the desired assertion. □

Let

$$H_{p,q,m}(z) = \sum_{l=0}^{m}U_l(z)V_l(z) \tag{5.33}$$

be the polynomial introduced in Sect. 4.4.

Lemma 5.5. *Suppose that $j\in\mathbb{Z}_+$ and $j\neq q-m$. Then*

$$\int_{\mathbb{S}^{4n-1}} e_{\nu,\eta}(|z|e_1)D^j(H_{p,q,m})(\eta)\,\mathrm{d}\omega(\eta) = 0, \tag{5.34}$$

where $e_1 = (1,0,\dots,0)\in\mathbb{S}^{4n-1}$ and

$$D = z_{n+1}\frac{\partial}{\partial z_1} - \overline{z}_1\frac{\partial}{\partial\overline{z}_{n+1}}.$$

Proof. It is clear that

$$\int_{\mathbb{S}^{4n-1}} e_{\nu,\eta}(|z|e_1)D^j(H_{p,q,m})(\eta)\,\mathrm{d}\omega(\eta)$$

$$= \frac{1}{2\pi}\int_{\mathbb{S}^{4n-1}}\mathrm{d}\omega(\eta)\int_{-\pi}^{\pi} e_{\nu,\tau_\theta(\eta)}(|z|e_1)D^j(H_{p,q,m})\big(\tau_\theta(\eta)\big)\,\mathrm{d}\theta, \tag{5.35}$$

where

$$\tau_\theta(\eta) = \big(\eta_1,\dots,\eta_n,\mathrm{e}^{\mathrm{i}\theta}\eta_{n+1},\eta_{n+2},\dots,\eta_{2n}\big).$$

The equality

$$e_{\nu,\eta}(|z|e_1) = \left(\frac{1-|z|^2}{1-2|z|\operatorname{Re}\eta_1 + |z|^2(|\eta_1|^2+|\eta_{n+1}|^2)}\right)^{\nu}$$

yields

$$e_{\nu,\tau_\theta(\eta)}(|z|e_1) = e_{\nu,\eta}(|z|e_1). \tag{5.36}$$

Next, (5.33) and (4.38) imply

$$D^j(H_{p,q,m})\big(\tau_\theta(\eta)\big) = \mathrm{e}^{\mathrm{i}\theta(j+m-q)} \sum_{l=0}^{m} U_l(\eta) D^j(V_l)(\eta). \tag{5.37}$$

Substituting (5.36) and (5.37) into (5.35), we derive (5.34). □

Passing to the proof of (5.26), consider the function

$$\psi_\eta(\xi) = \int_{\mathrm{Sp}(n-1)} P_\eta(\tau\xi)\,\mathrm{d}\tau, \quad \xi \in \mathbb{S}^{4n-1}, \tag{5.38}$$

where P_η is given by (4.63). According to Lemma 4.19,

$$\psi_\eta(\xi) = \sum_{j=0}^{p+q-2m} c_j(\eta) D^j(H_{p,q,m})(\xi)$$

with

$$c_j(\eta) = \int_{\mathbb{S}^{4n-1}} \psi_\eta(\xi)\overline{D^j(H_{p,q,m})(\xi)}\,\mathrm{d}\omega(\xi) \times \left(\int_{\mathbb{S}^{4n-1}} \left|D^j(H_{p,q,m})(\xi)\right|^2 \mathrm{d}\omega(\xi)\right)^{-1}. \tag{5.39}$$

In view of (5.38), (5.39), and (4.63),

$$c_j(\eta) = \left(\int_{\mathbb{S}^{4n-1}} \left|D^j(H_{p,q,m})(\xi)\right|^2 \mathrm{d}\omega(\xi)\right)^{-1} \overline{D^j(H_{p,q,m})(\eta)}.$$

Now using Lemmas 5.5, 4.20, and 4.21 and repeating the arguments in the proof of Proposition 5.2, we see that

$$\int_{\mathbb{S}^{4n-1}} e_{\nu,\eta}(z)H(\eta)\,\mathrm{d}\omega_{\mathrm{norm}}(\eta) = c\big(1-|z|^2\big)^{\nu} F\big(\nu+p+q-m, \nu+m-1; 2n+p+q; |z|^2\big)H(z)$$

for some $c \in \mathbb{C}$. Then by (5.10), (5.11), (5.19), and (5.29),

$$c = \big((2n)_{p+q}\big)^{-1}(\nu)_{p+q-m}(\nu-1)_m,$$

whence (5.26) follows. Thereby, the proposition is established. □

From (5.26) and Lemma 4.31 we obtain

Corollary 5.6. *Let* $z \in \mathbb{H}^n_{\mathbb{Q}}$ *and* $H \in \mathcal{H}_5^{n,k,m}$. *Then*

$$\int_{\mathbb{S}^{4n-1}} e_{\nu,\eta}(z)H(\eta)\,\mathrm{d}\omega_{\mathrm{norm}}(\eta) = \big((2n)_k\big)^{-1}(\nu)_{k-m}(\nu-1)_m\big(1-|z|^2\big)^{\nu} \times F\big(\nu+k-m,\nu+m-1;2n+k;|z|^2\big)H(z). \tag{5.40}$$

Furthermore, we have the following:

Corollary 5.7. *Spherical functions on* $\mathbb{H}^n_{\mathbb{Q}}$ *have the form*

$$\varphi_\lambda(z) = \int_{\mathbb{S}^{4n-1}} e_{\nu,\eta}(z)\,\mathrm{d}\omega_{\mathrm{norm}}(\eta).$$

The proof of this statement is the same as that of Corollary 5.1 (see (5.25)).

5.4 The Case of $\mathbb{H}^2_{\mathbb{C}a}$

In the rest of this chapter we retain the notation established in Sects. 2.4 and 4.6. In addition, for $\lambda \in \mathbb{C}$ and $\eta \in \mathbb{S}^{15}$, we set (see (2.65))

$$e_{\nu,\eta}(x) = \left(\frac{1-|x|^2}{\Psi_{\mathbb{C}a}(x,\eta)}\right)^{\nu}, \quad x \in B^{16}_{\mathbb{R}},$$

where

$$\nu = \nu(\lambda) = \frac{11 - \mathrm{i}\lambda}{2}.$$

In turns out that $e_{\nu,\eta}$ is an analogue of the exponential function for the Cayley hyperbolic plane $\mathbb{H}^2_{\mathbb{C}a}$. The following result is valid.

Proposition 5.6. *Let* $x \in \mathbb{H}^2_{\mathbb{C}a}$ *and* $H \in \mathcal{H}_6^{k,m}$. *Then*

$$(Le_{\nu,\eta})(x) = -\big(\lambda^2 + 11^2\big)e_{\nu,\eta}(x) \tag{5.41}$$

and

$$\int_{\mathbb{S}^{15}} e_{\nu,\eta}(x)H(\eta)\,\mathrm{d}\omega_{\mathrm{norm}}(\eta) = \big((8)_k\big)^{-1}(\nu)_{k-m}(\nu-3)_m\big(1-|x|^2\big)^{\nu} \times F\big(\nu+k-m,\nu+m-3;8+k;|x|^2\big)H(x). \tag{5.42}$$

Proof. Equality (5.41) follows from (2.73) by (2.66) and (2.67). Now using (4.110) and Lemma 4.39, we infer that

$$\int_{\mathbb{S}^{15}} e_{\nu,\eta}(x)H(\eta)\,\mathrm{d}\omega_{\mathrm{norm}}(\eta)$$
$$= c_1\big(1-|x|^2\big)^{\nu} F\big(\nu+k-m, \nu+m-3; 8+k; |x|^2\big)H(x), \tag{5.43}$$

where the constant c_1 does not depend on $H \in \mathcal{H}_6^{k,m}$ (see the proof of Proposition 5.2). In order to calculate c_1, we require an analog of relation (5.29).

Lemma 5.6. *The equality*

$$\big(H(\partial)e_{\nu,\eta}\big)(0) = 2^k(\nu)_{k-m}(\nu-3)_m H(\eta)$$

holds.

Proof. By (5.43), Lemma 5.1 and Corollary 5.2,

$$\big(H(\partial)e_{\nu,\eta}\big)(0) = c_2 H(\eta), \tag{5.44}$$

where $c_2 \in \mathbb{C}$ does not depend on $\eta \in \mathbb{S}^{15}$ and $H \in \mathcal{H}_6^{k,m}$. Let p_9 and $P_{k,m}$ be the polynomials defined in Sects. 1.1 and 4.6. Then according to (5.44) and (5.11),

$$c_2 = \big(P_{k,m}(\partial)f\big)(0)$$

with $f(x) = (1-2x_1+p_9(x))^{-\nu}$. By the definition of $P_{k,m}$,

$$c_2 = \sum_{\beta=0}^{[k/2]-m}(-1)^{\beta}\frac{15(k-2m)!}{(k-2m-2\beta)!(2\beta+1)!(2\beta+3)(2\beta+5)}$$
$$\times\left(\left(\frac{\partial}{\partial x_1}\right)^{k-2m-2\beta}\left(p_9(\partial)-\left(\frac{\partial}{\partial x_1}\right)^2\right)^{\beta}\big(p_9(\partial)\big)^m f\right)(0)$$
$$= \sum_{\beta=0}^{[k/2]-m}\sum_{i=\beta}^{m+\beta}(-1)^{\beta}\frac{15(k-2m)!}{(k-2m-2\beta)!(2\beta+1)!(2\beta+3)(2\beta+5)}\binom{m}{m+\beta-i}$$
$$\times\left(\left(\frac{\partial}{\partial x_1}\right)^{k-2i}\left(p_9(\partial)-\left(\frac{\partial}{\partial x_1}\right)^2\right)^{i} f\right)(0).$$

For $|2x_1-p_9(x)|<1$, we have

$$f(x) = \sum_{j=0}^{\infty}\sum_{p=0}^{j}\sum_{q=0}^{p}(-1)^{j-q}2^q\binom{j}{p}\binom{p}{q}\frac{(\nu)_j}{j!}x_1^{2p-q}\big(p_9(x)-x_1^2\big)^{j-p}.$$

Since

$$\left(\frac{\partial}{\partial x_1}\right)^i \left(x_1^j\right)(0) = i!\delta_{i,j}$$

and

$$\left(p_9(\partial) - \left(\frac{\partial}{\partial x_1}\right)^2\right)^i \left(p_9(x) - x_1^2\right)^j (0) = \frac{(2i+1)!(2i+3)(2i+5)}{15}\delta_{i,j},$$

we deduce

$$c_2 = \sum_{\beta=0}^{[k/2]-m} \sum_{i=\beta}^{m+\beta} \sum_{q=[(k+1)/2]-i}^{k-2i} (-1)^{k+\beta-i+q} \frac{(k-2m)!(k-2i)!}{(k-2m-2\beta)!(i+q)!} 2^{2i+2q-k} (\nu)_{i+q}$$
$$\times \frac{(2i+1)!(2i+3)(2i+5)}{(2\beta+1)!(2\beta+3)(2\beta+5)} \binom{m}{m+\beta-i} \binom{i+q}{q} \binom{q}{2q+2i-k}.$$

The computations below are based on the simple relation

$$(a+1-N)_N = (-1)^N(-a)_N \tag{5.45}$$

and the following combinatorial identities:

$$\sum_{j=0}^{N} 4^{N-j} \binom{N}{j} \frac{(a)_{N-j}(2b)_{2j}}{(b)_j} = \frac{(2a+2b+1)_{2N}}{(a+b+1)_N}, \tag{5.46}$$

$$\sum_{j=0}^{N} (-1)^j \binom{N}{j} \frac{(2a)_{2j}(b)_j}{(a)_j(2b)_{2j}} = 4^N \frac{(b-a)_N(b+1)_N}{(2b+1)_{2N}}, \tag{5.47}$$

$$\sum_{j=0}^{N} \binom{N}{j} \frac{(2a)_{2N-2j}(2b)_{2j}}{(a)_{N-j}(b)_j} = 4^N(a+b+1)_N. \tag{5.48}$$

Identities (5.46)–(5.48) are obtained by induction on N using (5.32). Owing to (5.45) and (5.46),

$$\sum_{q=[(k+1)/2]-i}^{k-2i} (-1)^q 4^q \binom{i+q}{q} \binom{q}{2q+2i-k} \frac{(\nu)_{i+q}}{(i+q)!}$$
$$= \frac{(-1)^k 4^{[(k+1)/2]-i}}{i!(k-2i)!} \frac{(\nu)_{[(k+1)/2]}(\nu)_i(2\nu)_{2[k/2]}}{(\nu)_{[k/2]}(2\nu)_{2i}}.$$

Hence,

$$c_2 = 2^{[(k+1)/2]-[k/2]} \frac{(\nu)_{[(k+1)/2]}(2\nu)_{2[k/2]}}{(\nu)_{[k/2]}} \sum_{\beta=0}^{[k/2]-m} \sum_{i=\beta}^{m+\beta} (-1)^{\beta-i} \frac{(\nu)_i}{(2\nu)_{2i}}$$

$$\times \frac{(2i+1)!(2i+3)(2i+5)}{(2\beta+1)!(2\beta+3)(2\beta+5)} \frac{m!(k-2m)!}{(k-2m-2\beta)!(m+\beta-i)!(i-\beta)!i!}.$$

Next, (5.47) implies

$$\sum_{i=\beta}^{m+\beta}(-1)^{\beta-i}\frac{(2i+1)!(2i+3)(2i+5)}{(2\beta+1)!(2\beta+3)(2\beta+5)}\frac{1}{(m+\beta-i)!(i-\beta)!i!}\frac{(\nu)_i}{(2\nu)_{2i}}$$
$$=\frac{4^m}{\beta!m!}\frac{(\nu-3)_m(\nu)_\beta(\nu+\beta+1)_m}{(2\nu)_{2\beta}(2\nu+2\beta+1)_{2m}}.$$

Consequently,

$$c_2 = 2^{[(k+1)/2]-[k/2]+2m}(k-2m)!\frac{(\nu-3)_m(\nu)_{[(k+1)/2]}(2\nu)_{2[k/2]}}{(\nu)_{[k/2]}}$$
$$\times \sum_{\beta=0}^{[k/2]-m}\frac{1}{\beta!(k-2m-2\beta)!}\frac{(\nu)_\beta(\nu+\beta+1)_m}{(2\nu)_{2\beta}(2\nu+2\beta+1)_{2m}}.$$

Finally, by (5.48) we find

$$c_2 = 2^k(\nu)_{k-m}(\nu-3)_m. \tag{5.49}$$

Combining (5.44) and (5.49), we arrive at the assertion of Lemma 5.6. □

The proof of (5.44) shows that $c_2 = 2^k(8)_k c_1$. Then in view of (5.49), $c_1 = ((8)_k)^{-1}(\nu)_{k-m}(\nu-3)_m$. This, together with (5.43), gives (5.42). Thus, Proposition 5.6 is proved. □

From (5.41) and Proposition 1.1 we can conclude:

Corollary 5.8. *Spherical functions on $\mathbb{H}^2_{\mathbb{C}a}$ are of the form*

$$\varphi_\lambda(x) = \int_{\mathbb{S}^{15}} e_{\nu,\eta}(x)\,\mathrm{d}\omega_{\mathrm{norm}}(\eta).$$

To end this section we give a proof of Proposition 2.19(iv) without invoking Maple.

Let $a \in \mathbb{C}a^2$, $|a| < 1$. In view of (5.41) and Proposition 2.19(ii), (iii),

$$L\big(e_{\nu,\eta}\circ \varkappa_a\big) = \big(Le_{\nu,\eta}\big)\circ \varkappa_a, \tag{5.50}$$

where $\varkappa_a$ is given by (2.63). Next, for $|x| < 1/\sqrt{2}$ and $H \in \mathcal{H}_6^{k,m}$, we have the following equalities:

$$\big(1-|x|^2\big)^\nu F\big(\nu+k-m, \nu+m-3; 8+k; |x|^2\big)H(x)$$
$$=\big(1-|x|^2\big)^{m-k}F\left(\nu+k-m, k-\nu-m+11; 8+k; \frac{|x|^2}{|x|^2-1}\right)H(x)$$

$$= \sum_{j=0}^{\infty} \frac{(\nu + k - m)_j (k - \nu - m + 11)_j}{(8 + k)_j \, j!} (-1)^j |x|^{2j} \big(1 - |x|^2\big)^{m-k-j} H(x), \tag{5.51}$$

$$H(x) = \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j \big((1+t)^{m-k}\big)\Big|_{t=0} |x|^{2j} \big(1 - |x|^2\big)^{m-k-j} H(x), \tag{5.52}$$

$$|x|^2 = \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j \left(\frac{t}{1+t}\right)\Big|_{t=0} \left(\frac{|x|^2}{1 - |x|^2}\right)^j. \tag{5.53}$$

In addition it is obvious that

$$x_i^2 - \frac{|x|^2}{16} \in \mathcal{H}_1^{16,2} \quad \text{for } i = 1, \dots, 16. \tag{5.54}$$

Taking (5.50) and Lemma 4.36 into account, we obtain from (5.42) and (5.51)–(5.54) that

$$L(f \circ \varkappa_a) = (Lf) \circ \varkappa_a \tag{5.55}$$

if $f(x) = x_i$ or $f(x) = x_i x_j$, $1 \leqslant i, j \leqslant 16$. Then by (1.43) relation (5.55) holds for every $f \in C^\infty(\mathbb{H}^2_{\mathbb{C}a})$. Hence, $\varkappa_a \in I(\mathbb{H}^2_{\mathbb{C}a})$ (see Sect. 1.2). This finishes the proof as $\tau_{a/|a|} \in I(\mathbb{H}^2_{\mathbb{C}a})$.

Part II
Transformations with Generalized Transmutation Property Associated with Eigenfunctions Expansions

This is perhaps the central part of the book from the point of view of the mathematical machinery. In it we develop the theory of transmutation operators as a key tool in the study of mean periodic functions on multidimensional domains.

Transmutation operators arise naturally from eigenfunctions expansions of Laplacians. In this part we will deal with Euclidean spaces, symmetric spaces of noncompact type, compact two-point homogeneous spaces, and the phase space associated to the Heisenberg group—the contexts in which there is already a well-established spectral theory of Laplacians.

In Chap. 6 we give some preliminary results about entire functions and distributions. Chapter 7 provides a brief introduction to the theory of special functions. Although these two chapters can be viewed as auxiliary, some of the results presented here are new.

Chapter 8 deals with a subject that is basically a topic in nonharmonic Fourier series and which, at first glance, may seem to have little to do with our principal concern in Part II, namely transmutation operators. Nevertheless, its main results will be essential later in applications of the theory of transmutation operators to mean periodic functions.

The main development of transmutation operators starts with Chap. 9, where we treat the case $\mathbb{R}^n,\ n \geqslant 2$. Chapters 10–12 are devoted to the case of symmetric spaces and the phase space. Here appear three types of expansions: (1) Bessel functions, (2) Jacobi functions (this further subdivides into two subcases), and (3) Laguerre functions. For each type of expansion, we define transmutation operators and investigate the following basic questions: the generalized homomorphism property with respect to suitable convolution algebras, support properties, the homeomorphism property with respect to suitable distribution spaces, explicit inversion formulas, the images of certain special functions, normative type estimates, connections with the dual Abel transform, and applications to positive definite functions. The generalized homomorphism property is the crucial one; it relates the mean periodicity on the spaces in question to that on $\mathbb{R}^1$, and allows many proofs in Parts III and IV to be carried out by reduction to the one-dimensional case.

Chapter 6
Preliminaries

In this chapter we discuss some properties of entire functions and distributions that will be used many times later.

Section 6.1 contains the necessary information concerning the theory of entire functions. First, we consider entire functions $f : \mathbb{C} \to \mathbb{C}$ satisfying the estimates

$$\int_{-\infty}^{\infty} \frac{\log_+ |f(t)|}{1+t^2}\,\mathrm{d}t < +\infty$$

and

$$|f(z)| \leqslant \gamma_1 (1+|z|)^{\gamma_2} \mathrm{e}^{\gamma_3 |z|}, \quad z \in \mathbb{C},$$

where the constants $\gamma_1, \gamma_2, \gamma_3 > 0$ are independent of z. For such functions, we present several facts about distribution of zeros, the Hadamard factorization property, and give an explicit formula for the indicator function. We then fix an arbitrary entire function f and investigate various properties of the entire function

$$a^{\lambda,\eta}(f,z) = \sum_{j=\eta}^{n_\lambda - 1} \frac{1}{\eta!(j-\eta)!} \left(\frac{(z-\lambda)^{n_\lambda}}{f(z)} \right)^{(j-\eta)} \Bigg|_{z=\lambda} \frac{f(z)}{(z-\lambda)^{n_\lambda - j}},$$

where λ is a zero of f, n_λ is the multiplicity of λ, and $\eta \in \{0, \dots, n_\lambda - 1\}$. It plays an important role for the study of biorthogonal decompositions below. The remainder of Sect. 6.1 is devoted to Cauchy-type estimates of holomorphic functions.

Section 6.2 deals with distributions on Euclidean spaces. The main results treated are the Titchmarsh theorem on supports of convolutions, the Paley–Wiener–Schwartz theorem, and the Ehrenpreis–Hörmander characterization of invertible distributions.

V.V. Volchkov, V.V. Volchkov, *Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group,* Springer Monographs in Mathematics,

6.1 Holomorphic Functions

We start with the basic notation concerning entire functions which will be used throughout this book. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. For $z \in \mathbb{C}$ and $\nu \in \mathbb{Z}_+$, we set

$$f^{\langle\nu\rangle}(z) = \begin{cases} f^{(\nu)}(z) & \text{if } z \neq 0, \\ f^{(2\nu)}(0) & \text{if } z = 0. \end{cases}$$

Let $\mathcal{Z}(f) = \{z \in \mathbb{C} : f(z) = 0\}$. Throughout the section we assume that f is nonzero and $\mathcal{Z}(f) \neq \varnothing$. For $\lambda \in \mathcal{Z}(f)$, let $n_\lambda(f)$ denote the multiplicity of zero λ. For brevity, we shall often write n_λ instead of $n_\lambda(f)$. Let $\lambda_1, \lambda_2, \ldots$ be the points of the set $\mathcal{Z}(f)$ arranged according to increasing modulus (for points with equal absolute values, the numbering is chosen arbitrarily). If E is a nonempty subset of $\mathcal{Z}(f)$ and X is a topological vector space, then for each function $u : E \to X$, we define

$$\sum_{\lambda \in E} u(\lambda) = \lim_{R \to +\infty} \sum_{\lambda_j \in E, |\lambda_j| < R} u(\lambda_j)$$

if the limit exists.

Let us now consider some properties of the set $\mathcal{Z}(f)$ under additional growth restrictions on f. For $\varepsilon \in (0, \pi)$, $R, t > 0$, we set

$$N_\pm(R, \varepsilon) = \big\{\lambda \in \mathcal{Z}(f) : 0 < |\lambda| < R, |\arg(\pm\lambda)| < \varepsilon\big\}$$

and

$$\log_+ t = \begin{cases} \log t & \text{if } t > 1, \\ 0 & \text{if } 0 < t \leqslant 1. \end{cases}$$

Proposition 6.1. *Suppose that*

$$\int_{-\infty}^{+\infty} \frac{\log_+ |f(t)|}{1 + t^2}\, \mathrm{d}t < +\infty$$

and

$$|f(z)| \leqslant \gamma_1 (1 + |z|)^{\gamma_2} \mathrm{e}^{\gamma_3 |z|}, \quad z \in \mathbb{C},$$

where the constants γ_1, γ_2, $\gamma_3 > 0$ are independent of z. Then the following assertions hold.

(i) *The series $\sum_{\lambda \in \mathcal{Z}(f)\backslash\{0\}} \lambda^{-1} n_\lambda$ converges. In addition,*

$$\sum_{\lambda \in \mathcal{Z}(f)\backslash\{0\}} \big(\big|\mathrm{Im}\big(\lambda^{-1}\big)\big| + |\lambda|^{-1-\varepsilon}\big) n_\lambda < +\infty \quad \textit{for each } \varepsilon > 0.$$

(ii) *The limit*

$$\lim_{r \to +\infty} \frac{\log |f(r\mathrm{e}^{\mathrm{i}\theta})|}{r} = \begin{cases} \gamma_+ \sin\theta & \textit{if } \theta \in (0, \pi), \\ \gamma_- |\sin\theta| & \textit{if } \theta \in (-\pi, 0) \end{cases}$$

exists for almost all $\theta \in [-\pi, \pi]$, *where the constants* γ_+ *and* γ_- *do not depend on* θ.

(iii) *If* $\varepsilon \in (0, \pi/2]$, *then*

$$\lim_{R\to+\infty} \frac{1}{R} \sum_{\lambda\in N_+(R,\varepsilon)} n_\lambda = \lim_{R\to+\infty} \frac{1}{R} \sum_{\lambda\in N_-(R,\varepsilon)} n_\lambda = \frac{\gamma_- + \gamma_+}{2\pi}.$$

In particular, if $\mathcal{Z}(f)$ *is infinite, then*

$$\lambda^{-1} n_\lambda \to 0 \quad \text{as } \lambda \to \infty, \lambda \in \mathcal{Z}(f). \tag{6.1}$$

(iv) *The function* f *has the form*

$$f(z) = cz^m e^{i\alpha z} \lim_{R\to+\infty} \prod_{\substack{\lambda\in\mathcal{Z}(f)\\ 0<|\lambda|<R}} \left(1 - \frac{z}{\lambda}\right)^{n_\lambda}, \quad z \in \mathbb{C},$$

where $c \in \mathbb{C}$, $m \in \mathbb{Z}_+$, $\alpha \in \mathbb{R}^1$.

For the proof of the proposition, we refer the reader to Levin [146], Lectures 16 and 17.

Proposition 6.2. *Let* $R > 0$, $f(0) \neq 0$, *and*

$$|f(z)| \leqslant \gamma_1 e^{\gamma_2 |z|}, \quad z \in \mathbb{C}, \tag{6.2}$$

where $\gamma_1, \gamma_2 > 0$ *are independent of* z. *Then*

$$\sum_{\lambda\in\mathcal{Z}(f), |\lambda|\leqslant R} n_\lambda \leqslant \log \frac{\gamma_1}{|f(0)|} + \gamma_2 e R.$$

Proof. For $t > 0$, we set

$$N_f(t) = \sum_{\lambda\in\mathcal{Z}(f), |\lambda|\leqslant t} n_\lambda.$$

Then the following Jensen formula is valid:

$$\int_0^r N_f(t) t^{-1}\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left| f\left(re^{i\theta}\right)\right| d\theta - \log |f(0)|, \quad r > 0.$$

Combining this with (6.2), we obtain

$$N_f(R) \leqslant \int_R^{eR} N_f(t) t^{-1}\, dt \leqslant \log \frac{\gamma_1}{|f(0)|} + \gamma_2 e R,$$

as contended. □

Next, assume that $\lambda \in \mathcal{Z}(f)$ and $\eta \in \{0, \dots, n_\lambda - 1\}$. We define the sequence $\{a_j^{\lambda,\eta}(f)\}$, $j = 0, \dots, n_\lambda - 1$, by

$$a_0^{\lambda,\eta}(f) = \frac{n_\lambda! \delta_{0,\eta}}{f^{(n_\lambda)}(\lambda)}, \tag{6.3}$$

$$a_j^{\lambda,\eta}(f) = \frac{n_\lambda!}{f^{(n_\lambda)}(\lambda)} \left(\frac{\delta_{j,\eta}}{j!} - \sum_{s=0}^{j-1} a_s^{\lambda,\eta}(f) \frac{f^{(n_\lambda - s + j)}(\lambda)}{(n_\lambda - s + j)!} \right), \quad j \geqslant 1, \tag{6.4}$$

where $\delta_{j,\eta}$ is the Kronecker symbol. Notice that

$$a_j^{\lambda,\eta}(f) = \begin{cases} 0 & \text{if } j < \eta, \\ \dfrac{n_\lambda!}{\eta! f^{(n_\lambda)}(\lambda)} & \text{if } j = \eta. \end{cases} \tag{6.5}$$

To study basic properties of the sequence $a_j^{\lambda,\eta}(f)$, consider the entire function

$$a^{\lambda,\eta}(f, z) = \sum_{j=0}^{n_\lambda - 1} a_j^{\lambda,\eta}(f) \frac{f(z)}{(z - \lambda)^{n_\lambda - j}}. \tag{6.6}$$

Proposition 6.3.

(i) *If* $\lambda, \mu \in \mathcal{Z}(f)$, $\eta \in \{0, \dots, n_\lambda - 1\}$, *and* $\nu \in \{0, \dots, n_\mu - 1\}$, *then*

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} \right)^\nu a^{\lambda,\eta}(f, z) \Big|_{z=\mu} = \delta_{\lambda,\mu} \delta_{\eta,\nu}. \tag{6.7}$$

(ii) *If* $j \geqslant \eta$, *then*

$$a_j^{\lambda,\eta}(f) = \frac{1}{\eta!(j - \eta)!} \left(\frac{(z - \lambda)^{n_\lambda}}{f(z)} \right)^{(j-\eta)} \Bigg|_{z=\lambda}. \tag{6.8}$$

In particular,

$$a_j^{\lambda,\eta}(f) = \frac{1}{\eta!} a_{j-\eta}^{\lambda,0}(f). \tag{6.9}$$

Proof. To prove (i) we write the function f as a power series

$$f(z) = \sum_{k=n_\mu}^{\infty} \frac{f^{(k)}(\mu)}{k!} (z - \mu)^k \tag{6.10}$$

and substitute (6.10) into (6.6). Then by (6.3) and (6.4) we obtain (6.7). Turning to (ii), one sees from (6.6) that

$$a^{\lambda,\eta}(f, z) \frac{(z - \lambda)^{n_\lambda}}{f(z)} = \sum_{j=0}^{n_\lambda - 1} a_j^{\lambda,\eta}(f)(z - \lambda)^j.$$

Hence,

$$a_j^{\lambda,\eta}(f) = \frac{1}{j!}\left(a^{\lambda,\eta}(f,z)\frac{(z-\lambda)^{n_\lambda}}{f(z)}\right)^{(j)}\Bigg|_{z=\lambda}.$$

This, together with (6.7), yields (6.8) and (6.9). □

Proposition 6.4. *If* $\eta = n_\lambda - 1$, *then*

$$(z-\lambda)a^{\lambda,\eta}(f,z) = a_\eta^{\lambda,\eta}(f)f(z). \tag{6.11}$$

Next, if $n_\lambda > 1$ *and* $\eta \leqslant n_\lambda - 2$, *then*

$$(z-\lambda)a^{\lambda,\eta}(f,z) - (\eta+1)a^{\lambda,\eta+1}(f,z) = a_{n_\lambda-1}^{\lambda,\eta}(f)f(z). \tag{6.12}$$

Proof. Relation (6.11) is a consequence of (6.6) and (6.5). Assume now that $n_\lambda > 1$ and $\eta \in \{0,\ldots,n_\lambda-2\}$. One derives from (6.6) that

$$u(z) - a_{n_\lambda-1}^{\lambda,\eta}(f) = \sum_{k=\eta+1}^{n_\lambda-1}\left(a_{k-1}^{\lambda,\eta}(f) - (\eta+1)a_k^{\lambda,\eta+1}(f)\right)(z-\lambda)^{k-n_\lambda}, \tag{6.13}$$

where

$$u(z) = \left((z-\lambda)a^{\lambda,\eta}(f,z) - (\eta+1)a^{\lambda,\eta+1}(f,z)\right)\big/ f(z). \tag{6.14}$$

Formulae (6.7) and (6.14) ensure us that the function u is entire. Moreover, it follows by (6.13) that

$$u(z) \to a_{n_\lambda-1}^{\lambda,\eta}(f) \quad \text{as } z \to \infty.$$

By Liouville's theorem, $u(z) = a_{n_\lambda-1}^{\lambda,\eta}(f)$ for each $z \in \mathbb{C}$. In view of (6.14), this brings us to (6.12). □

Proposition 6.5. *Let* $g(z) = \mathrm{i}(z-c)f(z)$, $c \in \mathbb{C}$, *and let* $j \in \{0,\ldots,n_\lambda-1\}$.

(i) *If* $c \neq \lambda$, *then*

$$a_j^{\lambda,0}(f) = \mathrm{i}(\lambda-c)a_j^{\lambda,0}(g) + \mathrm{i}a_{j-1}^{\lambda,0}(g),$$

where the number $a_{j-1}^{\lambda,0}(g)$ *is set to be equal to zero for* $j=0$.

(ii) *If* $c = \lambda$, *then*

$$a_j^{\lambda,0}(f) = \mathrm{i}a_j^{\lambda,0}(g).$$

Proof. This is immediate from Proposition 6.3(ii). □

To continue, we define

$$\sigma^{\lambda,\eta}(f) = \sum_{j=0}^{n_\lambda-1}\left|a_j^{\lambda,\eta}(f)\right|, \quad \sigma_\lambda(f) = \sigma^{\lambda,0}(f). \tag{6.15}$$

Proposition 6.6.

(i) $\sigma^{\lambda,\eta}(f) \leqslant \frac{1}{\eta!}\sigma_\lambda(f)$.

(ii) $|a^{\lambda,\eta}(f,z)| \leqslant \sigma^{\lambda,\eta}(f)\max_{|\zeta-z|=2}|f(\zeta)|$.

(iii) $\sigma_\lambda(f) \leqslant \frac{n_\lambda!}{|f^{(n_\lambda)}(\lambda)|}(1+\gamma(\lambda,f))^{n_\lambda-1}$, *where*

$$\gamma(\lambda, f) = \begin{cases} 0 & \text{if } n_\lambda = 1, \\ \dfrac{n_\lambda!}{|f^{(n_\lambda)}(\lambda)|} \max\limits_{n_\lambda+1\leqslant q\leqslant 2n_\lambda-1} \dfrac{|f^{(q)}(\lambda)|}{q!} & \text{if } n_\lambda > 1. \end{cases} \tag{6.16}$$

(iv) *If p is a polynomial of degree d and $p(\lambda) \neq 0$, then*

$$c_1(1+|\lambda|)^{-d}\sigma_\lambda(f) \leqslant \sigma_\lambda(fp) \leqslant c_2(1+|\lambda|)^{-d}\sigma_\lambda(f),$$

where the constants $c_1, c_2 > 0$ are independent of λ.

(v) *Let $\{\xi_{\lambda,\eta}\}$, $\lambda \in \mathcal{Z}(f)$, $\eta \in \{0,\dots,n_\lambda-1\}$, be a sequence of complex numbers such that*

$$\sum_{\lambda\in\mathcal{Z}(f)}\sum_{\eta=0}^{n_\lambda-1}\frac{|\xi_{\lambda,\eta}|\sigma^{\lambda,\eta}(f)}{1+|\lambda|} < +\infty. \tag{6.17}$$

Then the entire function

$$w(z) = \sum_{\lambda\in\mathcal{Z}(f)}\sum_{\eta=0}^{n_\lambda-1} a^{\lambda,\eta}(f,z)\xi_{\lambda,\eta} \tag{6.18}$$

satisfies

$$w^{(\nu)}(\mu) = \xi_{\mu,\nu}, \quad \mu \in \mathcal{Z}(f), \nu \in \{0,\dots,n_\mu-1\},$$

and the series in (6.18) *converges uniformly on each compact subset of $\mathbb{C}$. In addition,*

$$|w(z)| \leqslant |f(z)| \sum_{|\lambda-z|\geqslant 1}\sum_{\eta=0}^{n_\lambda-1}\frac{|\xi_{\lambda,\eta}|\sigma^{\lambda,\eta}(f)}{|\lambda-z|} + \max_{|\zeta-z|\leqslant 2}|f(\zeta)| \sum_{|\lambda-z|<1}\sum_{\eta=0}^{n_\lambda-1}|\xi_{\lambda,\eta}|\sigma^{\lambda,\eta}(f).$$

Proof. Estimate (i) follows at once from (6.15), (6.9), and (6.5). Next, let $z \in \mathbb{C}$. By the maximum-modulus principle we have

$$\left|\frac{f(z)}{(z-\lambda)^{n_\lambda-j}}\right| \leqslant \begin{cases} |f(z)| & \text{if } |z-\lambda| > 1, \\ \max\limits_{|\zeta-\lambda|=1}|f(\zeta)| & \text{if } |z-\lambda| \leqslant 1, \end{cases}$$

for each $j \in \{0,\dots,n_\lambda-1\}$. Thus,

$$\left|\frac{f(z)}{(z-\lambda)^{n_\lambda-j}}\right| \leqslant \max_{|\zeta-z|\leqslant 2}|f(\zeta)|,$$

and the validity of (ii) is obvious from (6.6) and (6.15). To show (iii) it is enough to consider the case $n_\lambda > 1$ (see (6.15) and (6.3)). For brevity, we set $\gamma = \gamma(\lambda, f)$ and $x_j = |a_j^{\lambda,0}(f)|$, $j \in \{0, \dots, n_\lambda - 1\}$. Relations (6.3) and (6.4) yield

$$x_j \leqslant \begin{cases} x_0 & \text{if } j = 0, \\ \gamma \sum_{s=0}^{j-1} x_s & \text{if } j \geqslant 1. \end{cases}$$

One checks by induction on j that $x_j \leqslant \gamma(1+\gamma)^{j-1} x_0$ for $j \geqslant 1$. Combining this with (6.15), we arrive at (iii). Next, for $d = 1$, the estimate in (iv) is clear from Proposition 6.5. In the general case statement (iv) follows by iteration. Finally, using (6.17), (6.7), and part (ii), we obtain (v). □

For the rest of the section, we suppose that f is even. Then $-\lambda \in \mathcal{Z}(f)$ and $n_{-\lambda} = n_\lambda$ for each $\lambda \in \mathcal{Z}(f)$. Applying (6.3) and (6.4), one concludes by induction on j that

$$a_j^{-\lambda,\eta}(f) = (-1)^{n_\lambda - j + \eta} a_j^{\lambda,\eta}(f), \quad j, \eta \in \{0, \dots, n_\lambda - 1\}. \tag{6.19}$$

For $\lambda \in \mathcal{Z}(f)$, $\lambda \neq 0$, $\eta \in \{0, \dots, n_\lambda - 1\}$, we set

$$b^{\lambda,\eta}(f, z) = a^{\lambda,\eta}(f, z) + a^{\lambda,\eta}(f, -z). \tag{6.20}$$

Formulae (6.20) and (6.6) show that

$$b^{\lambda,\eta}(f, z) = \sum_{j=0}^{n_\lambda - 1} b_j^{\lambda,\eta}(f) \frac{f(z)}{(z^2 - \lambda^2)^{n_\lambda - j}}, \tag{6.21}$$

where

$$b_j^{\lambda,\eta}(f) = 2 \sum_{k=0}^{j} \sum_{m=j-k}^{(n_\lambda - k)/2} a_k^{\lambda,\eta}(f) \binom{n_\lambda - k}{2m} \binom{m}{j-k} \lambda^{n_\lambda + k - 2j}.$$

Assume now that $0 \in \mathcal{Z}(f)$. Then the number n_0 is even, and for $\eta \in \{0, \dots, n_0/2 - 1\}$, we put

$$b^{0,\eta}(f, z) = \frac{1}{2}\big(a^{0,2\eta}(f, z) + a^{0,2\eta}(f, -z)\big) = \sum_{k=0}^{n_0/2-1} b_k^{0,\eta}(f) \frac{f(z)}{z^{n_0 - 2k}}, \tag{6.22}$$

where $b_k^{0,\eta}(f) = a_{2k}^{0,2\eta}(f)$. For each $\lambda \in \mathcal{Z}(f)$, we set

$$q(\lambda, f) = \begin{cases} n_\lambda - 1 & \text{if } \lambda \neq 0, \\ n_\lambda/2 - 1 & \text{if } \lambda = 0 \in \mathcal{Z}(f). \end{cases}$$

Let us now give analogues of Propositions 6.3–6.5 for the sequence $b_j^{\lambda,\eta}(f)$.

Proposition 6.7.

(i) *If* $\lambda, \mu \in \mathcal{Z}(f)$, $\eta \in \{0, \dots, n_\lambda - 1\}$, $\nu \in \{0, \dots, n_\mu - 1\}$, *then*

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\nu} b^{\lambda,\eta}(f, z)\bigg|_{z=\mu} = \begin{cases} 0 & \text{if } \lambda^2 \neq \mu^2, \\ (-1)^{\nu}\delta_{\nu,\eta} & \text{if } \lambda = -\mu \neq 0, \\ \delta_{\nu,\eta} & \text{if } \lambda = \mu \neq 0, \\ \delta_{\nu,\eta}\delta_{\nu,2[\nu/2]} & \text{if } \lambda = \mu = 0. \end{cases} \tag{6.23}$$

(ii) *If* $j \in \{0, \dots, q(\lambda, f)\}$, *then*

$$b_j^{\lambda,0}(f) = \frac{1}{j!}\left(\frac{(z-\lambda^2)^{q(\lambda,f)+1}}{f(\sqrt{z})}\right)^{(j)}\Bigg|_{z=\lambda^2}.$$

Proof. To prove (i) it is enough to repeat the argument in the proof of Proposition 6.3(i). Part (ii) follows from (6.21)–(6.23). □

Proposition 6.8.

(i) $(z^2 - \lambda^2) b^{\lambda,q(\lambda,f)}(f, z) = b_{q(\lambda,f)}^{\lambda,q(\lambda,f)}(f) f(z)$.

(ii) *If* $\lambda \neq 0$ *and* $q(\lambda, f) \geqslant 1$, *then*

$$\big(z^2 - \lambda^2\big) b^{\lambda,q(\lambda,f)-1}(f, z) - 2\lambda q(\lambda, f) b^{\lambda,q(\lambda,f)}(f, z) = b_{q(\lambda,f)}^{\lambda,q(\lambda,f)-1}(f) f(z).$$

(iii) *If* $\lambda \neq 0$ *and* $q(\lambda, f) \geqslant 2$, *then*

$$\begin{aligned}\big(z^2 - \lambda^2\big) b^{\lambda,\eta}(f, z) - 2\lambda(\eta + 1) b^{\lambda,\eta+1}(f, z) - (\eta + 2)(\eta + 1) b^{\lambda,\eta+2}(f, z)\\ = b_{q(\lambda,f)}^{\lambda,\eta}(f) f(z)\end{aligned}$$

for all $\eta \in \{0, \dots, q(\lambda, f) - 2\}$.

(iv) *If* $0 \in \mathcal{Z}(f)$ *and* $q(0, f) \geqslant 1$, *then*

$$z^2 b^{0,\eta}(f, z) - (2\eta + 2)(2\eta + 1) b^{0,\eta+1}(f, z) = b_{q(0,f)}^{0,\eta}(f) f(z)$$

for all $\eta \in \{0, \dots, q(0, f) - 1\}$.

The proof of this result reproduces the proof of Proposition 6.4, but instead of (6.6) we use (6.21) and (6.22).

Proposition 6.9. *Let* $g(z) = (c - z^2) f(z)$ *for some* $c \in \mathbb{C}$, *let* $\lambda \in \mathcal{Z}(f)$, *and assume that* $j \in \{0, \dots, q(\lambda, f)\}$.

(i) *If* $c \neq \lambda^2$, *then*

$$b_j^{\lambda,0}(f) = \big(c - \lambda^2\big) b_j^{\lambda,0}(g) - b_{j-1}^{\lambda,0}(g),$$

where the number $b_{j-1}^{\lambda,0}(g)$ *is set to be equal to zero for* $j = 0$.

(ii) *If* $c = \lambda^2$, *then* $b_j^{\lambda,0}(f) = -b_j^{\lambda,0}(g)$.

Proof. This result is a consequence of Proposition 6.7(ii). □

To conclude the section we establish upper estimates for derivatives of some holomorphic functions.

Proposition 6.10. *Let* $0 < r < R$, $z \in \mathbb{C}$, *and assume that* g *is holomorphic in* $\{\zeta \in \mathbb{C} : |\zeta - z| < R\}$. *Then*

$$\left|g^{(s)}(z)\right| \leqslant c\sqrt{s+1}\left(\frac{s+1}{\mathrm{e}r}\right)^s \max_{|\zeta - z| = r} |g(\zeta)|, \quad s \in \mathbb{Z}_+, \tag{6.24}$$

where the constant $c > 0$ *is independent of* z, s, r, R, g.

Proof. The Cauchy formula yields

$$g^{(s)}(z) = \frac{s!}{2\pi \mathrm{i}} \int_{|\zeta - z| = r} g(\zeta)(\zeta - z)^{-s-1}\,\mathrm{d}\zeta,$$

whence

$$\left|g^{(s)}(z)\right| \leqslant s! r^{-s} \max_{|\zeta - z| = r} |g(\zeta)|. \tag{6.25}$$

The required estimate is now clear from Stirling's formula. □

For $\varepsilon \in (0, 1)$, $\delta \geqslant 0$, we set

$$\mathcal{O}_{\varepsilon,\delta} = \left\{\zeta \in \mathbb{C} : |\zeta| > \delta, |\arg \zeta| < \pi - \arccos \varepsilon\right\}.$$

Proposition 6.11. *Let* $\varepsilon \in (0, 1)$, $\delta \geqslant 0$, *and assume that* g *is holomorphic in* $\mathcal{O}_{\varepsilon,\delta}$ *and*

$$|g(\zeta)| \leqslant c(1 + |\zeta|)^\alpha \left(\log(2 + |\zeta|)\right)^\beta \mathrm{e}^{\gamma \operatorname{Im} \zeta}, \quad \zeta \in \mathcal{O}_{\varepsilon,\delta},$$

where the constants $c > 0$, $\alpha, \beta, \gamma \in \mathbb{R}^1$, $\gamma \neq 0$, *are independent of* ζ. *Suppose that* $s \in \mathbb{N}$, $z \in \mathbb{C}$, $\operatorname{Re} z \geqslant 0$, *and* $|\gamma z| > \max\{s + \delta|\gamma|, s/\varepsilon\}$. *Then*

$$\left|g^{(s)}(z)\right| \leqslant c_1 \sqrt{s}|\gamma|^s (1 + |z|)^\alpha \left(\log(2 + |z|)\right)^\beta \mathrm{e}^{\gamma \operatorname{Im} z}, \tag{6.26}$$

where $c_1 > 0$ *depends only on* ε, α, β, c.

Proof. It is easy to verify that $\{\zeta \in \mathbb{C} : |\zeta - z| \leqslant s/|\gamma|\} \subset \mathcal{O}_{\varepsilon,\delta}$. Using now estimate (6.24) with $r = s/|\gamma|$, we obtain (6.26). □

Proposition 6.12. *Let* $\varepsilon \in (0, 1/2)$, $\delta \geqslant 0$, *and assume that* h *is holomorphic in* $\mathcal{O}_{\varepsilon,\delta}$. *Suppose that* $\alpha, \beta, \gamma \in \mathbb{R}^1$, $\gamma \neq 0$, *and*

$$\left|h(\zeta) - \zeta^\alpha \exp\left(\mathrm{i}(\beta + \gamma\zeta)\right)\right| \leqslant c(1 + |\zeta|)^{\alpha - 1} \mathrm{e}^{-\gamma \operatorname{Im} \zeta}, \quad \zeta \in \mathcal{O}_{\varepsilon,\delta}, \tag{6.27}$$

where $c > 0$ is independent of ζ. Let $s \in \mathbb{N}$, $z \in \mathbb{C}$, $\operatorname{Re} z \geqslant 0$, and $|\gamma z| > \max\{s + \delta|\gamma|, s/\varepsilon\}$. Then

$$\left|h^{(s)}(z) - (\mathrm{i}\gamma)^s z^\alpha \exp\left(\mathrm{i}(\beta + \gamma z)\right)\right| \leqslant c_1 s|\gamma|^s\left(1 + |\gamma|^{-1}\right)(1 + |z|)^{\alpha-1}\mathrm{e}^{-\gamma \operatorname{Im} z},$$

where $c_1 > 0$ depends only on ε, α, c.

Proof. Notice that

$$\left(z^\alpha \exp\left(\mathrm{i}(\beta + \gamma z)\right)\right)^{(s)} = (\mathrm{i}\gamma)^s z^\alpha \exp\left(\mathrm{i}(\beta + \gamma z)\right) + w(z), \tag{6.28}$$

where

$$w(z) = \sum_{q=1}^{s} \binom{s}{q} \left(z^\alpha\right)^{(q)} (\mathrm{i}\gamma)^{s-q} \exp\left(\mathrm{i}(\beta + \gamma z)\right).$$

Then one has

$$\begin{aligned} |\gamma|^{-s}|z|^{-\alpha}\mathrm{e}^{\gamma \operatorname{Im} z}|w(z)| &\leqslant \sum_{q=1}^{s} \binom{s}{q} |\gamma z|^{-q} \prod_{m=0}^{q-1} (|\alpha| + m) \\ &\leqslant 2^{|\alpha|} \sum_{q=1}^{s} \frac{s!}{(s-q)!}\left(\frac{2}{|\gamma z|}\right)^q \\ &\leqslant \frac{2^{|\alpha|+1}s}{|\gamma z|} \sum_{q=1}^{\infty} \left(\frac{2s}{|\gamma z|}\right)^{q-1}. \end{aligned}$$

Combining this with (6.27) and (6.28) and using Proposition 6.11 with $g(\zeta) = h(\zeta) - \zeta^\alpha \exp(\mathrm{i}(\beta + \gamma\zeta))$, we arrive at the desired result. □

Proposition 6.13. *Let $\varepsilon \in (0, 1/2)$, $\delta \geqslant 0$, and assume that h is holomorphic in $\mathcal{O}_{\varepsilon,\delta}$. Suppose that $\alpha, \beta \in \mathbb{R}^1$, $\gamma > 0$, and*

$$\left|h(\zeta) - \zeta^\alpha \cos(\beta + \gamma\zeta)\right| \leqslant c(1 + |\zeta|)^{\alpha-1}\mathrm{e}^{\gamma|\operatorname{Im}\zeta|}, \quad \zeta \in \mathcal{O}_{\varepsilon,\delta},$$

where $c > 0$ is independent of ζ. Let $s \in \mathbb{N}$, $z \in \mathbb{C}$, $\operatorname{Re} z \geqslant 0$ and $\gamma|z| > \max\left\{s + \delta\gamma, s/\varepsilon\right\}$. Then

$$\left|h^{(s)}(z) - \gamma^s z^\alpha \cos\left(\frac{\pi}{2}s + \beta + \gamma z\right)\right| \leqslant c_1 s\gamma^s\left(1 + \gamma^{-1}\right)(1 + |z|)^{\alpha-1}\mathrm{e}^{\gamma|\operatorname{Im} z|},$$

where $c_1 > 0$ depends only on ε, α, c.

Proof. The argument is quite parallel to the proof of Proposition 6.12. □

6.2 Distributions

In this section we assemble certain preliminary information concerning distributions on $\mathbb{R}^n$. First, we recall the definition and some basic properties of convolution. For all unproved statements, we refer to Hörmander [126], Vol. 1.

If $\mathcal{O}_j$ is a nonempty open set in $\mathbb{R}^{n_j}$, $j = 1, 2$, and if $g_j \in C(\mathcal{O}_j)$, then the function $g_1 \otimes g_2$ in $\mathcal{O}_1 \times \mathcal{O}_2 \subset \mathbb{R}^{n_1+n_2}$ defined by

$$(g_1 \otimes g_2)(x, y) = g_1(x)g_2(y), \quad x \in \mathcal{O}_1, y \in \mathcal{O}_2,$$

is called the *direct* (or *tensor*) *product* of g_1 and g_2. It can be proved that if $g_j \in \mathcal{D}'(\mathcal{O}_j)$, $j = 1, 2$, then there is a unique distribution $g \in \mathcal{D}'(\mathcal{O}_1 \times \mathcal{O}_2)$ such that

$$\langle g, u_1 \otimes u_2 \rangle = \langle g_1, u_1 \rangle \langle g_2, u_2 \rangle, \quad u_j \in \mathcal{D}(\mathcal{O}_j).$$

The distribution g is called the tensor product of g_1 and g_2, and one writes $g = g_1 \otimes g_2$.

The convolution $g_1 * g_2$ of two distributions $g_1, g_2 \in \mathcal{D}'(\mathbb{R}^n)$, one of which has compact support, is defined by the formula

$$\langle g_1 * g_2, u \rangle = \langle g_1 \otimes g_2, u(x + y) \rangle, \tag{6.29}$$

where $u \in \mathcal{D}(\mathbb{R}^n)$, $x, y \in \mathbb{R}^n$. Notice that $g_1 * g_2 = g_2 * g_1$ and

$$\frac{\partial}{\partial x_j}(g_1 * g_2) = \frac{\partial g_1}{\partial x_j} * g_2, \quad j \in \{1, \dots, n\}.$$

Let δ_0 be the Dirac measure supported at the origin in $\mathbb{R}^n$. Then

$$p\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)g = p\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)\delta_0 * g, \quad g \in \mathcal{D}'\big(\mathbb{R}^n\big)$$

for each polynomial p.

If $g_1 \in \mathcal{D}'(\mathbb{R}^n)$ and $g_2 \in \mathcal{E}'(\mathbb{R}^n)$, then

$$\langle g_1 * g_2, u \rangle = \langle g_2, \check{g}_1 * u \rangle, \quad u \in \mathcal{D}\big(\mathbb{R}^n\big),$$

where $\check{g}_1$ is defined by

$$\langle \check{g}_1, f(x) \rangle = \langle g_1, f(-x) \rangle, \quad f \in \mathcal{D}\big(\mathbb{R}^n\big).$$

In particular, for $g_1 \in \mathcal{D}'(\mathbb{R}^n)$ and $g_2 \in \mathcal{D}(\mathbb{R}^n)$, we have $g_1 * g_2 \in C^\infty(\mathbb{R}^n)$ and

$$(g_1 * g_2)(x) = \langle g_1, g_2(x - \cdot) \rangle.$$

For the case where $g_1 \in L^{1,\text{loc}}(\mathbb{R}^n)$ and $g_2 \in (\mathcal{E}' \cap L^1)(\mathbb{R}^n)$, one obtains $g_1 * g_2 \in L^{1,\text{loc}}(\mathbb{R}^n)$ and

$$(g_1 * g_2)(x) = \int_{\mathbb{R}^n} g_1(y)g_2(x - y)\, \mathrm{d}y.$$

We note also that if $g_1 \in \mathcal{D}'(\mathbb{R}^n)$, $g_2, g_3 \in \mathcal{E}'(\mathbb{R}^n)$, then

$$(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3).$$

Let A_1, A_2 be subsets in $\mathbb{R}^n$, and let

$$A_1 \pm A_2 = \{a_1 \pm a_2 : a_1 \in A_1, a_2 \in A_2\}.$$

Assume now that $g_2 \in \mathcal{E}'(\mathbb{R}^n)$ and that $\mathcal{O}_1$, $\mathcal{O}_2$ are nonempty open subsets of $\mathbb{R}^n$ such that $\mathcal{O}_2 - \operatorname{supp} g_2 \subset \mathcal{O}_1$. If $g_1 \in \mathcal{D}'(\mathcal{O}_1)$, then the convolution $g_1 * g_2$ is well defined by (6.29) as distribution in $\mathcal{D}'(\mathcal{O}_2)$. Moreover, if $g_1 = G|_{\mathcal{O}_1}$ for some $G \in \mathcal{D}'(\mathbb{R}^n)$, then

$$(G * g_2)|_{\mathcal{O}_2} = g_1 * g_2.$$

It is not hard to make sure that

$$\operatorname{supp} g_1 * g_2 \subset \operatorname{supp} g_1 + \operatorname{supp} g_2 \tag{6.30}$$

and that the map $g \to g * g_2$ is continuous from $\mathcal{D}'(\mathcal{O}_1)$ into $\mathcal{D}'(\mathcal{O}_2)$.

Next, for $r \geqslant 0$, we set

$$B_r = \{x \in \mathbb{R}^n : |x| < r\}, \qquad \overset{\bullet}{B}_r = \{x \in \mathbb{R}^n : |x| \leqslant r\}.$$

Let $\mathcal{O}$ be a nonempty open subset of $\mathbb{R}^n$, and let $f \in \mathcal{D}'(\mathcal{O})$. The *standard smoothing procedure* consists in the approximation of f by smooth functions of the form $f * w$, where $w \in \mathcal{D}(B_\varepsilon)$, and $\varepsilon > 0$ is sufficiently small. This is a very efficient approach because to study the properties of f, it often suffices to have information on the properties of $f * w$. It can be used to extend statements concerning smooth functions to distributions, particularly when translation invariant questions are concerned. For future use, we state the following result.

Theorem 6.1. *Let K be a nonempty compact subset of $\mathcal{O}$, let $\varepsilon > 0$, and let $K + \overset{\bullet}{B}_\varepsilon \subset \mathcal{O}$. We set*

$$v_\varepsilon(x) = \varepsilon^{-n} v(x/\varepsilon), \quad x \in \mathbb{R}^n,$$

where $v \in \mathcal{D}(\mathbb{R}^n)$, $v \geqslant 0$, $\operatorname{supp} v \subset \overset{\bullet}{B}_1$, and $\int_{\mathbb{R}^n} v(x)\,dx = 1$. Then the following assertions hold.

(i) *If $f \in \mathcal{D}'(\mathcal{O})$, then $f * v_\varepsilon \to f$ as $\varepsilon \to 0$ in the space $\mathcal{D}'(\mathcal{O})$.*
(ii) *If $f \in L^{p,\mathrm{loc}}(\mathcal{O})$ for some $p \in [1, +\infty)$, then $f * v_\varepsilon \to f$ as $\varepsilon \to 0$ in $L^p(K)$.*
(iii) *If $f \in C^m(\mathcal{O})$ for some $m \in \mathbb{Z}_+$, then $f * v_\varepsilon \to f$ as $\varepsilon \to 0$ in $C^m(K)$.*

For example, one can take the function

$$v(x) = \begin{cases} c \exp\big(1/(|x|^2 - 1)\big) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geqslant 1, \end{cases}$$

where

$$c = \left(\int_{B_1} \exp\left(1/\left(|x|^2 - 1\right)\right) \mathrm{d}x \right)^{-1}.$$

The convolution $f * v_\varepsilon$ is called the *regularization* of f. For the proof of Theorem 6.1, we refer the reader to Hörmander [126], Theorem 1.3.2.

The following Titchmarsh theorem of supports states that there is equality in (6.30) if g_1 and g_2 both have compact supports and if one takes convex hulls of the supports.

Theorem 6.2. *If* $g_1, g_2 \in \mathcal{E}'(\mathbb{R}^n)$, *then*

$$\operatorname{conv}\operatorname{supp} g_1 * g_2 = \operatorname{conv}\operatorname{supp} g_1 + \operatorname{conv}\operatorname{supp} g_2. \tag{6.31}$$

Here we write conv A *for convex hull of* $A \subset \mathbb{R}^n$, *that is,*

$$\operatorname{conv} A = \{\lambda_1 x + \lambda_2 y : x, y \in A, \lambda_1, \lambda_2 \geqslant 0, \lambda_1 + \lambda_2 = 1\}.$$

The standard proofs of this theorem depend on analytic function theory, see [126], Sect. 16.3.

Corollary 6.1. *Assume that* $f_1, f_2 \in L^1(0,1)$ *and*

$$\int_0^t f_1(u) f_2(t-u)\,\mathrm{d}u = 0 \quad \textit{for almost all } t \in (0,1).$$

Also let supp $f_1 \subset [\alpha, 1]$ *and* supp $f_2 \subset [\beta, 1]$ *for some* $\alpha, \beta \in [0,1]$. *Then* $\alpha + \beta \geqslant 1$.

Proof. We define $g_j \in \mathcal{E}'(\mathbb{R}^1)$ by letting $g_j = f_j$ on $(0,1)$ and $g_j = 0$ on $\mathbb{R}^1 \backslash (0,1)$, $j = 1, 2$. It follows by the hypothesis that

$$\operatorname{supp}\, g_1 * g_2 \subset [1, +\infty).$$

The required conclusion now follows from (6.31). □

To continue, let $T \in \mathcal{E}'(\mathbb{R}^n)$. The order of T will be denoted by ord T. We set

$$r_0(T) = \inf\{r > 0 : \operatorname{supp}\, T \subset B_r\}$$

and

$$r(T) = \inf\left\{r > 0 : \operatorname{supp}\, T \subset \mathring{B}_r(y) \text{ for some } y \in \mathbb{R}^n\right\},$$

where $\mathring{B}_r(y) = \{x \in \mathbb{R}^n : |x - y| \leqslant r\}$.

The entire function

$$\widehat{T}(\zeta) = \left\langle T(x), \mathrm{e}^{-\mathrm{i}(x_1\zeta_1 + \cdots + x_n\zeta_n)} \right\rangle \tag{6.32}$$

of variable $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$ is called the *Fourier–Laplace transform* of T. For each $j \in \{1, \ldots, n\}$, relation (6.32) yields

$$\mathrm{i}\zeta_j \widehat{T}(\zeta) = \widehat{g}(\zeta), \quad \text{where } g = \frac{\partial T}{\partial x_j}. \tag{6.33}$$

Therefore, if $T \in (\mathcal{E}' \cap C^m)(\mathbb{R}^n)$ for some $m \in \mathbb{Z}_+$, then

$$\left|\widehat{T}(\zeta)\right| \leqslant c(1+|\zeta|)^{-m} \mathrm{e}^{r_0(T)|\operatorname{Im}\zeta|}, \quad \zeta \in \mathbb{C}^n, \tag{6.34}$$

where $\operatorname{Im}\zeta = (\operatorname{Im}\zeta_1, \dots, \operatorname{Im}\zeta_n)$, and the constant $c > 0$ is independent of ζ. Next, it is elementary to see that

$$\widehat{T_1 * T_2} = \widehat{T_1}\widehat{T_2}, \quad T_1, T_2 \in \mathcal{E}'\big(\mathbb{R}^n\big). \tag{6.35}$$

For $f \in L^1(\mathbb{R}^n)$, we define the *Fourier transform* of f by the formula

$$\widehat{f}(\zeta) = \int_{\mathbb{R}^n} f(x) \mathrm{e}^{-\mathrm{i}\langle x,\zeta\rangle_{\mathbb{R}}} \, \mathrm{d}x, \quad \zeta \in \mathbb{R}^n.$$

It can be shown that if $f \in (L^1 \cap L^2)(\mathbb{R}^n)$, then $\widehat{f} \in L^2(\mathbb{R}^n)$. Moreover, there exists a linear homeomorphism $F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that

$$F(f) = \widehat{f} \quad \text{for } f \in \big(L^1 \cap L^2\big)\big(\mathbb{R}^n\big)$$

and

$$\|F(f)\|_{L^2(\mathbb{R}^n)} = (2\pi)^{n/2} \|f\|_{L^2(\mathbb{R}^n)} \quad \text{for } f \in L^2\big(\mathbb{R}^n\big)$$

(see [126], Sect. 7.1). We shall write $\widehat{f}$ instead of $F(f)$ for each $f \in L^2(\mathbb{R}^n)$. Relation (6.35) remains to be true, provided that $T_1, T_2 \in L^2(\mathbb{R}^n)$.

One of the most important properties of the map $T \to \widehat{T}$ is contained in the following Paley–Wiener–Schwartz theorem.

Theorem 6.3. *Let K be a nonempty convex compact subset of $\mathbb{R}^n$, and let*

$$H(\xi) = \sup_{x \in K} \langle x, \xi\rangle_{\mathbb{R}}, \quad \xi \in \mathbb{R}^n.$$

If $T \in \mathcal{E}'(\mathbb{R}^n)$ and $\operatorname{supp} T \subset K$, *then*

$$\left|\widehat{T}(\zeta)\right| \leqslant c_1(1+|\zeta|)^{\operatorname{ord} T} \mathrm{e}^{H(\operatorname{Im}\zeta)}, \quad \zeta \in \mathbb{C}^n,$$

where $c_1 > 0$ is independent of ζ. Conversely, if $f : \mathbb{C}^n \to \mathbb{C}$ is an entire function satisfying

$$|f(\zeta)| \leqslant c_2(1+|\zeta|)^{\gamma} \mathrm{e}^{H(\operatorname{Im}\zeta)}, \quad \zeta \in \mathbb{C}^n,$$

for some $c_2 > 0$ and $\gamma \in \mathbb{R}^1$, then there exists $T \in \mathcal{E}'(\mathbb{R}^n)$ such that $\widehat{T} = f$ and $\operatorname{supp} T \subset K$. *In addition,*

$$\operatorname{ord} T \leqslant \max\{0, 1+\gamma+n/2\}. \tag{6.36}$$

Proof. Apart from estimate (6.36), the proof of this result can be found in [126], Theorem 7.3.1. To prove (6.36) put $\alpha = (1+[\gamma+n/2])/2$. Define the functions g, h by the relations

$$g(x) = \widehat{T}(x)\big(1+|x|^2\big)^{-\alpha}, \quad h(x) = \mathrm{i}x_1\widehat{T}(x)\big(1+|x|^2\big)^{-\alpha-1/2}, \quad x \in \mathbb{R}^n.$$

First, consider the case where $\alpha \in \mathbb{Z}_+$. Then $g \in L^2(\mathbb{R}^n)$, and for each $\varphi \in \mathcal{D}(\mathbb{R}^n)$, one has

$$T * \varphi = (1 - \Delta)^{\alpha}(u * \varphi),$$

where $u \in L^2(\mathbb{R}^n)$ is defined by $\widehat{u} = g$. Since φ is arbitrary, this yields ord $T \leqslant 2\alpha$. Suppose now that $\alpha - 1/2 \in \mathbb{Z}_+$. Then $h \in L^2(\mathbb{R}^n)$ and

$$\frac{\partial T}{\partial x_1} * \varphi = (1 - \Delta)^{\alpha+1/2}(v * \varphi),$$

where $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $v \in L^2(\mathbb{R}^n)$, and $\widehat{v} = h$. Thus,

$$\operatorname{ord} \frac{\partial T}{\partial x_1} \leqslant 2\alpha + 1,$$

and hence (6.36) is valid. Finally, if $\alpha < 1/2$, then $\widehat{T}$ and T are in $L^2(\mathbb{R}^n)$, and estimate (6.36) is obvious. □

Corollary 6.2. *The following conditions on $T \in \mathcal{E}'(\mathbb{R}^n)$ are equivalent.*

(i) $r_0(T) = 0$.
(ii) $\widehat{T}$ *is a polynomial.*
(iii) $T = p\big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\big)\delta_0$, *where $p : \mathbb{R}^n \to \mathbb{C}$ is a polynomial.*

Proof. This corollary results from using Theorem 6.3, Liouville's theorem, and the definition of $\widehat{T}$. □

We now turn to the problem of inverting the Fourier–Laplace transform.

Proposition 6.14. *Let $T \in \mathcal{E}'(\mathbb{R}^n)$, $k \in \mathbb{Z}_+$, and assume that*

$$\big|\widehat{T}(\zeta)\big| \leqslant c(1 + |\zeta|)^{-n-k-1}e^{R|\operatorname{Im}\zeta|}, \quad \zeta \in \mathbb{C}^n, \tag{6.37}$$

where $c > 0$ and $R \geqslant 0$ are independent of ζ. Then $T \in (\mathcal{E}' \cap C^k)(\mathbb{R}^n)$, $r_0(T) \leqslant R$, and

$$T(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \widehat{T}(\zeta)e^{i(x_1\zeta_1+\dots+x_n\zeta_n)}\,d\zeta, \quad x \in \mathbb{R}^n. \tag{6.38}$$

Proof. Estimate (6.37) ensures us that $\widehat{T} \in (L^1 \cap L^2)(\mathbb{R}^n)$. Hence, $T \in (\mathcal{E}' \cap L^2)(\mathbb{R}^n)$, and (6.38) holds (see Stein and Weiss [203], Chap. 1, Corollary 1.21). Using now (6.37), (6.38), and Theorem 6.3, we deduce that $T \in (\mathcal{E}' \cap C^k)(\mathbb{R}^n)$ and $r_0(T) \leqslant R$. □

Next, we say that a distribution $T \in \mathcal{E}'(\mathbb{R}^n)$ is *invertible* if there exists a constant $\alpha > 0$ such that

$$\sup\big\{\big|\widehat{T}(\zeta)\big| : \zeta \in \mathbb{C}^n, |\zeta - \xi| < \alpha\log(2 + |\xi|)\big\} > (\alpha + |\xi|)^{-\alpha}, \quad \xi \in \mathbb{R}^n.$$

Let $\operatorname{Inv}(\mathbb{R}^n)$ denote the set of all invertible distributions. Estimate (6.34) shows that if $U \in \operatorname{Inv}(\mathbb{R}^n)$ and $T \in \mathcal{D}(\mathbb{R}^n)$, then $U + T \in \operatorname{Inv}(\mathbb{R}^n)$. The class $\operatorname{Inv}(\mathbb{R}^n)$ plays an essential role in the theory of convolution equations. Here we restrict ourselves only to the following result.

Theorem 6.4. *Let $T \in \mathcal{E}'(\mathbb{R}^n)$. Then the following statements are equivalent.*

(i) $T \in \mathrm{Inv}(\mathbb{R}^n)$.
(ii) *If $U \in \mathcal{E}'(\mathbb{R}^n)$ and the function $\widehat{U}/\widehat{T}$ is entire, then $\widehat{U} = \widehat{T}\widehat{V}$ for some $V \in \mathcal{E}'(\mathbb{R}^n)$.*
(iii) *The convolution equation*

$$f * T = g \tag{6.39}$$

has a solution $f \in \mathcal{D}'(\mathbb{R}^n)$ (respectively, $f \in C^\infty(\mathbb{R}^n)$) for every $g \in \mathcal{D}'(\mathbb{R}^n)$ (respectively, $g \in C^\infty(\mathbb{R}^n)$).
(iv) *Equation (6.39) with $g = \delta_0$ has a solution $f \in \mathcal{D}'(\mathbb{R}^n)$.*

For a proof, we refer to Hörmander [126], Sect. 16.6. The reader can find more information there.

Corollary 6.3. *If $U \in \mathcal{E}'(\mathbb{R}^n)$, $T \in \mathrm{Inv}(\mathbb{R}^n)$, and the function $\widehat{U}/\widehat{T}$ is entire, then $r_0(U) \geqslant r_0(T)$.*

The proof follows at once from Theorems 6.4 and 6.2.

Chapter 7
Some Special Functions

Spherical functions and their generalizations will play a decisive role in our study of transmutation operators. For two-point homogeneous spaces and the phase space of the Heisenberg group, they are expressed in terms of classical special functions. For Euclidean space, they are Bessel functions. For rank one symmetric spaces, they are Jacobi functions. For the phase space, they are Laguerre functions. In this chapter we define these functions and discuss some their properties which are needed in studying the expansions in terms of them. In particular, we present various differentiation formulas, integral representations, and asymptotic estimates. The key formulas are the Koornwinder integral representations for Jacobi functions and their analogues for the Kummer confluent hypergeometric function. These formulas generalize the classical Mehler–Dirichlet representation for Legendre functions and make it possible to obtain Paley–Wiener-type theorems for the corresponding integral transforms (see Chaps. 11 and 12 below).

7.1 Cylindrical Functions

In the section we will survey basic properties of cylindrical functions used in this book. The proof of all properties given here can be found in Watson [250].

Let $\nu \in \mathbb{R}^1$ and $z \in \mathbb{C}\backslash(-\infty, 0]$. Recall that the function

$$J_\nu = \left(\frac{z}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m}}{m!\Gamma(\nu + m + 1)} \tag{7.1}$$

is called the *Bessel function* of order ν. Since

$$J_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu + 1)} + O\big(|z|^{2+\nu}\big) \quad \text{as } z \to 0,$$

V.V. Volchkov, V.V. Volchkov, *Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group,* Springer Monographs in Mathematics,

the functions J_ν and $J_{-\nu}$ are linearly independent, provided that $\nu \notin \mathbb{Z}$. If $\nu \in \mathbb{N}$, formula (7.1) yields

$$(-1)^\nu J_\nu(z) = J_{-\nu}(z) = J_\nu(-z).$$

The *Neumann function* of order $\nu \in \mathbb{R}^1$ is defined by the equality

$$N_\nu(z) = \lim_{\mu\to\nu} \frac{J_\mu(z)\cos(\mu\pi) - J_{-\mu}(z)}{\sin(\mu\pi)}. \tag{7.2}$$

For $\nu \in \mathbb{Z}_+$, it follows by (7.1) and (7.2) that

$$N_\nu(z) = \frac{2}{\pi} J_\nu(z)\Big(\log\frac{z}{2} + \gamma\Big) - \frac{1}{\pi}\sum_{m=0}^{\nu-1} \frac{(\nu-m-1)!}{m!}\Big(\frac{z}{2}\Big)^{-\nu+2m}$$
$$-\frac{1}{\pi}\sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{\nu+2m}}{m!(m+\nu)!}\Bigg(\sum_{k=1}^{\nu+m}\frac{1}{k} + \sum_{k=1}^{m}\frac{1}{k}\Bigg),$$

where

$$\gamma = \lim_{q\to+\infty}\Bigg(\sum_{m=1}^{q}\frac{1}{m} - \log q\Bigg)$$

is Euler's constant.

Relations (7.1) and (7.2) show that

$$\frac{\mathrm{d}}{\mathrm{d}z}\big(z^\nu J_\nu(z)\big) = z^\nu J_{\nu-1}(z), \qquad \frac{\mathrm{d}}{\mathrm{d}z}\big(z^{-\nu} J_\nu(z)\big) = -z^{-\nu} J_{\nu+1}(z), \tag{7.3}$$

and the same formulae are valid for the Neumann functions.

By the definitions of J_ν and N_ν we see that these functions satisfy the Bessel differential equation

$$z^2 f'' + zf' + \big(z^2 - \nu^2\big) f = 0, \quad z \in \mathbb{C}\backslash(-\infty, 0]. \tag{7.4}$$

Thus, the general solution of (7.4) has the form

$$f = c_1 J_\nu + c_2 N_\nu, \quad c_1, c_2 \in \mathbb{C}. \tag{7.5}$$

This function is called a *cylindrical function* of order ν. Next, the Lommel–Hankel formula

$$J_\nu(z)N_{\nu+1}(z) - J_{\nu+1}(z)N_\nu(z) = -2/(\pi z) \tag{7.6}$$

is a consequence of (7.3)–(7.5).

If $\nu > -1/2$, (7.1) leads to the Poisson integral

$$J_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi}\,\Gamma\big(\nu+1/2\big)}\int_{-1}^{1} \mathrm{e}^{\mathrm{i}zt}\big(1-t^2\big)^{\nu-1/2}\,\mathrm{d}t. \tag{7.7}$$

Assume now that $\alpha \in \{-1, 1\}$, $\nu \geqslant 0$, and let $\varepsilon \in (0, \pi)$ be fixed. Then

$$J_\nu(z) + \mathrm{i}\alpha N_\nu(z) = \sqrt{\frac{2}{\pi z}} \exp\left(\mathrm{i}\alpha\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)\right)\left(1 + O\left(\frac{1}{|z|}\right)\right) \tag{7.8}$$

as $z \to \infty$, $|\arg z| \leqslant \pi - \varepsilon$. The following result generalizes (7.8) for the case where $\varepsilon = \pi/2$.

Proposition 7.1. *Let* $\alpha \in \{-1, 1\}$, $\nu \geqslant 0$, $\beta \in \mathbb{R}^1$, $s \in \mathbb{Z}_+$, $\theta > 2$, *and suppose that* $z \in \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta \geqslant 0,\ |\zeta| > \theta s + 1\}$. *Then*

$$\left(z^\beta\left(J_\nu(z) + \mathrm{i}\alpha N_\nu(z)\right)\right)^{(s)} = \sqrt{\frac{2}{\pi}}(\mathrm{i}\alpha)^s z^{\beta-1/2} \exp\left(\mathrm{i}\alpha\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)\right)$$
$$+ O\left((s+1)(1+|z|)^{\beta-3/2} \exp(-\alpha \operatorname{Im} z)\right),$$

where the constant in the quantity O *depends only on* ν, β, θ.

The proof follows at once from (7.8) and Proposition 6.12.

To go further, for $\nu \in \mathbb{R}^1$ and $z \in \mathbb{C}\backslash(-\infty, 0]$, we set

$$\mathbf{I}_\nu(z) = z^{-\nu} J_\nu(z), \qquad \mathbf{N}_\nu(z) = z^{-\nu} N_\nu(z).$$

According to (7.1), the function $\mathbf{I}_\nu$ admits holomorphic extension to $\mathbb{C}$. We now assemble certain information on the zeros of $\mathbf{I}_\nu$ needed later.

Let $\nu > -1$. Then $\mathbf{I}_\nu$ has infinitely many zeroes. Moreover, all the zeros of $\mathbf{I}_\nu$ are real, simple, and the set of these zeros is symmetric with respect to $z = 0$. Next, $\mathbf{I}_\nu(z)$ is a transcendental quantity, provided that $\nu \in \mathbf{Q}$ and z is an algebraic number other than zero. In particular, this shows that $\mathbf{I}_\nu$ and $\mathbf{I}_{\nu+m}$ have no common zero when $\nu \in \mathbf{Q}$, $\nu > -1$, $m \in \mathbb{N}$. Let $\lambda_1, \lambda_2, \dots$ be the sequence of all positive zeros of $\mathbf{I}_\nu$ numbered in the ascending order. It can be shown that

$$\lambda_m = \pi\left(m + \frac{\nu}{2} - \frac{1}{4}\right) - \frac{4\nu^2 - 1}{\pi(m + \frac{\nu}{2} - \frac{1}{4})}$$
$$- \frac{(4\nu^2 - 1)(28\nu^2 - 31)}{384\pi^3(m + \frac{\nu}{2} - \frac{1}{4})^3} + O\left(\frac{1}{m^5}\right) \quad \text{as } m \to \infty. \tag{7.9}$$

In addition, the following orthogonality relations hold:

$$\int_0^1 t J_\nu(\lambda_k t) J_\nu(\lambda_m t)\, \mathrm{d}t = \begin{cases} 0 & \text{if } k \neq m, \\ \dfrac{1}{2} J_{\nu+1}^2(\lambda_m) & \text{if } k = m. \end{cases} \tag{7.10}$$

7.2 Jacobi Functions

Let $\alpha, \beta, \lambda \in \mathbb{C}$, $t \in (0, +\infty)$, and let

$$\Delta_{\alpha,\beta}(t) = \left(e^t - e^{-t}\right)^{2\alpha+1}\left(e^t + e^{-t}\right)^{2\beta+1}. \tag{7.11}$$

Consider the differential equation

$$(L_{\alpha,\beta} f)(t) = -\left(\lambda^2 + \gamma^2\right) f(t), \tag{7.12}$$

where

$$(L_{\alpha,\beta} f)(t) = \frac{1}{\Delta_{\alpha,\beta}(t)} \frac{d}{dt}\left(\Delta_{\alpha,\beta}(t) \frac{d f(t)}{dt}\right), \quad \gamma = \alpha + \beta + 1.$$

By the change of variable $z = -(\sinh t)^2$ equation (7.12) reduces to a hypergeometric differential equation with parameters $(\gamma + i\lambda)/2$, $(\gamma - i\lambda)/2$, $\alpha + 1$ (cf. [73, 2.1 (1)]). Thus, for $(-\alpha) \notin \mathbb{N}$, the function

$$\varphi_\lambda^{(\alpha,\beta)}(t) = F\left(\frac{1}{2}(\gamma + i\lambda), \frac{1}{2}(\gamma - i\lambda); \alpha + 1; -(\sinh t)^2\right) \tag{7.13}$$

satisfies (7.12). Here, as usual, the hypergeometric function $F(a, b; c; z)$ denotes the unique analytic continuation for $z \notin [1, +\infty)$ of the series

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad |z| < 1.$$

Relation (7.13) shows that there exists $\varepsilon = \varepsilon(\alpha, \beta) > 0$ such that for all $\lambda \in \mathbb{C}$,

$$\left|\varphi_\lambda^{(\alpha,\beta)}(t) - 1\right| \leqslant \frac{1}{2} \quad \text{if } t \in \left[0, \varepsilon(1 + |\lambda|)^{-1}\right]. \tag{7.14}$$

Next, by the definition of $\varphi_\lambda^{(\alpha,\beta)}(t)$ we see that $(\Gamma(\alpha + 1))^{-1}\varphi_\lambda^{(\alpha,\beta)}(t)$ is an entire function of α, β, and λ. In addition,

$$\varphi_\lambda^{(\alpha,\beta)}(0) = 1, \qquad \frac{d}{dt}\varphi_\lambda^{(\alpha,\beta)}(t)\Big|_{t=0} = 0, \tag{7.15}$$

and

$$\varphi_\lambda^{(\alpha,\beta)}(t) = \varphi_{-\lambda}^{(\alpha,\beta)}(t), \qquad \overline{\varphi_\lambda^{(\alpha,\beta)}(t)} = \varphi_{\overline{\lambda}}^{(\overline{\alpha},\overline{\beta})}(t). \tag{7.16}$$

For $i\lambda \notin \mathbb{N}$, another solution of (7.12) (cf. [73, 2.9 (9)]) is given by the function

$$\Phi_\lambda^{(\alpha,\beta)}(t) = \left(e^t - e^{-t}\right)^{i\lambda-\gamma} F\left(\frac{1}{2}(\gamma - 2\alpha - i\lambda), \frac{1}{2}(\gamma - i\lambda); 1 - i\lambda; -(\sinh t)^2\right). \tag{7.17}$$

For each fixed $t > 0$, as function of λ, $\Phi_\lambda^{(\alpha,\beta)}(t)$ is holomorphic in $\mathbb{C}\backslash\{-\mathrm{i}\mathbb{N}\}$.

The functions $\varphi_\lambda^{(\alpha,\beta)}(t)$ and $\Phi_\lambda^{(\alpha,\beta)}(t)$ are called *Jacobi functions* of the first and second kind, respectively. In this section we investigate various properties of these functions.

Proposition 7.2. *The following formulae are valid.*

(i) $$\varphi_\lambda^{(\alpha,\beta)}(t) = (\cosh t)^{\mathrm{i}\lambda-\gamma} F\left(\frac{1}{2}(\gamma+\mathrm{i}\lambda), \frac{1}{2}(\gamma+\mathrm{i}\lambda)-\beta, 1+\alpha, (\tanh t)^2\right).$$

(ii) $$\big(\Gamma(\alpha+1)\big)^{-1}\frac{\mathrm{d}\varphi_\lambda^{(\alpha,\beta)}(t)}{\mathrm{d}t} = -\frac{1}{4}\big(\lambda^2+\gamma^2\big)\big(\Gamma(\alpha+2)\big)^{-1}(\sinh 2t)\varphi_\lambda^{(\alpha+1,\beta+1)}(t).$$

(iii) $$\big(\Gamma(\alpha+2)\big)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\big((\sinh 2t)^{-1}\Delta_{\alpha+1,\beta+1}(t)\varphi_\lambda^{(\alpha+1,\beta+1)}(t)\big) = 16\big(\Gamma(\alpha+1)\big)^{-1}\Delta_{\alpha,\beta}(t)\varphi_\lambda^{(\alpha,\beta)}(t).$$

(iv) $$\frac{(\cosh t)^2}{\Gamma(\alpha+1)}\varphi_\lambda^{(\alpha,\beta)}(t) = -\frac{(\sinh t)^2}{4\Gamma(\alpha+3)}\big((\alpha-\beta+3)^2+\lambda^2\big)\varphi_\lambda^{(\alpha+2,\beta-2)}(t) + \frac{1}{\Gamma(\alpha+2)}\big(\alpha+1+(\alpha-\beta+2)(\sinh t)^2\big) \times \varphi_\lambda^{(\alpha+1,\beta-1)}(t).$$

Proof. Relations (i)–(iv) are direct consequences from the well-known properties of the hypergeometric function (see Erdélyi (ed.) [73], Chap. 2, formulas 2.8 (20), 2.8 (27), 2.8 (30), and 2.9 (4)). □

Proposition 7.3.

(i) *If* $\mu, \nu \in \mathbb{C}$ *and* $\operatorname{Re}\alpha > -1$, *then*

$$\big(\mu^2-\nu^2\big)\int_0^t \Delta_{\alpha,\beta}(\xi)\varphi_\mu^{(\alpha,\beta)}(\xi)\varphi_\nu^{(\alpha,\beta)}(\xi)\,\mathrm{d}\xi = \Delta_{\alpha,\beta}(t)\left(\varphi_\nu^{(\alpha,\beta)}(t)\frac{\mathrm{d}}{\mathrm{d}t}\varphi_\mu^{(\alpha,\beta)}(t) - \varphi_\mu^{(\alpha,\beta)}(t)\frac{\mathrm{d}}{\mathrm{d}t}\varphi_\nu^{(\alpha,\beta)}(t)\right). \tag{7.18}$$

(ii) *If* $\operatorname{Re}\alpha > -1/2$, *then*

$$\varphi_\lambda^{(\alpha,\beta)}(t) = k(\alpha,\beta,t)\int_0^t \cos(\lambda x)(\cosh 2t - \cosh 2x)^{\alpha-\frac{1}{2}} \times F\left(\alpha+\beta, \alpha-\beta; \alpha+\frac{1}{2}; \frac{\cosh t - \cosh x}{2\cosh t}\right)\mathrm{d}x, \tag{7.19}$$

where

$$k(\alpha,\beta,t)=\frac{\Gamma(\alpha+1)2^{-\alpha+3/2}}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}(\sinh t)^{-2\alpha}(\cosh t)^{-\alpha-\beta}.$$

(iii) *For each* $p\in\mathbb{Z}_+$ *there exists* $\gamma_0>0$ *such that, for all* $t>0$ *and all* $\lambda\in\mathbb{C}$,

$$\left|\big(\Gamma(\alpha+1)\big)^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^p\varphi_\lambda^{(\alpha,\beta)}(t)\right|\leqslant\gamma_0(1+|\lambda|)^{k+p}(1+t)\mathrm{e}^{(|\mathrm{Im}\,\lambda|-\mathrm{Re}\,\gamma)t},$$

where $k=0$ *if* $\mathrm{Re}\,\lambda>-1/2$ *and* $k=[1/2-\mathrm{Re}\,\alpha]$ *if* $\mathrm{Re}\,\alpha\leqslant-1/2$.

Proof. Taking in (7.12) $f=\varphi_\lambda^{(\alpha,\beta)}$ and putting $\lambda=\mu,\nu$, we get

$$\begin{aligned}&\big(\mu^2-\nu^2\big)\Delta_{\alpha,\beta}(t)\varphi_\mu^{(\alpha,\beta)}(t)\varphi_\nu^{(\alpha,\beta)}(t)\\&\quad=\frac{\mathrm{d}}{\mathrm{d}t}\left(\Delta_{\alpha,\beta}(t)\left(\varphi_\nu^{(\alpha,\beta)}(t)\frac{\mathrm{d}}{\mathrm{d}t}\varphi_\mu^{(\alpha,\beta)}(t)-\varphi_\mu^{(\alpha,\beta)}(t)\frac{\mathrm{d}}{\mathrm{d}t}\varphi_\nu^{(\alpha,\beta)}(t)\right)\right).\end{aligned}$$

This yields (i). For the proof of (ii) and (iii), we refer the reader to Koornwinder [138], Sect. 2. □

Proposition 7.4. *Let* $0<a<b$, $t\in[a,b]$, *and*

$$c(\alpha,\beta,t)=\frac{\sqrt{\pi}(\sinh t)^{\alpha+1/2}(\cosh t)^{\beta+1/2}}{2^{\alpha+1/2}\Gamma(\alpha+1)}.$$

Then, for every $\varepsilon\in(0,\pi)$,

$$\begin{aligned}c(\alpha,\beta,t)\varphi_\lambda^{(\alpha,\beta)}(t)=&\frac{\cos(\lambda t-\frac{\pi}{4}(2\alpha+1))}{\lambda^{\alpha+1/2}}\\&+\frac{(\frac{1}{4}-\alpha^2)(\cosh t)^2+(\frac{1}{4}-\beta^2)(\sinh t)^2}{\sinh 2t}\\&\times\frac{\sin(\lambda t-\frac{\pi}{4}(2\alpha+1))}{\lambda^{\alpha+3/2}}+O\left(\frac{\mathrm{e}^{t|\mathrm{Im}\,\lambda|}}{\lambda^{\alpha+5/2}}\right)\end{aligned}\tag{7.20}$$

and

$$c(\alpha,\beta,t)\frac{\mathrm{d}}{\mathrm{d}\lambda}\varphi_\lambda^{(\alpha,\beta)}(t)=-t\frac{\sin(\lambda t-\frac{\pi}{4}(2\alpha+1))}{\lambda^{\alpha+1/2}}+O\left(\frac{\mathrm{e}^{t|\mathrm{Im}\,\lambda|}}{\lambda^{\alpha+3/2}}\right)$$

as $\lambda\to\infty$, $|\arg\lambda|\leqslant\pi-\varepsilon$, *where the constants in* O *depend only on* α,β,a,b, *and* ε.

Proof. If $\mathrm{Re}\,\alpha>1/2$, then the desired result follows from Proposition 7.3(ii) and asymptotic expansion of Fourier integrals (see Riekstyn'sh [182], Chap. 10.3). In the general case Proposition 7.2(iv) is applicable. □

For $\mathrm{i}\lambda\notin\mathbb{N}$, we define the sequence $\Gamma_m(\lambda)\in\mathbb{C}$, $m=0,1,\ldots$, as follows. Let $\Gamma_0(\lambda)=1$, and let

$$k(k-\mathrm{i}\lambda)\Gamma_{2k}(\lambda)=\sum_{n=0}^{k-1}(2n-\mathrm{i}\lambda+\gamma)\big(\alpha-\beta+(2\beta+1)\varepsilon_{k,n}\big)\Gamma_{2n}(\lambda),\quad k\in\mathbb{N},$$

where

$$\varepsilon_{k,n}=\begin{cases}0 & \text{if } k-n \text{ is odd,}\\ 1 & \text{otherwise.}\end{cases}$$

If m is odd, we set $\Gamma_m(\lambda)=0$.

Suppose that $E\subset\mathbb{C}$ is one of the following sets:

(a) E is a compact contained in $\mathbb{C}\backslash\{-\mathrm{i}\mathbb{N}\}$;
(b) $E=\{z\in\mathbb{C}:\operatorname{Im} z\geqslant-\zeta|\operatorname{Re} z|\}$ for some $\zeta\geqslant 0$.

It can be verified (see Flensted–Jensen [76], Lemma 7) that there exist constants $\gamma_1,\gamma_2>0$ such that

$$|\Gamma_m(\lambda)|\leqslant\gamma_1(1+m)^{\gamma_2}\quad\text{for all } m\in\mathbb{Z}_+,\lambda\in E. \tag{7.21}$$

Furthermore, for any $\lambda\in\mathbb{C}\backslash\{-\mathrm{i}\mathbb{N}\}$,

$$\Phi_\lambda^{(\alpha,\beta)}(t)=\mathrm{e}^{(\mathrm{i}\lambda-\gamma)t}\sum_{m=0}^{\infty}\Gamma_m(\lambda)\mathrm{e}^{-mt}, \tag{7.22}$$

where the expansion converges in $C^\infty(0,+\infty)$ (see (7.21) and [76]).

Next, let $p\in\mathbb{Z}_+$. Applying Proposition 6.10, we infer from (7.21) that there exists $\gamma_3>0$ such that

$$\left|\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^p\Gamma_m(\lambda)\right|\leqslant\gamma_3(1+m)^{\gamma_2}\quad\text{for all } m\in\mathbb{Z}_+,\lambda\in E.$$

This, together with (7.22), gives

$$\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^p\Phi_\lambda^{(\alpha,\beta)}(t)=(\mathrm{i}t)^p\mathrm{e}^{(\mathrm{i}\lambda-\gamma)t}\big(1+O\big(\mathrm{e}^{-2t}\big)\big)\quad\text{as } t\to+\infty \tag{7.23}$$

for each $\lambda\in\mathbb{C}\backslash\{-\mathrm{i}\mathbb{N}\}$.

Proposition 7.5. *Let* $(-\alpha)\notin\mathbb{N}$. *Then the following assertions hold.*

(i) *If* $\mathrm{i}\lambda\notin\mathbb{Z}$, *then*

$$\varphi_\lambda^{(\alpha,\beta)}(t)=c_{\alpha,\beta}(\lambda)\Phi_\lambda^{(\alpha,\beta)}(t)+c_{\alpha,\beta}(-\lambda)\Phi_{-\lambda}^{(\alpha,\beta)}(t),$$

where

$$c_{\alpha,\beta}(\lambda)=\frac{2^{\gamma-\mathrm{i}\lambda}\Gamma(\alpha+1)\Gamma(\mathrm{i}\lambda)}{\Gamma(\frac{1}{2}(\gamma+\mathrm{i}\lambda))\Gamma(\frac{1}{2}(\gamma+\mathrm{i}\lambda)-\beta)}. \tag{7.24}$$

(ii) *If* $\mathrm{i}\lambda \notin \mathbb{N}$*, then*

$$\Delta_{\alpha,\beta}(t)\left(\varphi_\lambda^{(\alpha,\beta)}(t)\frac{\mathrm{d}}{\mathrm{d}t}\Phi_\lambda^{(\alpha,\beta)}(t) - \Phi_\lambda^{(\alpha,\beta)}(t)\frac{\mathrm{d}}{\mathrm{d}t}\varphi_\lambda^{(\alpha,\beta)}(t)\right) = 2\mathrm{i}\lambda c_{\alpha,\beta}(-\lambda). \tag{7.25}$$

(iii) *If* $\alpha \pm \beta \in (-1, +\infty)$*, then there exists* $\gamma_4 > 0$ *such that*

$$\left(\varphi_\lambda^{(\alpha,\beta)}(t)\frac{\mathrm{d}}{\mathrm{d}t}\Phi_\lambda^{(\alpha,\beta)}(t) - \Phi_\lambda^{(\alpha,\beta)}(t)\frac{\mathrm{d}}{\mathrm{d}t}\varphi_\lambda^{(\alpha,\beta)}(t)\right)^{-1} \leqslant \gamma_4 \Delta_{\alpha,\beta}(t)(1+|\lambda|)^{\alpha-1/2}. \tag{7.26}$$

(iv) *If* $\alpha \geqslant 0$*,* $\mathrm{i}\lambda \notin \mathbb{N}$ *and* $\lambda c_{\alpha,\beta}(-\lambda) \neq 0$*, then*

$$\Phi_\lambda^{(\alpha,\beta)}(t) = \frac{\mathrm{i}\lambda c_{\alpha,\beta}(-\lambda)}{2^{2\gamma}} u_\alpha(t)\big(1+o(1)\big) \quad \text{as } t \to 0,$$

where

$$u_\alpha(t) = \begin{cases} -\alpha^{-1}t^{-2\alpha} & \text{if } \alpha > 0, \\ 2\log t & \text{if } \alpha = 0. \end{cases}$$

In addition,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_\lambda^{(\alpha,\beta)}(t) = \frac{\mathrm{i}\lambda c_{\alpha,\beta}(-\lambda)}{2^{2\gamma-1}} t^{-1-2\alpha}\big(1+o(1)\big) \quad \text{as } t \to 0.$$

Proof. Concerning (i) and (ii), see Koornwinder [138], formulae (2.5) and (2.6), and Flensted-Jensen [76], Lemma 8. To prove (iii) it is enough to apply (7.24), (7.25), and Stirling's formula. As for (iv), take $\varepsilon > 0$ such that (7.14) is satisfied. If $\zeta = (1+|\lambda|)^{-1}\varepsilon$, equality (7.25) allows us to write

$$\Phi_\lambda^{(\alpha,\beta)}(t) = \varphi_\lambda^{(\alpha,\beta)}(t)\left(\frac{\Phi_\lambda^{(\alpha,\beta)}(\zeta)}{\varphi_\lambda^{(\alpha,\beta)}(\zeta)} - 2\mathrm{i}\lambda c_{\alpha,\beta}(-\lambda)\int_t^\zeta \frac{\mathrm{d}\xi}{\Delta_{\alpha,\beta}(\xi)(\varphi_\lambda^{(\alpha,\beta)}(\xi))^2}\right) \tag{7.27}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_\lambda^{(\alpha,\beta)}(t) = \frac{2\mathrm{i}\lambda c_{\alpha,\beta}(-\lambda)}{\Delta_{\alpha,\beta}(t)\varphi_\lambda^{(\alpha,\beta)}(t)} + \left(\frac{\mathrm{d}}{\mathrm{d}t}\varphi_\lambda^{(\alpha,\beta)}(t)\right) \times \left(\frac{\Phi_\lambda^{(\alpha,\beta)}(\zeta)}{\varphi_\lambda^{(\alpha,\beta)}(\zeta)} - 2\mathrm{i}\lambda c_{\alpha,\beta}(-\lambda)\int_t^\zeta \frac{\mathrm{d}\xi}{\Delta_{\alpha,\beta}(\xi)(\varphi_\lambda^{(\alpha,\beta)}(\xi))^2}\right), \tag{7.28}$$

where $t \in (0, \zeta]$. Furthermore,

$$\varphi_\lambda^{(\alpha,\beta)}(t) = 1 + O\big(t^2\big) \quad \text{as } t \to 0 \tag{7.29}$$

in view of (7.15) and (7.13). To complete the proof we have only to combine (7.27)–(7.29) and (7.11). □

Corollary 7.1. *Let* $(-\alpha) \notin \mathbb{N}$, $\lambda \in \mathbb{C}\backslash\{0\}$, $\operatorname{Im}\lambda \geqslant 0$, $p \in \mathbb{Z}_+$, *and let*

$$h_p(\lambda, t) = c_{\alpha,\beta}(\lambda)\mathrm{e}^{(\mathrm{i}\lambda-\gamma)t}(\mathrm{i}t)^p.$$

Then

$$\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^p \varphi_\lambda^{(\alpha,\beta)}(t) = h_p(\lambda, t)\left(1 + O\left(\frac{1}{t}\right)\right) + (-1)^p h_p(-\lambda, t)\left(1 + O\left(\frac{1}{t}\right)\right) \quad \text{as } t \to +\infty.$$

Next, let $\tau_p^{\alpha,\beta} = 2(2p+1)^{-1}\lambda c_{\alpha,\beta}(\lambda)|_{\lambda=0} \neq 0$. *Then*

$$\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^{2p} \varphi_\lambda^{(\alpha,\beta)}(t)\bigg|_{\lambda=0} = \tau_p^{\alpha,\beta}\mathrm{e}^{-\gamma t}(\mathrm{i}t)^{2p+1}\left(1 + O\left(\frac{1}{t}\right)\right) \quad \text{as } t \to +\infty.$$

Proof. This follows from Proposition 7.5(i) and (7.23). □

For the rest of the section, we assume that

$$\alpha \geqslant 0, \quad \alpha \geqslant \beta \geqslant -\frac{1}{2}, \quad r > 0. \tag{7.30}$$

The Taylor expansion of $F(\alpha+\beta, \alpha-\beta; \alpha+\frac{1}{2}; z)$ at the origin implies, by (7.30), that

$$F\left(\alpha+\beta, \alpha-\beta; \alpha+\frac{1}{2}; z\right) > 0 \quad \text{if } z \in [0, 1/2]. \tag{7.31}$$

Proposition 7.6. *For fixed* α, β, r, *the following statements are valid.*

(i) *The function* $\varphi_\lambda^{(\alpha,\beta)}(r)$ *has infinitely many zeros. All the zeros of* $\varphi_\lambda^{(\alpha,\beta)}(r)$ *are real and simple; they are located symmetrically relative to* $\lambda = 0$. *In addition,* $\varphi_\lambda^{(\alpha,\beta)}(r) > 0$, *provided that* $\mathrm{i}\lambda \in \mathbb{R}^1$.

(ii) *Let* $\lambda_l = \lambda_l(\alpha, \beta, r)$, $l = 1, 2, \dots$, *be the sequence of all positive zeros of* $\varphi_\lambda^{(\alpha,\beta)}(r)$ *numbered in the ascending order, and suppose that* $0 < r_1 \leqslant r \leqslant r_2$. *Then*

$$r\lambda_l = \pi\left(\frac{2\alpha+3}{4} + l + q(r, \alpha, \beta)\right) + \frac{(1/4-\alpha^2)(\cosh r)^2 + (1/4-\beta^2)(\sinh r)^2}{\lambda_l \sinh 2r} + O\left(\frac{1}{\lambda_l^3}\right),$$

where $q(r, \alpha, \beta) \in \mathbb{Z}$ *does not depend on l, and the constant in O depends only on* α, β, r_1, r_2.

Proof. To show (i), first assume that $i\lambda \in \mathbb{R}^1$. Then $\varphi_\lambda^{(\alpha,\beta)}(r) > 0$ because of (7.19) and (7.31). Suppose now that $\lambda \in \mathbb{C}$ and $\varphi_\lambda^{(\alpha,\beta)}(r) = 0$. We claim that $\lambda \in \mathbb{R}^1$ and

$$\frac{\mathrm{d}}{\mathrm{d}z}\varphi_z^{(\alpha,\beta)}(r)\bigg|_{z=\lambda} \neq 0.$$

Assume that $\lambda \notin \mathbb{R}^1$. By the above argument, $i\lambda \notin \mathbb{R}^1$, whence $\lambda^2 \neq \overline{\lambda}^2$. Putting $\mu = \overline{\nu} = \lambda$ in (7.18) and taking (7.16) into account, we get

$$\int_0^r \Delta_{\alpha,\beta}(\xi)\left|\varphi_\lambda^{(\alpha,\beta)}(\xi)\right|^2 \mathrm{d}\xi = 0, \tag{7.32}$$

which is impossible. Now assume that

$$\frac{\mathrm{d}}{\mathrm{d}z}\varphi_z^{(\alpha,\beta)}(r)\bigg|_{z=\lambda} = 0.$$

Putting $\mu = \lambda$ in (7.18) and letting $\nu \to \lambda$, one obtains (7.32) once again. Thus, all the zeros of $\varphi_\lambda^{(\alpha,\beta)}(r)$ are real and simple. Since $\varphi_\lambda^{(\alpha,\beta)}(r)$ is even, this completes the proof of (i).

Let us prove (ii). Propositions 6.2 and 7.4 lead to the conclusion that if $R > 0$, then the number of zeros of $\varphi_\lambda^{(\alpha,\beta)}(r)$ in $[-R, R]$ does not exceed $\gamma_5(1 + R)$, where $\gamma_5 > 0$ depends only on r_1, r_2, α, β. Combining this with Proposition 7.4, we infer that

$$r\lambda_l = \pi\left(\frac{2\alpha + 3}{4} + l + q(r, \alpha, \beta)\right) + \varepsilon_l(r, \alpha, \beta), \tag{7.33}$$

where $q(r, \alpha, \beta) \in \mathbb{Z}$, and $\varepsilon_l(r, \alpha, \beta) \to 0$ uniformly with respect to $r \in [r_1, r_2]$ as $l \to +\infty$. Next, using (7.19) and (7.31), one concludes that

$$\lambda_l \geqslant \frac{\pi}{2r} \geqslant \frac{\pi}{2r_2} \quad \text{for all } l. \tag{7.34}$$

Putting $\lambda = \lambda_l$ in (7.20) and applying (7.33), (7.34), we arrive at (ii). □

Let us now define

$$N_{\alpha,\beta}(r) = \left\{\lambda > 0 : \varphi_\lambda^{(\alpha,\beta)}(r) = 0\right\}.$$

The proof of the following fact will occupy the remainder of this section.

Theorem 7.1. *Let $k \in \mathbb{N}$, $k > \alpha + 3/2$, and assume that a function $u : [0, r] \to \mathbb{C}$ enjoys the following properties:*

(1) $L_{\alpha,\beta}^s u \in C^2[0, r]$ *for* $s = 0, 1, \ldots, k-1$, *and* $L_{\alpha,\beta}^k u \in C[0, r]$.
(2) $(L_{\alpha,\beta}^s u)(r) = 0$ *for* $s = 0, 1, \ldots, k-1$.

For $\lambda \in N_{\alpha,\beta}(r)$*, we set*

$$c_\lambda(u) = \left(\int_0^r \Delta_{\alpha,\beta}(t)\big(\varphi_\lambda^{(\alpha,\beta)}(t)\big)^2\,\mathrm{d}t\right)^{-1}\int_0^r \Delta_{\alpha,\beta}(t)u(t)\varphi_\lambda^{(\alpha,\beta)}(t)\,\mathrm{d}t. \tag{7.35}$$

Then

$$c_\lambda(u) = O\big(\lambda^{2\alpha+2-2k}\big) \quad \textit{as } \lambda \to +\infty \tag{7.36}$$

and

$$u(t) = \sum_{\lambda\in N_{\alpha,\beta}(r)} c_\lambda(u)\varphi_\lambda^{(\alpha,\beta)}(t), \quad t \in [0,r], \tag{7.37}$$

where the series in (7.37) *converges to* u *in the space* $C^s[0,r]$ *with* $s < 2k-2\alpha-3$*.*

To prove the theorem a couple of lemmas will be needed.

Lemma 7.1. *Let* $g \in L^1[0,r]$ *and*

$$f(z) = \int_0^r \Delta_{\alpha,\beta}(t)g(t)\varphi_z^{(\alpha,\beta)}(t)\,\mathrm{d}t, \quad z \in \mathbb{C}. \tag{7.38}$$

If $f(\lambda) = 0$ *for all* $\lambda \in N_{\alpha,\beta}(r)$*, then* $g = 0$*.*

Proof. Consider the function $w(z) = f(z)/\varphi_z^{(\alpha,\beta)}(r)$. By Propositions 7.6(i) and 7.3(iii), w is an even entire function whose order does not exceed 1. Let $A = \{z \in \mathbb{C} : |\mathrm{Im}\,z| = |\mathrm{Re}\,z|\}$. By an appeal to Proposition 7.4 we find from (7.38) and (7.20) that

$$|w(z)| \leqslant \gamma_5(1+|z|)^{\alpha+1/2} \times \left(\mathrm{e}^{-r|\mathrm{Im}\,z|/2} + \mathrm{e}^{-r|\mathrm{Im}\,z|}\left|\int_{r/2}^r \Delta_{\alpha,\beta}(t)g(t)\varphi_z^{(\alpha,\beta)}(t)\mathrm{d}t\right|\right), \quad z \in A, \tag{7.39}$$

where γ_5 is independent of z. Using now (7.20) with $t \in [r/2, r]$, one sees from the Riemann–Lebesgue lemma that

$$\int_{r/2}^r \Delta_{\alpha,\beta}(t)g(t)\varphi_z^{(\alpha,\beta)}(t)\,\mathrm{d}t = o\big(z^{-\alpha-1/2}\mathrm{e}^{r|\mathrm{Im}\,z|}\big)$$

as $z \to \infty$, $z \in A$. Then, thanks to (7.39), $w(z) \to 0$ as $z \to \infty$ along the straight lines $\mathrm{Im}\,z = \pm\mathrm{Re}\,z$. By the Phragmén–Lindelöf principle $w = 0$, and so $f(\mathrm{i}(\gamma + 2m)) = 0$ for all $m \in \mathbb{Z}_+$. Going to the definition of $\varphi_\lambda^{(\alpha,\beta)}$, we find from (7.38) that

$$\int_0^r \Delta_{\alpha,\beta}(t)g(t)(\sinh t)^{2m}\,\mathrm{d}t = 0, \quad m \in \mathbb{Z}_+.$$

The desired statement is now clear. □

Lemma 7.2. *Let* $\lambda, \mu \in N_{\alpha,\beta}(r)$ *and*

$$\delta(\lambda, \mu) = \int_0^r \Delta_{\alpha,\beta}(t)\varphi_\lambda^{(\alpha,\beta)}(t)\varphi_\mu^{(\alpha,\beta)}(t)\,dt. \tag{7.40}$$

Then $\delta(\lambda, \mu) = 0$ *if* $\lambda \neq \mu$ *and*

$$\delta(\lambda, \lambda) > \gamma_6 \lambda^{-2(\alpha+1)},$$

where $\gamma_6 > 0$ *does not depend on* λ.

Proof. For $\lambda \neq \mu$, the assertion follows from Proposition 7.3(i). Next, let us choose $\varepsilon > 0$ such that for all $\lambda \in N_{\alpha,\beta}(r)$, condition (7.14) is fulfilled. If $\zeta = (1 + \lambda)^{-1}\varepsilon < r$, we have

$$\delta(\lambda, \lambda) > \int_0^\zeta \Delta_{\alpha,\beta}(t)\big(\varphi_\lambda^{(\alpha,\beta)}(t)\big)^2\,dt.$$

Combining this with (7.14) and (7.11), one obtains the desired estimate for $\delta(\lambda, \lambda)$. □

Proof of Theorem 7.1. For $\lambda \in N_{\alpha,\beta}(r)$, let $\delta(\lambda, \lambda)$ be defined by (7.40). Bearing Proposition 7.2(ii), (iii) in mind and integrating in (7.35) by part, we get

$$c_\lambda(u) = \frac{(-1)^k}{\delta(\lambda, \lambda)(\lambda^2 + \gamma^2)^k} \int_0^r \Delta_{\alpha,\beta}(t)\big(L_{\alpha,\beta}^k u\big)(t)\varphi_\lambda^{(\alpha,\beta)}(t)\,dt$$

(see properties (1) and (2) of the function u). Once Proposition 7.3(iii) and Lemma 7.2 are established, it is easy to find (7.36). Due to Proposition 7.3(iii), the series in (7.37) converges in $C^s[0, r]$ for each $s < 2k - 2\alpha - 3$. Hence, the function

$$g(t) = u(t) - \sum_{\lambda \in N_{\alpha,\beta}(r)} c_\lambda(u)\varphi_\lambda^{(\alpha,\beta)}(t)$$

satisfies

$$\int_0^r \Delta_{\alpha,\beta}(t)g(t)\varphi_\lambda^{(\alpha,\beta)}(t)\,dt = 0$$

for all $\lambda \in N_{\alpha,\beta}(r)$ (see Lemma 7.2). Now it follows by Lemma 7.1 that $g = 0$, yielding the proof. □

7.3 Extension of the Jacobi Polynomials

A Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ ($n \in \mathbb{Z}_+, \alpha, \beta \in (-1, +\infty)$) is given by

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} F\Big(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\Big).$$

The map $n \to P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$ extends to a holomorphic function on $\mathbb{C}$ defined by the same formula with n replaced by $\lambda \in \mathbb{C}$. Here we study the main properties of this function.

Consider, for $\alpha, \beta, \lambda \in \mathbb{C}$ and $0 < t < \pi/2$, the differential equation

$$(l_{\alpha,\beta} f)(t) = \big((\alpha+\beta+1)^2 - \lambda^2\big) f(t), \tag{7.41}$$

where

$$(l_{\alpha,\beta} f)(t) = \frac{1}{A_{\alpha,\beta}(t)} \frac{\mathrm{d}}{\mathrm{d}t}\left(A_{\alpha,\beta}(t) \frac{\mathrm{d}f(t)}{\mathrm{d}t}\right)$$

with $A_{\alpha,\beta}(t) = (\sin t)^{2\alpha+1}(\cos t)^{2\beta+1}$. By substituting $x = \sin^2 t$ into (7.41) a hypergeometric differential equation is obtained with parameters $(\alpha+\beta+1+\lambda)/2$, $(\alpha+\beta+1-\lambda)/2$, $\alpha+1$ (see Erdélyi (ed.) [73, 2.1 (1)]). Hence, if $\alpha \neq -1, -2, -3, \ldots$ then the function

$$\varphi_{\lambda,\alpha,\beta}(t) = F\left(\frac{\alpha+\beta+1+\lambda}{2}, \frac{\alpha+\beta+1-\lambda}{2}; \alpha+1; \sin^2 t\right) \tag{7.42}$$

is the solution of (7.41) which satisfies

$$\varphi_{\lambda,\alpha,\beta}(0) = 1, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\varphi_{\lambda,\alpha,\beta}(t)\bigg|_{t=0} = 0. \tag{7.43}$$

Application of [73, 2.8 (20) and 2.8 (27)] gives for this solution the differentiation formulas

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_{\lambda,\alpha,\beta}(t) = \frac{(\alpha+\beta+1)^2 - \lambda^2}{4(\alpha+1)} \sin(2t)\varphi_{\lambda,\alpha+1,\beta+1}(t) \tag{7.44}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(A_{\alpha+1/2,\beta+1/2}(t)\varphi_{\lambda,\alpha+1,\beta+1}(t)\big) = 2(\alpha+1)A_{\alpha,\beta}(t)\varphi_{\lambda,\alpha,\beta}(t). \tag{7.45}$$

Next, taking in (7.41) $f = \varphi_{\lambda,\alpha,\beta}$ and putting $\lambda = \mu, \nu$, we get

$$\begin{aligned}&\big(\mu^2 - \nu^2\big)A_{\alpha,\beta}(t)\varphi_{\nu,\alpha,\beta}(t)\varphi_{\mu,\alpha,\beta}(t)\\ &\quad = \frac{\mathrm{d}}{\mathrm{d}t}\left(A_{\alpha,\beta}(t)\left(\varphi_{\mu,\alpha,\beta}(t)\frac{\mathrm{d}}{\mathrm{d}t}\varphi_{\nu,\alpha,\beta}(t) - \varphi_{\nu,\alpha,\beta}(t)\frac{\mathrm{d}}{\mathrm{d}t}\varphi_{\mu,\alpha,\beta}(t)\right)\right).\end{aligned}$$

Consequently, for $\operatorname{Re}\alpha > -1$,

$$\begin{aligned}&\big(\mu^2 - \nu^2\big)\int_0^t A_{\alpha,\beta}(x)\varphi_{\nu,\alpha,\beta}(x)\varphi_{\mu,\alpha,\beta}(x)\,\mathrm{d}x\\ &\quad = A_{\alpha,\beta}(t)\left(\varphi_{\mu,\alpha,\beta}(t)\frac{\mathrm{d}}{\mathrm{d}t}\varphi_{\nu,\alpha,\beta}(t) - \varphi_{\nu,\alpha,\beta}(t)\frac{\mathrm{d}}{\mathrm{d}t}\varphi_{\mu,\alpha,\beta}(t)\right).\end{aligned} \tag{7.46}$$

Now we present an analogue of formula (7.19) for the function $\varphi_{\lambda,\alpha,\beta}(t)$.

Proposition 7.7. *If* $\operatorname{Re}\alpha > -1/2$, *then*

$$\begin{aligned}\varphi_{\lambda,\alpha,\beta}(t) = {} & \frac{2^{\alpha+1/2}\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}(\sin t)^{-2\alpha}(\cos t)^{-\beta-1/2} \\ & \times \int_0^t \cos(\lambda x)(\cos x - \cos t)^{\alpha-1/2} \\ & \times F\left(\frac{1}{2}+\beta, \frac{1}{2}-\beta; \alpha+\frac{1}{2}; \frac{\cos t - \cos x}{2\cos t}\right) \mathrm{d}x. \end{aligned} \tag{7.47}$$

To prove Proposition 7.7 we require three lemmas. Put

$$R_\lambda^{(\alpha,\beta)}(\cos 2t) = \varphi_{2\lambda+\alpha+\beta+1,\alpha,\beta}(t). \tag{7.48}$$

Lemma 7.3. *For* $\operatorname{Re}(\alpha+\beta) < 0$, $\operatorname{Re}(\lambda+\alpha+\beta+1) > 0$, *we have*

$$\begin{aligned} R_\lambda^{(\alpha,\beta)}(\cos 2t) = {} & \frac{2\Gamma(\lambda+1)}{\Gamma(-\alpha-\beta)\Gamma(\lambda+\alpha+\beta+1)}(\cos t)^{-2\alpha-2\beta-2} \\ & \times \int_0^t \left(\frac{\sin x}{\sin t}\right)^{2\lambda} \frac{(\tan x)^{2\alpha+2\beta+1}}{(\tan^2 t - \tan^2 x)^{\alpha+\beta+1}} R_\lambda^{(\alpha,-\alpha)}(\cos 2x)\,\mathrm{d}x. \end{aligned} \tag{7.49}$$

Proof. It follows from Askey and Fitch [6, (2.10)] that for $x > 0$, $\operatorname{Re}\mu > 0$, and $\operatorname{Re} b > 0$,

$$\Gamma(b)x^{b+\mu-1}F(a,b;c;-x) = \frac{\Gamma(b+\mu)}{\Gamma(\mu)}\int_0^x t^{b-1}F(a,b+\mu;c;-t)(x-t)^{\mu-1}\,\mathrm{d}t. \tag{7.50}$$

When $x = \tan^2 t$, $a = \lambda+\alpha+1$, $b = \lambda+\alpha+\beta+1$, $c = \alpha+1$, and $\mu = -\alpha-\beta$, (7.50) becomes

$$\begin{aligned} & (\cos t)^{-2\lambda-2\alpha-2\beta-2}F\big(\lambda+\alpha+1, \lambda+\alpha+\beta+1; \alpha+1; -\tan^2 t\big) \\ & = \frac{2\Gamma(\lambda+1)}{\Gamma(-\alpha-\beta)\Gamma(\lambda+\alpha+\beta+1)}(\cos t)^{-2\alpha-2\beta-2}\int_0^t \left(\frac{\sin x}{\sin t}\right)^{2\lambda} \\ & \quad \times \frac{(\tan x)^{2\alpha+2\beta+1}}{(\tan^2 t - \tan^2 x)^{\alpha+\beta+1}}(\cos x)^{-2\lambda-2} \\ & \quad \times F\big(\lambda+\alpha+1, \lambda+1; \alpha+1; -\tan^2 x\big)\,\mathrm{d}x. \end{aligned} \tag{7.51}$$

Using (7.51) and the identity

$$F(a,b;c;z) = (1-z)^{-b}F\left(c-a,b;c;\frac{z}{z-1}\right) \tag{7.52}$$

(see [73, 2.9 (4)]), we arrive at (7.49). □

Lemma 7.4. *Suppose that* $\operatorname{Re}\alpha > -1/2$, $-1 < \operatorname{Re}(\alpha+\beta) < 0$, *and* $\operatorname{Re}\lambda \geqslant 0$. *Then*

$$\left|R_\lambda^{(\alpha,\beta)}(\cos 2t)\right| \leqslant c_1(1+|\lambda|)^{-\operatorname{Re}(\alpha+\beta)}\mathrm{e}^{2t|\operatorname{Im}\lambda|}, \tag{7.53}$$

where c_1 is independent of λ.

Proof. By the Mehler–Dirichlet formula [73, 3.7 (27) and 3.4 (6)]

$$R_\lambda^{(\alpha,-\alpha)}(\cos 2x) = \frac{2^{1/2-\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}(\sin x)^{-2\alpha} \times \int_0^{2x}\big(\cos y - \cos(2x)\big)^{\alpha-1/2}\cos\left(\left(\lambda+\frac{1}{2}\right)y\right)\mathrm{d}y.$$

Therefore,

$$\begin{aligned}\left|R_\lambda^{(\alpha,-\alpha)}(\cos 2x)\right| &\leqslant c_2\frac{\mathrm{e}^{2x|\operatorname{Im}\lambda|}}{(\sin x)^{2\operatorname{Re}\alpha}}\int_0^{2x}\big(\cos y-\cos(2x)\big)^{\operatorname{Re}\alpha-1/2}\,\mathrm{d}y\\ &\leqslant c_2\frac{\mathrm{e}^{2x|\operatorname{Im}\lambda|}}{(\sin x)^{2\operatorname{Re}\alpha}\sqrt{1+\cos 2x}}\\ &\quad\times\int_{\cos(2x)}^1\big(y-\cos(2x)\big)^{\operatorname{Re}\alpha-1/2}(1-y)^{-1/2}\,\mathrm{d}y\\ &= c_3\frac{\mathrm{e}^{2x|\operatorname{Im}\lambda|}}{\cos x},\end{aligned} \tag{7.54}$$

where c_2, c_3 depend only on α. From (7.54) and (7.49) we conclude that

$$\left|R_\lambda^{(\alpha,\beta)}(\cos 2t)\right| \leqslant c_4\left|\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\alpha+\beta+1)}\right|\mathrm{e}^{2t|\operatorname{Im}\lambda|},$$

where c_4 does not depend on λ. This proves (7.53), since

$$\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\alpha+\beta+1)} = \lambda^{-\alpha-\beta}\big(1+O\big(\lambda^{-1}\big)\big)$$

(see [73, 1.18 (4)]). □

Lemma 7.5. *Let f be an entire function. Assume that for some $a \in (0,\pi)$,*

$$f(\lambda) = O\big(\mathrm{e}^{a|\lambda|}\big) \quad \text{if } \operatorname{Re}\lambda \geqslant 0 \tag{7.55}$$

and $f(\lambda)=0$ if $\lambda \in \mathbb{Z}_+$. Then $f \equiv 0$.

Lemma 7.5 is due to Carlson. A proof can be found in Titchmarsh [212, Chap. 5]. For other results of a similar nature and their generalizations, see Levin [145, Chap. 4, Sect. 5; 146, Lecture 21], Agmon [2], Gel'fond [82, Chap. 2, Theorem 7], and the references there.

Proof of Proposition 7.7. Let $-1 < \operatorname{Re}(\alpha+\beta) < 0$. Set

$$f(\lambda) = R_\lambda^{(\alpha,\beta)}(\cos 2t) - g(2\lambda+\alpha+\beta+1),$$

where $g(\lambda)$ is the function on the right-hand side of (7.47). According to Koornwinder [138, (5.8)], $f(\lambda) = 0$ for $\lambda \in \mathbb{Z}_+$. In addition, owing to (7.53), the function f satisfies (7.55). Hence, by Lemma 7.5

$$R_\lambda^{(\alpha,\beta)}(\cos 2t) = g(2\lambda+\alpha+\beta+1). \tag{7.56}$$

Because f is an entire function with respect to β, equality (7.56) holds for all $\beta \in \mathbb{C}$ by analytic continuation. Thereby the proposition is established (see (7.48)). □

Corollary 7.2. *Let* $\operatorname{Re}\alpha > -1/2$*,* $\beta \in \mathbb{C}$*, and let*

$$c(\alpha) = \begin{cases} [\operatorname{Re}\alpha + 1/2] & \text{if } \operatorname{Re}\alpha + 1/2 \notin \mathbb{N}, \\ [\operatorname{Re}\alpha] & \text{if } \operatorname{Re}\alpha + 1/2 \in \mathbb{N}. \end{cases} \tag{7.57}$$

Then for any integer $0 \leqslant s \leqslant c(\alpha)$*, there exists a positive constant* $c(\alpha,\beta,s)$ *such that*

$$|\varphi_{\lambda,\alpha,\beta}(t)| \leqslant \frac{c(\alpha,\beta,s)}{(\sin t)^{2s}(\cos t)^{s+\operatorname{Re}\beta+|\operatorname{Re}\beta|+1/2}} \frac{e^{t|\operatorname{Im}\lambda|}}{|\lambda|^s} \tag{7.58}$$

for all $t \in (0, \pi/2)$ *and all* $\lambda \in \mathbb{C}\backslash\{0\}$.

Proof. For brevity, we set

$$f(y) = (y-\cos t)^{\alpha-1/2} F\left(\frac{1}{2}+\beta, \frac{1}{2}-\beta; \alpha+\frac{1}{2}; \frac{\cos t - y}{2\cos t}\right).$$

Repeated integration by parts gives

$$\int_{-t}^{t} e^{i\lambda x} f(\cos x)\, dx = \left(\frac{i}{\lambda}\right)^s \int_{-t}^{t} e^{i\lambda x} \left(\frac{d}{dx}\right)^s \big(f(\cos x)\big)\, dx.$$

Hence, in view of (7.47),

$$|\varphi_{\lambda,\alpha,\beta}(t)| \leqslant \frac{2^{\operatorname{Re}\alpha-1/2}}{\sqrt{\pi}} \left|\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1/2)}\right| (\sin t)^{-2\operatorname{Re}\alpha} (\cos t)^{-\operatorname{Re}\beta-1/2}$$
$$\times \frac{e^{t|\operatorname{Im}\lambda|}}{|\lambda|^s} \int_{-t}^{t} \left|\left(\frac{d}{dx}\right)^s \big(f(\cos x)\big)\right| dx. \tag{7.59}$$

Next, owing to Proposition 7.3(iii),

$$\left|F\left(\frac{1}{2}+\beta+j,\frac{1}{2}-\beta+j;\alpha+\frac{1}{2}+j;\frac{\cos t-\cos x}{2\cos t}\right)\right|$$
$$\leqslant c_5\left(\frac{\cos x}{\cos t}\right)^{|\mathrm{Re}\,\beta|},\quad j\in\mathbb{Z}_+, \tag{7.60}$$

where c_5 depends only on α,β,j. Combining (7.59), (7.60), and (7.54), we obtain the desired estimate. □

Proposition 7.8. *Let $\alpha,\beta\in\mathbb{C}$. Suppose also that $a,b\in(0,\pi/2)$ are fixed and $t\in[a,b]$. Then for every $\varepsilon\in(0,\pi)$,*

$$\frac{\sqrt{\pi}(\sin t)^{\alpha+1/2}(\cos t)^{\beta+1/2}}{2^{\alpha+1/2}\Gamma(\alpha+1)}\varphi_{\lambda,\alpha,\beta}(t)$$
$$=\frac{\cos(\lambda t-\frac{\pi}{4}(2\alpha+1))}{\lambda^{\alpha+1/2}}+\frac{(\frac{1}{4}-\alpha^2)\cot t+(\beta^2-\frac{1}{4})\tan t}{2}$$
$$\times\frac{\sin(\lambda t-\frac{\pi}{4}(2\alpha+1))}{\lambda^{\alpha+3/2}}+O\left(\frac{e^{t|\mathrm{Im}\,\lambda|}}{\lambda^{\alpha+5/2}}\right) \tag{7.61}$$

and

$$\frac{\sqrt{\pi}(\sin t)^{\alpha+1/2}(\cos t)^{\beta+1/2}}{2^{\alpha+1/2}\Gamma(\alpha+1)}\frac{d}{d\lambda}\varphi_{\lambda,\alpha,\beta}(t)$$
$$=-t\frac{\sin(\lambda t-\frac{\pi}{4}(2\alpha+1))}{\lambda^{\alpha+1/2}}+O\left(\frac{e^{t|\mathrm{Im}\,\lambda|}}{\lambda^{\alpha+3/2}}\right) \tag{7.62}$$

as $\lambda\to\infty$, $|\arg\lambda|\leqslant\pi-\varepsilon$, where the constants in O depend only on α,β,a,b, and ε.

Proof. For $\mathrm{Re}\,\alpha>-1/2$ and $\beta\in\mathbb{C}$, the desired statement follows from Proposition 7.7 and the asymptotic expansion of Fourier integrals (see Riekstyn'sh [182], Chap. 2, Theorem 10.2). Now applying the formula

$$\frac{\cos^2 t}{\Gamma(\alpha+1)}\varphi_{\lambda,\alpha,\beta}(t)=\frac{\sin^2 t}{4\Gamma(\alpha+3)}\big((\alpha-\beta+3)^2-\lambda^2\big)\varphi_{\lambda,\alpha+2,\beta-2}(t)$$
$$+\frac{1}{\Gamma(\alpha+2)}\big(\alpha+1-(\alpha-\beta+2)\sin^2 t\big)\varphi_{\lambda,\alpha+1,\beta-1}(t)$$

(see [73, 2.8 (30)]), we obtain (7.61) and (7.62) in general. □

For the rest of the section, we assume that $\alpha\in(-1/2,+\infty)$, $\beta\in\mathbb{R}^1$, and $r\in(0,\pi/2)$. Let us describe some properties of the set $\{\lambda\in\mathbb{C}:\varphi_{\lambda,\alpha,\beta}(r)=0\}$.

Proposition 7.9.

(i) *The function* $\varphi_{\lambda,\alpha,\beta}(r)$ *has infinitely many zeroes. All the zeroes of* $\varphi_{\lambda,\alpha,\beta}(r)$ *are real, simple, and the set of these zeroes is symmetric with respect to* $\lambda = 0$.

(ii) *Let* $\lambda_l = \lambda_l(\alpha, \beta, r)$, $l \in \mathbb{N}$, *be the sequence of all positive zeroes of* $\varphi_{\lambda,\alpha,\beta}(r)$ *numbered in the ascending order and assume that* $0 < a \leqslant r \leqslant b < \pi/2$. *Then*

$$r\lambda_l = \pi\left(\frac{2\alpha+3}{4} + l + q(r,\alpha,\beta)\right) + \frac{(1/4-\alpha^2)\cot r + (\beta^2 - 1/4)\tan r}{2\lambda_l} + O\left(\frac{1}{\lambda_l^3}\right),$$

where $q(r,\alpha,\beta)$ *belongs to* $\mathbb{Z}$ *and does not depend on* l, *and the constant in* O *depends only on* α, β, a, b.

This proposition can be proved in much the same way as Proposition 7.6 (see (7.42)–(7.47) and (7.61)).

We define

$$\mathcal{N}_{\alpha,\beta}(r) = \{\lambda > 0 \colon \varphi_{\lambda,\alpha,\beta}(r) = 0\}. \tag{7.63}$$

Proposition 7.10. *Let* $k \in \mathbb{N}$, $k > \alpha + (3 - c(\alpha))/2$ (*see* (7.57)). *Suppose that a function* u *satisfies the following conditions:*

(1) $l^s_{\alpha,\beta}u \in C^2[0,r]$ *if* $s = 0, 1, \ldots, k-1$, *and* $l^k_{\alpha,\beta}u \in C[0,r]$;

(2) $(l^s_{\alpha,\beta}u)(r) = 0$, $s = 0, 1, \ldots, k-1$.

For $\lambda \in \mathcal{N}_{\alpha,\beta}(r)$, *put*

$$c_\lambda(u) = \left(\int_0^r A_{\alpha,\beta}(t)\varphi^2_{\lambda,\alpha,\beta}(t)\,\mathrm{d}t\right)^{-1} \int_0^r A_{\alpha,\beta}(t)u(t)\varphi_{\lambda,\alpha,\beta}(t)\,\mathrm{d}t.$$

Then

$$c_\lambda(u) = O\big(\lambda^{2\alpha+2-2k-c(\alpha)}\big) \quad \text{as } \lambda \to +\infty$$

and

$$u(t) = \sum_{\lambda\in\mathcal{N}_{\alpha,\beta}(r)} c_\lambda(u)\varphi_{\lambda,\alpha,\beta}(t), \quad t \in [0,r], \tag{7.64}$$

where the series in (7.64) *converges to* u *in the space* $C^s[0,r]$ *with* $s < k - \alpha - (3 - c(\alpha))/2$.

The proof of Proposition 7.10 is similar to that of Theorem 7.1 (see (7.44), (7.45), (7.58) and Lemmas 7.6 and 7.7 below).

Lemma 7.6. *Assume that* $v \in L^1[0,r]$ *and*

$$\int_0^r A_{\alpha,\beta}(t)v(t)\varphi_{\lambda,\alpha,\beta}(t)\,\mathrm{d}t = 0$$

for all $\lambda \in \mathcal{N}_{\alpha,\beta}(r)$. *Then* $v = 0$.

Lemma 7.7. *Let* $\lambda, \mu \in \mathcal{N}_{\alpha,\beta}(r)$*, and let*

$$\delta(\lambda, \mu) = \int_0^r A_{\alpha,\beta}(t)\varphi_{\lambda,\alpha,\beta}(t)\varphi_{\mu,\alpha,\beta}(t)\,\mathrm{d}t.$$

Then $\delta(\lambda, \mu) = 0$ *if* $\lambda \neq \mu$*, and* $\delta(\lambda, \lambda)\lambda^{2\alpha+2} > c_6$*, where* $c_6 > 0$ *does not depend on* λ*.*

In view of (7.42), (7.58), (7.61) and Proposition 7.9, to prove these lemmas we can essentially use the same arguments as in Lemmas 7.1 and 7.2. We leave the details for the reader.

7.4 Confluent Hypergeometric Functions

In this section we present concepts and facts concerning confluent hypergeometric functions which will be used in the sequel.

Let $a, \zeta \in \mathbb{C}$, $b \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$. Set

$$ {}_1F_1(a; b; \zeta) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{\zeta^k}{k!}. \tag{7.65}$$

The function ${}_1F_1$ is said to be the *Kummer confluent hypergeometric function*. The *Tricomi confluent hypergeometric function* $\Psi(a, b; \zeta)$ is defined for $a, b \in \mathbb{C}$ and $\zeta \in \mathbb{C} \setminus \{0\}$ as follows:

$$\Psi(a, b; \zeta) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)}{}_1F_1(a; b; \zeta) + \frac{\Gamma(b-1)}{\Gamma(a)}\zeta^{1-b}{}_1F_1(a-b+1; 2-b; \zeta)$$

if $b \notin \mathbb{Z}$,

$$\begin{aligned}\Psi(a, b; \zeta) &= \lim_{\beta \to b} \Psi(a, \beta; \zeta) \\ &= (-1)^b \frac{\partial}{\partial \beta}\Bigg(\frac{{}_1F_1(a; \beta; \zeta)}{\Gamma(\beta)\Gamma(a-\beta+1)} \\ &\qquad - \frac{\zeta^{1-\beta}{}_1F_1(a-\beta+1; 2-\beta; \zeta)}{\Gamma(a)\Gamma(2-\beta)}\Bigg)\Bigg|_{\beta=b}\end{aligned}$$

if $b \in \mathbb{N}$, and

$$\Psi(a, b; \zeta) = \zeta^{1-\beta}\Psi(a-b+1, 2-b; \zeta)$$

if $b \in \mathbb{Z}\setminus\mathbb{N}$.

Finally, the functions

$$M_{\alpha,\beta}(\zeta) = \zeta^{\beta+1/2}\mathrm{e}^{-\zeta/2}{}_1F_1(1/2-\alpha+\beta; 1+2\beta; \zeta),$$

$$W_{\alpha,\beta}(\zeta) = \zeta^{\beta+1/2}e^{-\zeta/2}\Psi(1/2-\alpha+\beta, 1+2\beta; \zeta)$$

are termed the *Whittaker confluent hypergeometric functions* (see Erdélyi (ed.) [73], Chap. 6).

Note that for $b \in \mathbb{N}$,

$$\begin{aligned}\Psi(a,b;\zeta) = \frac{(-1)^b}{\Gamma(b)\Gamma(a-b+1)}\Bigg(& {}_1F_1(a;b;\zeta)\log\zeta \\ & + \sum_{k=0}^{\infty}\frac{(a)_k}{(b)_k}\big(\psi(a+k)-\psi(k+1)-\psi(b+k)\big)\frac{\zeta^k}{k!}\Bigg) \\ & + \sum_{k=0}^{b-2}\frac{\Gamma(b-1)}{\Gamma(a)}\frac{(a-b+1)_k}{(2-b)_k}\frac{\zeta^{k-b+1}}{k!},\end{aligned} \tag{7.66}$$

where ψ is the logarithmic derivative of the gamma function, and the second sum in (7.66) is set to be equal to zero for $b = 1$.

The functions ${}_1F_1(a;b;\zeta)$ and $\Psi(a,b;\zeta)$ satisfy the equation

$$\zeta u''(\zeta) + (b-\zeta)u'(\zeta) - au(\zeta) = 0. \tag{7.67}$$

Analogously the functions $M_{\alpha,\beta}(\zeta)$ and $W_{\alpha,\beta}(\zeta)$ satisfy the equation

$$v''(\zeta) + \left(-\frac{1}{4} + \frac{\alpha}{\zeta} + \frac{1/4-\beta^2}{\zeta^2}\right)v(\zeta) = 0. \tag{7.68}$$

Equations (7.67) and (7.68) are called *confluent hypergeometric equations*.

Consider some elementary relations for ${}_1F_1$ and Ψ. We have [73, Chap. 6]:

$$\begin{aligned}& b(b-1){}_1F_1(a;b-1;\zeta) - b(b-1+\zeta){}_1F_1(a;b;\zeta) \\ & \quad + (b-a)\zeta\,{}_1F_1(a;b+1;\zeta) = 0,\end{aligned} \tag{7.69}$$

$$(a-b+1){}_1F_1(a;b;\zeta) - a\,{}_1F_1(a+1;b;\zeta) + (b-1){}_1F_1(a;b-1;\zeta) = 0, \tag{7.70}$$

$$b\,{}_1F_1(a;b;\zeta) - b\,{}_1F_1(a-1;b;\zeta) - \zeta\,{}_1F_1(a;b+1;\zeta) = 0, \tag{7.71}$$

$$(b-a-1)\Psi(a,b-1;\zeta) - (b-1+\zeta)\Psi(a,b;\zeta) + \zeta\Psi(a,b+1;\zeta) = 0,$$

$$\Psi(a,b;\zeta) - a\Psi(a+1,b;\zeta) - \Psi(a,b-1;\zeta) = 0,$$

$$(b-a)\Psi(a,b;\zeta) - \zeta\Psi(a,b+1;\zeta) + \Psi(a-1,b;\zeta) = 0.$$

In addition,

$$\left(\frac{d}{d\zeta}\right)^k {}_1F_1(a;b;\zeta) = \frac{(a)_k}{(b)_k}{}_1F_1(a+k;b+k;\zeta) \tag{7.72}$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}\zeta}\right)^k \Psi(a,b;\zeta) = (-1)^k (a)_k \Psi(a+k,b+k;\zeta). \tag{7.73}$$

Next, taking in (7.68) $v = v_{\alpha,\beta}$, where $v_{\alpha,\beta}$ is the Whittaker function with parameters α, β, and putting $\alpha = \alpha_1, \alpha_2$, we find that

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\big(v_{\alpha_1,\beta}(\zeta)v'_{\alpha_2,\beta}(\zeta) - v'_{\alpha_1,\beta}(\zeta)v_{\alpha_2,\beta}(\zeta)\big) = \frac{\alpha_1-\alpha_2}{\zeta} v_{\alpha_1,\beta}(\zeta)v_{\alpha_2,\beta}(\zeta). \tag{7.74}$$

Hence,

$$\begin{aligned} &u_{a_2,b}(\zeta)\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}\big(\zeta^{b/2}\mathrm{e}^{-\zeta/2}u_{a_1,b}(\zeta)\big) - u_{a_1,b}(\zeta)\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}\big(\zeta^{b/2}\mathrm{e}^{-\zeta/2}u_{a_2,b}(\zeta)\big) \\ &\quad = (a_1-a_2)\zeta^{b/2-1}\mathrm{e}^{-\zeta/2}u_{a_1,b}(\zeta)u_{a_2,b}(\zeta), \end{aligned}$$

where

$$u_{a,b}(\zeta) = {}_1F_1(a;b;\zeta).$$

Let $a_i \in \mathbb{C}$, $\operatorname{Re} b_i > 0$, $i = 1, 2$. Invoking the Laplace transform, one can prove that

$$\begin{aligned} &\int_0^x \frac{t^{b_1-1}}{\Gamma(b_1)} {}_1F_1(a_1;b_1;t)\frac{(x-t)^{b_2-1}}{\Gamma(b_2)} {}_1F_1(a_2;b_2;x-t)\,\mathrm{d}t \\ &\quad = \frac{x^{b_1+b_2-1}}{\Gamma(b_1+b_2)} {}_1F_1(a_1+a_2;b_1+b_2;x) \end{aligned} \tag{7.75}$$

(see [73, 6.10 (15)]). Formula (7.75) is called an *addition integral theorem* for the function ${}_1F_1$.

We shall now present a useful expansion of ${}_1F_1(a;b;\zeta)$ into a series in Bessel functions due to Tricomi [73, 6.4 (11)]. Set

$$A_k(\mu,\lambda) = \frac{1}{k!}\left(\frac{\mathrm{d}}{\mathrm{d}\zeta}\right)^k \big(\mathrm{e}^{2\mu\zeta}(1-\zeta)^{\mu-\lambda}(1+\zeta)^{-\mu-\lambda}\big)\Big|_{\zeta=0}, \quad k \in \mathbb{Z}_+.$$

The functions $A_k(\mu,\lambda)$ possess the following properties:

(a)

$$A_0(\mu,\lambda) = 1, \qquad A_1(\mu,\lambda) = 0, \qquad A_2(\mu,\lambda) = \lambda, \tag{7.76}$$

and

$$(k+1)A_{k+1}(\mu,\lambda) = (k+2\lambda-1)A_{k-1}(\mu,\lambda) - 2\mu A_{k-2}(\mu,\lambda) \quad \text{for } k \geqslant 2;$$

(b)

$$A_k(\mu,\lambda) = \sum_{m=0}^{[k/3]} a_{m,k}(\lambda)\mu^m, \tag{7.77}$$

where

$$a_{m,k}(\lambda) = \frac{1}{m!}\left(\frac{\mathrm{d}}{\mathrm{d}\mu}\right)^m \bigl(A_k(\mu,\lambda)\bigr)\Big|_{\mu=0};$$

(c) $A_k(-\mu,\lambda) = (-1)^k A_k(\mu,\lambda)$.

Tricomi proved that for all $\zeta \in \mathbb{C}$,

$$\mathrm{e}^{-\zeta/2}{}_1F_1(a;b;\zeta) = \Gamma(b)(c\zeta)^{(1-b)/2}\sum_{k=0}^{\infty} A_k\left(c,\frac{b}{2}\right)\left(\frac{\zeta}{4c}\right)^{k/2} J_{k+b-1}\bigl(2\sqrt{c\zeta}\bigr), \tag{7.78}$$

where

$$c = \frac{b}{2} - a. \tag{7.79}$$

We shall use relation (7.78) in order to obtain the following result.

Proposition 7.11. *Let* $\operatorname{Re} b > 1/2$ *and* $x > 0$. *Then*

$$x^{b-1}\mathrm{e}^{-x/2}{}_1F_1(a;b;x) = \int_0^{2\sqrt{x}} \cos\bigl(\sqrt{c}t\bigr)k_b(x,t)\,\mathrm{d}t, \tag{7.80}$$

where c *is given by* (7.79), *and*

$$k_b(x,t) = \frac{\Gamma(b)}{\sqrt{\pi}2^{2b-3}}\sum_{k=0}^{\infty}\frac{1}{\Gamma(b+k-1/2)8^k} \times \sum_{m=0}^{[k/3]}(-1)^m a_{m,k}(b/2)\frac{\mathrm{d}^{2m}}{\mathrm{d}t^{2m}}\Bigl(\bigl(4x-t^2\bigr)^{k-3/2+b}\Bigr). \tag{7.81}$$

Here $k_b(x,t)(4x-t^2)^{3/2-b}$ *is infinitely differentiable for* $(x,t) \in \mathbb{R}^2$.

Proof. It is not hard to see that

$$\frac{\mathrm{d}^{2m}}{\mathrm{d}t^{2m}}\Bigl(\bigl(4x-t^2\bigr)^{k-3/2+b}\Bigr) = \sum_{j=0}^{m} c_{j,m,k}(b)t^{2j}\bigl(4x-t^2\bigr)^{k-3/2+b-m-j},$$

where

$$|c_{j,m,k}(b)| \leqslant \bigl(2(k+|b|+3m+2)\bigr)^{2m}, \quad 0 \leqslant j \leqslant m. \tag{7.82}$$

In addition, using induction on k, we easily derive from (7.76) and (7.77) the inequality

$$|a_{m,k}(b/2)| \leqslant (|b|+2)^k, \quad 0 \leqslant m \leqslant [k/3]. \tag{7.83}$$

Estimates (7.82) and (7.83) show that function (7.81) is well defined and the function $k_b(x,t)(4x-t^2)^{3/2-b}$ belongs to $C^\infty(\mathbb{R}^2)$. Next, in view (7.7) and (7.78),

$$x^{b-1}\mathrm{e}^{-x/2}{}_1F_1(a;b;x) = \frac{\Gamma(b)}{\sqrt{\pi}2^{2b-3}}\sum_{k=0}^{\infty}\frac{1}{\Gamma(b+k-1/2)8^k}A_k(c,b/2) \times \int_0^{2\sqrt{x}}\cos\bigl(\sqrt{c}t\bigr)\bigl(4x-t^2\bigr)^{k-3/2+b}\,\mathrm{d}t. \tag{7.84}$$

Repeated integration by parts gives

$$c^j\int_0^{2\sqrt{x}}\cos\bigl(\sqrt{c}t\bigr)\bigl(4x-t^2\bigr)^{k-3/2+b}\,\mathrm{d}t = (-1)^j\int_0^{2\sqrt{x}}\cos\bigl(\sqrt{c}t\bigr)\frac{\mathrm{d}^{2j}}{\mathrm{d}t^{2j}}\Bigl(\bigl(4x-t^2\bigr)^{k-3/2+b}\Bigr)\,\mathrm{d}t \tag{7.85}$$

for $0 \leqslant j \leqslant k/2$. By (7.77) and (7.85),

$$A_k(c,b/2)\int_0^{2\sqrt{x}}\cos\bigl(\sqrt{c}t\bigr)\bigl(4x-t^2\bigr)^{k-3/2+b}\,\mathrm{d}t = \int_0^{2\sqrt{x}}\cos\bigl(\sqrt{c}t\bigr)\sum_{m=0}^{[k/3]}(-1)^m a_{m,k}(b/2)\frac{\mathrm{d}^{2m}}{\mathrm{d}t^{2m}}\Bigl(\bigl(4x-t^2\bigr)^{k-3/2+b}\Bigr)\,\mathrm{d}t. \tag{7.86}$$

Substituting (7.86) into (7.84) and taking (7.82) and (7.83) into account, we obtain (7.80). □

Consider now asymptotic formulae for a confluent hypergeometric function.

Proposition 7.12. *Let $b \in \mathbb{C}$ be fixed. and let $0 < x_1 \leqslant x \leqslant x_2 < \infty$. Then for each $\varepsilon \in (0,\pi)$,*

$$\frac{\sqrt{\pi}\mathrm{e}^{-x/2}{}_1F_1(a;b;x)}{\Gamma(b)} = \frac{\cos(2\sqrt{cx}-\frac{\pi}{4}(2b-1))}{(\sqrt{cx})^{b-1/2}} + \frac{1}{4}\Bigl(\frac{x^2}{3}-\frac{(2b-3)(2b-1)}{4}\Bigr) \times \frac{\sin(2\sqrt{cx}-\frac{\pi}{4}(2b-1))}{(\sqrt{cx})^{b+1/2}} + O\Bigl(\frac{\mathrm{e}^{|\mathrm{Im}\,(2\sqrt{cx})|}}{(\sqrt{cx})^{b+3/2}}\Bigr) \tag{7.87}$$

as $c \to \infty$, $|\arg\sqrt{cx}| \leqslant \pi-\varepsilon$,

$$c^{\frac{1}{4}-c}x^{\frac{b}{2}-\frac{1}{4}}\mathrm{e}^{c-\frac{x}{2}}\Psi(a,b;x) = \sqrt{2}\cos\Bigl(\pi c-2\sqrt{cx}-\frac{\pi}{4}\Bigr) + O\Bigl(\frac{\mathrm{e}^{|\mathrm{Im}\,(\pi c-2\sqrt{cx})|}}{|c|^{1/2}}\Bigr) \tag{7.88}$$

as $c \to \infty$, $|\arg c| \leqslant \pi-\varepsilon$, and

$$c^{\frac{1}{4}-c}x^{\frac{b}{2}-\frac{1}{4}}\mathrm{e}^{c-\frac{x}{2}}\Psi(a,b;x) = \frac{1}{2}(1+\mathrm{i})\mathrm{e}^{-\mathrm{i}\pi c+2\mathrm{i}\sqrt{cx}} + O\Bigl(\frac{\mathrm{e}^{\mathrm{Im}(\pi c-2\sqrt{cx})}}{|c|^{1/2}}\Bigr) \tag{7.89}$$

as $c \to \infty$, $|\pi-\arg c| \leqslant \pi-\varepsilon$, where the constants in O depend only on b, x_1, x_2, ε.

Proof. For $\operatorname{Re} b > 1/2$, relation (7.87) follows from Proposition 7.11 and the asymptotic expansion of Fourier integrals (see Riekstyn'sh [182], Chap. 2, Theorem 10.2). Using now (7.69), we get (7.87) in general. Equalities (7.88) and (7.89) are contained in [73], Chap. 6, Sect. 6.13. □

In the case where x or x^{-1} are unbounded as $a \to \infty$, a behavior of confluent hypergeometric functions is more complicated. We require the following asymptotic formula [73, 6.13 (26)]:

$$(c\zeta)^{b/2-1/2} e^{-\zeta/2} {}_1F_1(a; b; \zeta) = \Gamma(b) J_{b-1}\left(2\sqrt{c\zeta}\right) + c^{-3/4} \zeta^{5/4} O\left(e^{|\operatorname{Im}(2\sqrt{c\zeta})|}\right). \tag{7.90}$$

Here b, $\arg \zeta$, and $\arg c$ are bounded, $c \to \infty$, and $\zeta = O(|c|^{\alpha})$ for some $\alpha < 1/3$. Also note that, according to [73, 6.13 (15)],

$${}_1F_1(a; b; \zeta) = \Gamma(b) J_{b-1}\left(2\sqrt{c\zeta}\right) e^{\zeta/2} (c\zeta)^{1/2-b/2} + O\left(|c|^{-1}\right) \tag{7.91}$$

when b, $\arg \zeta$, and $\arg c$ are bounded, $c \to \infty$, and $\zeta = O(|c|^{-1})$.

For other results in this direction, see [73], Chap. 6, Sect. 6.13.

Chapter 8
Exponential Expansions

This chapter is the important first step in our study of mean periodic functions on subsets of the real line.

We shall study in the sequel mean periodic functions with respect to various classes of distributions. In Sect. 8.1 the reader is acquainted with the main classes of distributions handled throughout this book. As a rule, they may be characterized by means of conditions on zeros of Fourier transforms.

Section 8.2 contains a detailed discussion of biorthogonal systems concerning exponential polynomials. If $T \in \mathcal{E}'(\mathbb{R}^1)$, $T \neq 0$, and λ is a zero of multiplicity n_λ for the Fourier transform $\widehat{T}$, then the functions

$$(\mathrm{i}t)^\eta \mathrm{e}^{\mathrm{i}\lambda t}, \quad \eta \in \{0, \dots, n_\lambda - 1\} \tag{8.1}$$

are the solutions of the convolution equation $f * T = 0$. Utilizing the classical Paley–Wiener–Schwartz theorem, we introduce the distributions $T_{\lambda,\eta} \in \mathcal{E}'(\mathbb{R}^1)$ satisfying the biorthogonality conditions

$$\left\langle T_{\mu,\nu}, (-\mathrm{i}t)^\eta \mathrm{e}^{-\mathrm{i}\lambda t} \right\rangle = \delta_{\lambda,\mu}\delta_{\eta,\nu}.$$

After proving basic recurrent relations, we give in Sect. 8.2 a proof of the crucial and nontrivial fact about the completeness of the system $\{T_{\lambda,\eta}\}$.

In Sect. 8.3 we investigate expansions over systems (8.1) and $\{T_{\lambda,\eta}\}$. In a number of cases we prove sufficient conditions for convergence of these expansions.

Another principal object of study in this chapter is the distribution ζ_T defined in Sect. 8.4. It can often serve as an example on the basis of which the sharpness in many theorems can be tested. In particular, using properties of ζ_T, we solve the Lyubich problem on zeros of mean periodic functions. The distributions $T_{\lambda,\eta}$ and ζ_T will be of crucial importance for us in Part III.

V.V. Volchkov, V.V. Volchkov, *Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group,* Springer Monographs in Mathematics,

8.1 Main Classes of Distributions

The object of this section is to introduce some classes of one-dimensional distributions which are used throughout this book.

Let $T \in \mathcal{E}'(\mathbb{R}^1)$, $T \neq 0$. Then the Fourier transform

$$\widehat{T}(z) = \langle T, e^{-izt} \rangle, \quad z \in \mathbb{C},$$

is a nonzero entire function of variable z. It is clear from Theorem 6.3, Corollary 6.2, and Proposition 6.1 that the set $\mathcal{Z}(\widehat{T})$ is infinite if and only if $r(T) > 0$. Moreover, in order that $\mathcal{Z}(\widehat{T}) = \varnothing$, it is necessary and sufficient that $T = c\delta_0(\cdot + h)$ for some $c \in \mathbb{C}\backslash\{0\}$, $h \in \mathbb{R}^1$.

Let $\mathcal{Z}(\widehat{T}) \neq \varnothing$, and let p be a polynomial such that the function $\widehat{T}/p$ is entire. Then the equation

$$p\left(-i\frac{d}{dt}\right)U = T \tag{8.2}$$

has a solution $U \in \mathcal{E}'(\mathbb{R}^1)$ such that $r(U) = r(T)$ (see Theorem 6.3). In particular this is true if T is odd and $p(z) = z$ (in this case $0 \in \mathcal{Z}(\widehat{T})$). In view of what has been said above, if $r(T) > 0$, then we can choose the polynomial p so that $U \in (L^1 \cap \mathcal{E}')(\mathbb{R}^1)$. Let d_T denote the smallest of degrees of such polynomials p. Theorem 6.3 shows that $\widehat{T}/p \in L^2(\mathbb{R}^1)$ for each polynomial p of degree $1 + \operatorname{ord} T$ such that the function $\widehat{T}/p$ is entire. Hence, (8.2) is satisfied with $U \in (L^2 \cap \mathcal{E}')(\mathbb{R}^1)$, and thus

$$\operatorname{ord} T \leqslant d_T \leqslant 1 + \operatorname{ord} T. \tag{8.3}$$

Assume now that $\lambda \in \mathcal{Z}(\widehat{T})$. We set

$$m(\lambda, T) = n_\lambda - 1,$$

where, as usual, $n_\lambda = n_\lambda(\widehat{T})$ denotes the multiplicity of the zero λ of $\widehat{T}$. We note from Proposition 6.1(iii) that if $\mathcal{Z}(\widehat{T})$ is infinite, then

$$m(\lambda, T) = o(|\lambda|) \quad \text{as } \lambda \to \infty. \tag{8.4}$$

Theorem 6.3 and Proposition 6.1(i) show that

$$\sum_{\lambda \in \mathcal{Z}(\widehat{T})} \frac{m(\lambda, T)}{(1 + |\lambda|)^{1+\varepsilon}} < +\infty \quad \text{for each } \varepsilon > 0. \tag{8.5}$$

As in Sect. 6.1, we set

$$\sigma_\lambda(\widehat{T}) = \sum_{j=0}^{m(\lambda,T)} |a_j^{\lambda,0}(\widehat{T})|.$$

Let $\mathfrak{M}(\mathbb{R}^1)$ be the collection of all nonzero distributions $T \in \mathcal{E}'(\mathbb{R}^1)$ such that

$$\sup_{\lambda\in\mathcal{Z}(\widehat{T})} \frac{|\operatorname{Im}\lambda| + m(\lambda, T) + \log(1+\sigma_\lambda(\widehat{T}))}{\log(2+|\lambda|)} < +\infty. \tag{8.6}$$

Here and below we assume tacitly that

$$\sup_{\lambda\in\mathcal{Z}(\widehat{T})} u(\lambda) < +\infty \quad \text{if } \mathcal{Z}\big(\widehat{T}\big) = \varnothing \tag{8.7}$$

for each function $u : \mathbb{C} \to \mathbb{R}^1$. Therefore, $\delta_0 \in \mathfrak{M}(\mathbb{R}^1)$.

Having (8.7) in mind, now define

$$\mathfrak{U}\big(\mathbb{R}^1\big) = \Big\{T \in \mathcal{E}'\big(\mathbb{R}^1\big) : T \neq 0, \ \sup_{\lambda\in\mathcal{Z}(\widehat{T})} m(\lambda, T) < +\infty\Big\}$$

and $\mathfrak{N}(\mathbb{R}^1) = \mathfrak{M}(\mathbb{R}^1) \cap \mathfrak{U}(\mathbb{R}^1)$. Next, we write $\mathfrak{E}(\mathbb{R}^1)$ for the set of all $T \in \mathcal{E}'(\mathbb{R}^1)$ such that $T \neq 0$ and

$$\sup_{\lambda\in\mathcal{Z}(\widehat{T})} \frac{\log((1+\sigma_\lambda(\widehat{T}))(1+|\lambda|)) + n_\lambda \log n_\lambda}{1+|\operatorname{Im}\lambda|} < +\infty. \tag{8.8}$$

Suppose now that $\alpha > 0$. Let $\mathfrak{G}_\alpha(\mathbb{R}^1)$ be the set of all nonzero $T \in \mathcal{E}'(\mathbb{R}^1)$ with the following property: in order that $T \in \mathfrak{G}_\alpha(\mathbb{R}^1)$, it is necessary and sufficient that either the set $\mathcal{Z}(\widehat{T})$ is not infinite or

$$|\operatorname{Im}\lambda| + m(\lambda, T) + \log\big(1+\sigma_\lambda\big(\widehat{T}\big)\big) = o\big(|\lambda|^{1/\alpha}\big) \quad \text{as } \lambda \to \infty, \ \lambda \in \mathcal{Z}\big(\widehat{T}\big). \tag{8.9}$$

By the definition we see that

$$\mathfrak{N}\big(\mathbb{R}^1\big) \subset \mathfrak{M}\big(\mathbb{R}^1\big) \subset \mathfrak{G}_\alpha\big(\mathbb{R}^1\big) \quad \text{for each } \alpha > 0.$$

It can be shown that $\mathfrak{N}(\mathbb{R}^1) \neq \mathfrak{M}(\mathbb{R}^1)$ and $\mathfrak{M}(\mathbb{R}^1) \neq \mathfrak{G}_\alpha(\mathbb{R}^1)$ for all α. The classes $\mathfrak{N}(\mathbb{R}^1)$ and $\mathfrak{E}(\mathbb{R}^1)$ are broad enough. In many cases conditions (8.6), (8.8), and (8.9) can be verified by using asymptotic expansions of the Fourier integrals (see, for example, V.V. Volchkov [225], Part I, Theorem 6.1, and Riekstyn'sh [182], Chap. 10.3). For instance, if $\beta, \gamma \in (-1, +\infty)$, $h \in C^\infty[-r, r]$, and $h^{(\mu)}(-r)h^{(\nu)}(r) \neq 0$ for some $\mu, \nu \in \mathbb{Z}_+$ then the function

$$T(t) = \begin{cases} (r+t)^\beta (r-t)^\gamma h(t) & \text{if } |t| < r, \\ 0 & \text{if } |t| \geqslant r \end{cases}$$

is in the class $\mathfrak{N}(\mathbb{R}^1)$.

To check (8.6) or (8.8) it is useful to know when the condition

$$\sup_{\lambda\in\mathcal{Z}(\widehat{T})} \frac{\log(1+\sigma_\lambda(\widehat{T}))}{\log(2+|\lambda|)} < +\infty \tag{8.10}$$

is valid. In this connection the following simple results are worth recording.

Proposition 8.1. *Let* $T \in \mathcal{E}'(\mathbb{R}^1)$, $T \neq 0$, $\mathcal{Z}(\widehat{T}) \neq \varnothing$, *and*

$$\sup_{\lambda \in \mathcal{Z}(\widehat{T})} \frac{m(\lambda, T)}{\log(2 + |\lambda|)} < +\infty.$$

Assume that for each $\lambda \in \mathcal{Z}(\widehat{T})$, *there exist* $r_\lambda > 0$ *such that*

$$r_\lambda < \min\bigl\{|\mu - \lambda| : \mu \in \mathcal{Z}\bigl(\widehat{T}\bigr), \mu \neq \lambda\bigr\}, \qquad \sup_{\lambda \in \mathcal{Z}(\widehat{T})} r_\lambda < +\infty,$$

and

$$\bigl|\widehat{T}(z)\bigr| > (2 + |\lambda|)^{-\gamma}, \quad \textit{provided that } |z - \lambda| = r_\lambda,$$

where the constant $\gamma > 0$ *is independent of* λ. *Then* (8.10) *is true.*

Proof. For $\lambda \in \mathcal{Z}(\widehat{T})$, define

$$g(\zeta) = (\zeta - \lambda)^{n_\lambda} \big/ \widehat{T}(\zeta), \quad |\zeta - \lambda| \leqslant r_\lambda.$$

Applying (6.25) with $z = \lambda$, $r = r_\lambda$, $s \in \{0, \dots, m(\lambda, T)\}$, we see from (6.8) and (6.15) that (8.10) holds. □

Proposition 8.2. *Let* $T \in \mathfrak{U}(\mathbb{R}^1)$, $\mathcal{Z}(\widehat{T}) \neq \varnothing$, *and*

$$\sup_{\lambda \in \mathcal{Z}(\widehat{T})} \frac{|\operatorname{Im} \lambda|}{\log(2 + |\lambda|)} < +\infty.$$

Then (8.10) *is equivalent to the estimate*

$$\bigl|\widehat{T}^{(n_\lambda)}(\lambda)\bigr| > (2 + |\lambda|)^{-\gamma}, \quad \lambda \in \mathcal{Z}\bigl(\widehat{T}\bigr), \tag{8.11}$$

where $\gamma > 0$ *is independent of* λ.

Proof. Equalities (6.5) and (6.15) ensure us that (8.10) implies (8.11). The converse result follows from (6.24), Proposition 6.6(iii), and Theorem 6.3. □

To go further, denote by $\mathfrak{R}(\mathbb{R}^1)$ the set of all nonzero $T \in \mathcal{E}'(\mathbb{R}^1)$ such that

$$\mathcal{Z}\bigl(\widehat{T}\bigr) \neq \varnothing, \qquad \sum_{\lambda \in \mathcal{Z}(\widehat{T})} \sigma_\lambda\bigl(\widehat{T}\bigr) < +\infty, \tag{8.12}$$

and

$$\sup_{\lambda \in \mathcal{Z}(\widehat{T})} \frac{|\operatorname{Im} \lambda|}{1 + |\operatorname{Re} \lambda|} < +\infty. \tag{8.13}$$

Proposition 8.3. *Let* $T \in \mathcal{E}'(\mathbb{R}^1)$, $T \neq 0$, *and let* (8.13) *hold. Assume that*

$$\sup_{\lambda \in \mathcal{Z}(\widehat{T})} \sigma_\lambda\bigl(\widehat{T}\bigr)(1 + |\lambda|)^{-\gamma} < +\infty$$

for some $\gamma \geqslant 0$. *Then*

$$p\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)T \in \mathfrak{R}(\mathbb{R}^1)$$

for each polynomial p *of degree* $d \geqslant \gamma$.

Proof. Using (6.33) and Proposition 6.1(i), we see from Proposition 6.6(iv) that (8.12) is fulfilled with $p(\frac{\mathrm{d}}{\mathrm{d}t})T$ instead of T. Hence the proposition. □

We now recall from Sect. 6.1 that for each $T \in \mathrm{Inv}(\mathbb{R}^1)$, the equation

$$f * T = \delta_0 \tag{8.14}$$

has a solution $f \in \mathcal{D}'(\mathbb{R}^1)$. Denote by $\mathrm{Inv}_+(\mathbb{R}^1)$ (respectively $\mathrm{Inv}_-(\mathbb{R}^1)$) the set of all distributions $T \in \mathrm{Inv}(\mathbb{R}^1)$ such that (8.14) is satisfied for some $f \in \mathcal{D}'(\mathbb{R}^1)$ with $\mathrm{supp}\, f \subset [0, +\infty)$ (respectively $\mathrm{supp}\, f \subset (-\infty, 0]$). Notice that if $T \in \mathrm{Inv}_+(\mathbb{R}^1)$ is even, then $T \in \mathrm{Inv}_+(\mathbb{R}^1) \cap \mathrm{Inv}_-(\mathbb{R}^1)$.

The property

$$\mathfrak{M}(\mathbb{R}^1) \subset \mathrm{Inv}_+(\mathbb{R}^1) \cap \mathrm{Inv}_-(\mathbb{R}^1)$$

will later be proved as Theorem 8.5. On the other hand, one can easily show that $\mathfrak{M}(\mathbb{R}^1) \neq \mathrm{Inv}(\mathbb{R}^1)$. In fact, by (6.34), the distribution

$$T = \delta_0 + \varphi$$

is in the class $\mathrm{Inv}(\mathbb{R}^1)$ for each $\varphi \in \mathcal{D}(\mathbb{R}^1)$. However, for $\varphi \neq 0$, (6.34) shows that

$$\frac{|\mathrm{Im}\,\lambda|}{\log(2 + |\lambda|)} \to +\infty \quad \text{as } \lambda \to \infty,\ \lambda \in \mathcal{Z}(\widehat{T}),$$

and hence $T \notin \mathfrak{M}(\mathbb{R}^1)$.

Next, let $0 < R \leqslant +\infty$, and let $\mathfrak{W} = \mathfrak{W}(-R, R)$ be an arbitrary subset of $\mathcal{D}'(-R, R)$. Then we write $\mathfrak{W}_\natural = \mathfrak{W}_\natural(-R, R)$ for the set of all even distributions in the class $\mathfrak{W}$.

Assume now that $-\infty \leqslant a < b \leqslant +\infty$ and $\alpha > 0$. Let $\mathrm{QA}(a, b)$ (respectively $G^\alpha(a, b)$) denote the set of all functions $f \in C^\infty(a, b)$ such that for each $[a', b'] \subset (a, b)$,

$$\sum_{\nu=1}^{\infty}\left(\inf_{q \geqslant \nu}\left(\int_{a'}^{b'}\left|f^{(q)}(t)\right|\mathrm{d}t\right)^{1/q}\right)^{-1} = +\infty \tag{8.15}$$

(respectively

$$\int_{a'}^{b'}\left|f^{(q)}(t)\right|\mathrm{d}t \leqslant c^q(1 + q)^{\alpha q}, \quad q = 1, 2, \ldots, \tag{8.16}$$

where the constant $c > 0$ is independent of q). For the case where $-\infty < a < b < +\infty$, we shall write $f \in \mathrm{QA}[a, b]$ (respectively $f \in G^\alpha[a, b]$) if $f \in C^\infty[a, b]$ and condition (8.15) (respectively (8.16)) holds with $[a', b'] = [a, b]$.

For future use, we need the following result.

Lemma 8.1. *Let* $\{M_q\}_{q=1}^{\infty}$ *and* $\{M'_q\}_{q=1}^{\infty}$ *be sequences of positive numbers and assume that* $a > 0$, $l \in \mathbb{Z}_+$.

(i) *If* $\sum_{\nu=1}^{\infty}(\inf_{q\geqslant\nu} M_q^{1/q})^{-1} = +\infty$ *and* $M'_q \leqslant a^q(1+M_{q+l})$ *for each* $q \in \mathbb{N}$, *then*

$$\sum_{\nu=1}^{\infty}\Big(\inf_{q\geqslant\nu}\big(M'_q\big)^{1/q}\Big)^{-1} = +\infty.$$

(ii) *If* $\sum_{\nu=1}^{\infty}(\inf_{q\geqslant\nu} M_q^{1/q})^{-1} < +\infty$ *and* $M_q \leqslant a^q(1+M'_{q+l})$ *for all* $q \in \mathbb{N}$, *then*

$$\sum_{\nu=1}^{\infty}\Big(\inf_{q\geqslant\nu}\big(M'_q\big)^{1/q}\Big)^{-1} < +\infty.$$

The proof of this lemma follows from [225, Part I, Lemma 2.1].

To close we consider the following version of the Denjoy–Carleman theorem.

Theorem 8.1. *Let* $-\infty \leqslant a < b \leqslant +\infty$, *and let* $t_0 \in (a,b)$.

(i) *If* $f \in \mathrm{QA}(a,b)$ *and* $f^{(k)}(t_0) = 0$ *for all* $k \in \mathbb{Z}_+$ *then* $f = 0$.
(ii) *For each sequence* $\{M_q\}_{q=0}^{\infty}$ *of positive numbers satisfying*

$$\sum_{\nu=1}^{\infty}\Big(\inf_{q\geqslant\nu} M_q^{1/q}\Big)^{-1} < +\infty,$$

there exists a nonzero $f \in \mathcal{D}(a,b)$ *such that* $f \geqslant 0$ *and* $|f^{(q)}(t)| \leqslant M_q$ *for all* $t \in (a,b)$, $q \in \mathbb{Z}_+$. *Moreover, if* $b = -a$, *then* f *can be chosen even.*

Proof. To prove (i), first notice that

$$f^{(k)}(t) = \int_{t_0}^{t} f^{(k+1)}(\xi)\,\mathrm{d}\xi$$

for all $t \in (a,b)$, $k \in \mathbb{Z}_+$. By assumption on f and Lemma 8.1(i) this yields

$$\sum_{\nu=1}^{\infty}\bigg(\inf_{q\geqslant\nu}\Big(\max_{t\in[c,d]}\big|f^{(q)}(t)\big|\Big)^{1/q}\bigg)^{-1} = +\infty$$

for each $[c,d] \subset (a,b)$. Now the Denjoy–Carleman theorem [126, Theorem 1.3.8] amounts to $f = 0$ on (a,b). Finally, part (ii) is a consequence of [126, Theorem 1.3.5] (see also [126, the proof of Theorem 1.3.8]). □

8.2 Biorthogonal Systems. General Completeness Results

Throughout this section we assume that

$$T \in \mathcal{E}'\big(\mathbb{R}^1\big), \quad T \neq 0, \quad \mathcal{Z}\big(\widehat{T}\big) \neq \varnothing \quad \text{and} \quad \operatorname{supp} T \subset [-r(T), r(T)]. \tag{8.17}$$

Notice that if $T \in (\mathcal{E}' \cap L^1)(\mathbb{R}^1)$, then it follows by (8.17) that $r(T) > 0$. In the sequel we shall use this fact without saying this explicitly.

By Theorem 6.3,

$$\big|\widehat{T}(z)\big| \leqslant \gamma_1(1+|z|)^{\gamma_2} e^{r(T)|\operatorname{Im} z|}, \quad z \in \mathbb{C}, \tag{8.18}$$

where $\gamma_1 > 0$ and $\gamma_2 \in \mathbb{R}^1$ are independent of z.

The following result contains certain information concerning the behavior of $\widehat{T}$ at infinity.

Proposition 8.4. *The limit*

$$\lim_{r \to +\infty} \frac{\log |\widehat{T}(re^{i\theta})|}{r} = r(T)|\sin\theta| \tag{8.19}$$

exists for almost all $\theta \in [-\pi, \pi]$.

Proof. In view of Corollary 6.2, we can restrict ourselves only to the case $r(T) > 0$. Now define

$$h_T(\theta) = \limsup_{r \to +\infty} \frac{\log |\widehat{T}(re^{i\theta})|}{r}, \quad \theta \in [-\pi, \pi].$$

It is known that $h_T \in C[-\pi, \pi]$ (see Levin [146], Lecture 8). In addition, $h_T(\pi/2) = h_T(-\pi/2) = r(T)$ (see Theorem 6.3). Proposition 6.1(ii) now implies that $\gamma_+ = \gamma_- = r(T)$, and hence (8.19) holds. □

Let $\lambda \in \mathcal{Z}(\widehat{T})$ and $\eta \in \{0, \dots, m(\lambda, T)\}$. Using (6.6) and Proposition 6.6(ii), we see from Theorem 6.3 that there exists $T_{\lambda,\eta} \in \mathcal{E}'(\mathbb{R}^1)$ such that

$$r_0(T_{\lambda,\eta}) = r(T_{\lambda,\eta}) = r(T) \tag{8.20}$$

and

$$\widehat{T}_{\lambda,\eta}(z) = a^{\lambda,\eta}\big(\widehat{T}, z\big), \quad z \in \mathbb{C}. \tag{8.21}$$

Similarly, there exists $T^{\lambda,\eta} \in \mathcal{E}'(\mathbb{R}^1)$ such that

$$r_0\big(T^{\lambda,\eta}\big) = r\big(T^{\lambda,\eta}\big) = r(T) \tag{8.22}$$

and

$$\widehat{T^{\lambda,\eta}}(z)(z-\lambda)^{\eta+1} = \widehat{T}(z), \quad z \in \mathbb{C}. \tag{8.23}$$

In this section we shall study basic properties of the distributions $T_{\lambda,\eta}$ and $T^{\lambda,\eta}$.

Proposition 8.5.

(i) *If* $\mu \in \mathcal{Z}(\widehat{T})$ *and* $\nu \in \{0, \dots, m(\mu, T)\}$, *then we have the following biorthogonality condition:*

$$\left\langle T_{\lambda,\eta}, (-\mathrm{i}t)^{\nu}\mathrm{e}^{-\mathrm{i}\mu t}\right\rangle = \widehat{T}^{(\nu)}_{\lambda,\eta}(\mu) = \delta_{\lambda,\mu}\delta_{\eta,\nu}. \tag{8.24}$$

(ii) $\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - \lambda\right)^{m(\lambda,T)+1} T_{\lambda,\eta} = \sum_{j=0}^{m(\lambda,T)} a_j^{\lambda,\eta}\big(\widehat{T}\big)\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - \lambda\right)^{j} T.$

(iii) $T_{\lambda,\eta} = \sum_{j=0}^{m(\lambda,T)} a_j^{\lambda,\eta}\big(\widehat{T}\big) T^{\lambda,m(\lambda,T)-j}.$

Proof. Assertion (i) follows at once from Proposition 6.3(i). Parts (ii) and (iii) are direct consequences of (8.21), (8.23), and (6.6). □

Proposition 8.6. *Let* $\psi \in \mathcal{E}'(\mathbb{R}^1)$, $\psi \neq 0$. *Then*

$$(T * \psi)^{\lambda,\eta} = T^{\lambda,\eta} * \psi$$

for all $\lambda \in \mathcal{Z}(\widehat{T})$ *and* $\eta \in \{0, \dots, m(\lambda, T)\}$. *In particular,*

$$\left(p\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)T\right)^{\lambda,\eta} = p\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\left(T^{\lambda,\eta}\right)$$

for each nonzero polynomial p.

Proof. Since $\widehat{T * \psi} = \widehat{T} \cdot \widehat{\psi}$, the desired statement is obvious from (8.23). □

We now find some recursion relations for the distributions $T_{\lambda,\eta}$ and $T^{\lambda,\eta}$.

Proposition 8.7. *For each* $\lambda \in \mathcal{Z}(\widehat{T})$,

$$\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - \lambda\right) T_{\lambda,m(\lambda,T)} = a^{\lambda,m(\lambda,T)}_{m(\lambda,T)}\big(\widehat{T}\big) T \tag{8.25}$$

and

$$\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - \lambda\right) T^{\lambda,0} = T. \tag{8.26}$$

In addition, if $m(\lambda, T) \geqslant 1$, *then*

$$T'_{\lambda,\eta} - \mathrm{i}\lambda T_{\lambda,\eta} - \mathrm{i}(\eta+1)T_{\lambda,\eta+1} = \mathrm{i}a^{\lambda,\eta}_{m(\lambda,T)}\big(\widehat{T}\big) T \tag{8.27}$$

and

$$\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - \lambda\right) T^{\lambda,\eta+1} = T^{\lambda,\eta} \tag{8.28}$$

for all $\eta \in \{0, \dots, m(\lambda, T) - 1\}$.

Proof. This follows from (8.21), (8.23), (6.33), and Proposition 6.4. □

Corollary 8.1. *If T is a distribution of order q then $T^{\lambda,\eta}$ is a distribution of order* $\max\{0,\dots,q-\eta-1\}$.

Proof. Formulae (8.26) and (8.28) yield

$$\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\eta+1}\left(\mathrm{e}^{-\mathrm{i}\lambda t}T^{\lambda,\eta}\right)=\mathrm{e}^{-\mathrm{i}\lambda t}T.$$

Since $\operatorname{ord}(\mathrm{e}^{-\mathrm{i}\lambda t}\Psi)=\operatorname{ord}\Psi$ for each $\Psi\in\mathcal{E}'(\mathbb{R}^1)$, this, together with Bremermann [41, Sect. 4.7], brings us to the desired statement. □

Proposition 8.8. *Suppose that there exist $U\in\mathcal{E}'(\mathbb{R}^1)$ and $c\in\mathbb{C}$ such that $T=U'-\mathrm{i}cU$. Then $\mathcal{Z}(\widehat{U})\subset\mathcal{Z}(\widehat{T})$ and*

$$U_{\lambda,0}=T_{\lambda,0}-\mathrm{i}a^{\lambda,0}_{m(\lambda,T)}(\widehat{T})U$$

for each $\lambda\in\mathcal{Z}(\widehat{U})$.

Proof. This result is a consequence of (8.21), (6.33), and Proposition 6.5. □

The following propositions will be useful in several places later.

Proposition 8.9. *If $T\in\mathfrak{R}(\mathbb{R}^1)$, then*

$$\sum_{\lambda\in\mathcal{Z}(\widehat{T})}T_{\lambda,0}=\delta_0,\tag{8.29}$$

where the series converges unconditionally in $\mathcal{D}'(\mathbb{R}^1)$.

Proof. Let $\varphi\in\mathcal{D}(\mathbb{R}^1)$, and let $\psi\in C^\infty(\mathbb{R}^1)$ be defined by $\widehat{\psi}=\varphi$. Then

$$\sup_{t\in\mathbb{R}^1}|\psi(t)|(1+|t|)^\alpha<+\infty\quad\text{for each }\alpha>0.\tag{8.30}$$

Bearing (8.21) in mind, one has

$$\langle T_{\lambda,0},\varphi\rangle=\int_{-\infty}^{\infty}\psi(t)a^{\lambda,0}\big(\widehat{T},t\big)\,\mathrm{d}t,\quad \lambda\in\mathcal{Z}\big(\widehat{T}\big).$$

Using now (8.12), (8.30), (8.18), and Proposition 6.6(ii), we see that the series in (8.29) converges unconditionally in $\mathcal{D}'(\mathbb{R}^1)$ to some distribution $f\in\mathcal{E}'(\mathbb{R}^1)$. Moreover,

$$\widehat{f}(z)=\sum_{\lambda\in\mathcal{Z}(\widehat{T})}a^{\lambda,0}\big(\widehat{T},z\big),\quad z\in\mathbb{C},\tag{8.31}$$

and the function $u=(\widehat{f}-1)/\widehat{T}$ is entire (see (8.12) and Proposition 6.6(v)). Relations (8.31), (6.6), (8.12), and (8.13) show that there exists $\varepsilon\in(0,\pi/2)$ such that

$(\widehat{f}/\widehat{T})(re^{i\theta}) \to 0$ as $r \to +\infty$ for each $\theta \in (-\pi, \pi)$ satisfying $|\pi/2 - |\theta|| < \varepsilon$. This, together with Proposition 8.4 and the Phragmén–Lindelöf theorem, ensures that u is bounded in $\mathbb{C}$. Since $u(re^{i\theta}) \to 0$ as $r \to +\infty$ for some $\theta \in (-\pi, \pi)$, we conclude by Liouville's theorem that $u = 0$. Thus, $\widehat{f} = 1$, which completes the proof. □

Proposition 8.10. *Suppose that* $r(T) > 0$, $f \in C^\infty[-r(T), r(T)]$, *and*

$$\langle T, f^{(\nu)}(-\cdot)\rangle = 0 \tag{8.32}$$

for all $\nu \in \mathbb{Z}_+$. *Assume that*

$$T = \left(\frac{d}{dt} - i\lambda_1\right)^{s_1} \dots \left(\frac{d}{dt} - i\lambda_l\right)^{s_l} Q, \tag{8.33}$$

where $\{\lambda_1, \dots, \lambda_l\}$ *is a set of distinct complex numbers,* $s_1, \dots, s_l \in \mathbb{N}$, *and* $Q \in \mathcal{E}'(\mathbb{R}^1)$. *Let*

$$g(t) = f(t) - \sum_{j=1}^{l} \sum_{\eta=m(\lambda_j,T)+1-s_j}^{m(\lambda_j,T)} \langle T_{\lambda_j,\eta}, f(-\cdot)\rangle (it)^\eta e^{i\lambda_j t}, \quad t \in [-r(T), r(T)]. \tag{8.34}$$

Then the following assertions hold.

(i) supp $Q \subset [-r(T), r(T)]$ *and* $s_j \leqslant m(\lambda_j, T) + 1$ *for each* $j \in \{1, \dots, l\}$.
(ii) $\langle Q, g^{(\nu)}(-\cdot)\rangle = 0$ *for all* $\nu \in \mathbb{Z}_+$.
(iii) *If* $\lambda \in \mathcal{Z}(\widehat{T})\backslash\{\lambda_1, \dots, \lambda_l\}$, *then* $\lambda \in \mathcal{Z}(\widehat{T}) \cap \mathcal{Z}(\widehat{Q})$, $m(\lambda, T) = m(\lambda, Q)$, *and*

$$\langle Q_{\lambda,\eta}, g^{(\nu)}(-\cdot)\rangle = \langle T_{\lambda,\eta}, f^{(\nu)}(-\cdot)\rangle \tag{8.35}$$

for all $\eta \in \{0, \dots, m(\lambda, T)\}$ *and* $\nu \in \mathbb{Z}_+$.

Proof. Part (i) readily follows from (8.33). To prove (ii) we can assume, without loss of generality, that $l = s_1 = 1$. The general case reduces to this one via iteration. We have $T = (\frac{d}{dt} - i\lambda_1)Q$, whence $Q = \gamma T_{\lambda_1, m(\lambda_1,T)}$, where $\gamma = -i(a^{\lambda_1, m(\lambda_1,T)}_{m(\lambda_1,T)})^{-1}$ (see (8.25)). In addition,

$$g(t) = f(t) - \langle T_{\lambda_1,m(\lambda_1,T)}, f(-\cdot)\rangle (it)^{m(\lambda_1,T)} e^{i\lambda_1 t}, \quad t \in [-r(T), r(T)].$$

Assuming $j \in \mathbb{Z}_+$ and using Proposition 8.5(i), one concludes that

$$\begin{aligned} \langle Q, g^{(j)}(-\cdot)\rangle &= \gamma\langle T_{\lambda_1,m(\lambda_1,T)}, f^{(j)}(-\cdot)\rangle - \gamma(i\lambda_1)^j \langle T_{\lambda_1,m(\lambda_1,T)}, f(-\cdot)\rangle \\ &= \langle Q, f^{(j)}(-\cdot)\rangle - (i\lambda_1)^j \langle Q, f(-\cdot)\rangle. \end{aligned} \tag{8.36}$$

However, relation (8.32) yields

$$\left\langle \left(\frac{d}{dt} - i\lambda_1\right) Q, f^{(\nu)}(-\cdot)\right\rangle = 0, \quad \nu \in \mathbb{Z}_+.$$

Therefore,

$$\langle Q, f^{(j)}(-\cdot)\rangle = (\mathrm{i}\lambda_1)^j \langle Q, f(-\cdot)\rangle, \quad j \in \mathbb{Z}_+.$$

Combining this with (8.36), we arrive at (ii).

Next, let $\lambda \in \mathcal{Z}(\widehat{T})\backslash\{\lambda_1, \dots, \lambda_l\}$. Formula (8.33) implies that $\lambda \in \mathcal{Z}(\widehat{T}) \cap \mathcal{Z}(\widehat{Q})$ and $m(\lambda, T) = m(\lambda, Q)$. Applying Proposition 8.8, assertion (ii), and (8.34), we infer from (8.24) that

$$\langle Q_{\lambda,0}, g^{(\nu)}(-\cdot)\rangle = \langle T_{\lambda,0}, g^{(\nu)}(-\cdot)\rangle = \langle T_{\lambda,0}, f^{(\nu)}(-\cdot)\rangle \tag{8.37}$$

for each $\nu \in \mathbb{Z}_+$. This proves (8.35) for the case where $m(\lambda, T) = 0$. Assume now that $m(\lambda, T) \geqslant 1$ and let $\eta \in \{0, \dots, m(\lambda, T) - 1\}$. In view of (ii) and (8.27),

$$\mathrm{i}(\eta + 1)\langle Q_{\lambda,\eta+1}, g^{(\nu)}(-\cdot)\rangle = \langle Q_{\lambda,\eta}, g^{(\nu+1)}(-\cdot)\rangle - \mathrm{i}\lambda\langle Q_{\lambda,\eta}, g^{(\nu)}(-\cdot)\rangle. \tag{8.38}$$

By a similar way we obtain

$$\mathrm{i}(\eta + 1)\langle T_{\lambda,\eta+1}, f^{(\nu)}(-\cdot)\rangle = \langle T_{\lambda,\eta}, f^{(\nu+1)}(-\cdot)\rangle - \mathrm{i}\lambda\langle T_{\lambda,\eta}, f^{(\nu)}(-\cdot)\rangle. \tag{8.39}$$

Comparing (8.38) with (8.39) and using (8.37), we deduce (8.35) by induction on η. Hence the proposition. □

To continue, for $z \in \mathbb{C}$, $m \in \mathbb{Z}_+$, let us define the functions $\mathrm{e}^{z,m}$, $\mathrm{e}_+^{z,m} \in L^{1,\mathrm{loc}}(\mathbb{R}^1)$ by the formulae

$$\mathrm{e}^{z,m}(t) = (\mathrm{i}t)^m \mathrm{e}^{\mathrm{i}zt}, \qquad \mathrm{e}_+^{z,m}(t) = \begin{cases} \mathrm{e}^{z,m}(t) & \text{if } t > 0, \\ 0 & \text{if } t \leqslant 0. \end{cases} \tag{8.40}$$

Proposition 8.11. *One has*

$$T^{\lambda,\eta} = \frac{\mathrm{i}}{\eta!}\mathrm{e}_+^{\lambda,\eta} * T. \tag{8.41}$$

In particular, if $k \in \mathbb{Z}_+$ and $T \in (\mathcal{E}' \cap L_k^1)(\mathbb{R}^1)$, then

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j T^{\lambda,\eta}(t) = \frac{\mathrm{i}^{\eta+1}}{\eta!}\int_{-r(T)}^t T^{(j)}(\xi)(t-\xi)^\eta \mathrm{e}^{\mathrm{i}\lambda(t-\xi)}\,\mathrm{d}\xi \tag{8.42}$$

for all $j \in \{0, \dots, k\}$ and $t \in \mathbb{R}^1$.

Proof. If $T \in (\mathcal{E}' \cap L_k^1)(\mathbb{R}^1)$, relation (8.42) follows from (8.26) and (8.28). Let us pass to the case of general T. For each $\psi \in \mathcal{D}(\mathbb{R}^1)$, $\psi \neq 0$, we obtain $T*\psi \in \mathcal{D}(\mathbb{R}^1)$ and

$$T^{\lambda,\eta} * \psi = (T * \psi)^{\lambda,\eta} = \frac{\mathrm{i}}{\eta!}\mathrm{e}_+^{\lambda,\eta} * T * \psi$$

(see (8.40), (8.42), and Proposition 8.6). Since $\psi \in \mathcal{D}(\mathbb{R}^1)$ could be arbitrary, this proves (8.41). □

Corollary 8.2. *If* $k \in \mathbb{Z}_+$ *and* $T \in (\mathcal{E}' \cap L^1_k)(\mathbb{R}^1)$, *then* $T^{\lambda,\eta}, T_{\lambda,\eta} \in C^k(\mathbb{R}^1)$. *Moreover,*

$$\left\| \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j T^{\lambda,\eta} \right\|_{C[-r(T),r(T)]} \leqslant \frac{\left(2r(T)\right)^\eta}{\eta!} \left\| T^{(j)} \right\|_{L^1[-r(T),r(T)]} \tag{8.43}$$

and

$$\left\| \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j T_{\lambda,\eta} \right\|_{C[-r(T),r(T)]} \leqslant \mathrm{e}^{2r(T)} \sigma^{\lambda,\eta}(\widehat{T}) \left\| T^{(j)} \right\|_{L^1[-r(T),r(T)]} \tag{8.44}$$

for each $j \in \{0, \dots, k\}$.

Proof. If $\operatorname{Im}\lambda \geqslant 0$, then (8.43) is an easy consequence of (8.42). Next, let us rewrite (8.42) as

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j T^{\lambda,\eta}(t) = -\frac{\mathrm{i}^{\eta+1}}{\eta!} \int_t^{r(T)} T^{(j)}(\xi)(t-\xi)^\eta \mathrm{e}^{\mathrm{i}\lambda(t-\xi)}\,\mathrm{d}\xi.$$

This yields (8.43) when $\operatorname{Im}\lambda < 0$. Using now Proposition 8.5(iii), (6.15), and (8.43), we arrive at (8.44). □

Our next task is to study the values $\langle T^{\lambda,\eta}, f\rangle$ and $\langle T_{\lambda,\eta}, f\rangle$ for some smooth functions f.

Proposition 8.12. *Let* T *be a distribution of order* q *on* $\mathbb{R}^1$, *let* $r(T) > 0$, *and let* $f \in C^{q+s}[-r(T), r(T)]$ *for some* $s \in \mathbb{N}$. *Assume that*

$$\left\langle T, f^{(\nu)} \right\rangle = 0 \quad \textit{for each } \nu \in \{0, \dots, s-1\}. \tag{8.45}$$

Then

$$\left\langle T^{\lambda,\eta}, f \right\rangle = \left(\frac{\mathrm{i}}{\lambda}\right)^s \sum_{k=0}^{\eta} \binom{s+k-1}{k} \frac{\langle T^{\lambda,\eta-k}, f^{(s)}\rangle}{(-\lambda)^k} \tag{8.46}$$

for all $\lambda \in \mathcal{Z}(\widehat{T})\backslash\{0\}$ *and* $\eta \in \{0, \dots, m(\lambda, T)\}$.

Proof. Suppose that $\lambda \in \mathcal{Z}(\widehat{T})$, $\lambda \neq 0$. Relations (8.45) and (8.26) yield

$$\left\langle T^{\lambda,0}, f^{(\nu)} \right\rangle = \frac{\mathrm{i}}{\lambda} \left\langle T^{\lambda,0}, f^{(\nu+1)} \right\rangle, \quad \nu \in \{0, \dots, s-1\}.$$

Hence,

$$\left\langle T^{\lambda,0}, f \right\rangle = \left(\frac{\mathrm{i}}{\lambda}\right)^s \left\langle T^{\lambda,0}, f^{(s)} \right\rangle, \tag{8.47}$$

which proves (8.46) for $\eta = 0$. Assume now that $m(\lambda, T) \geqslant 1$ and let $\eta \in \{1, \dots, m(\lambda, T)\}$. By induction on η we find from (8.28) that

$$\left\langle T^{\lambda,\eta}, f \right\rangle = \left(\frac{\mathrm{i}}{\lambda}\right)^s \left\langle T^{\lambda,\eta}, f^{(s)} \right\rangle - \frac{1}{\lambda} \sum_{l=0}^{s-1} \left(\frac{\mathrm{i}}{\lambda}\right)^l \left\langle T^{\lambda,\eta-1}, f^{(l)} \right\rangle. \tag{8.48}$$

Next, one checks by induction on $s = 1, 2, \dots$ that

$$\sum_{j=0}^{s-1} \binom{j+k-1}{k-1} = \binom{s+k-1}{k}$$

for each $k \in \mathbb{N}$. Using now induction on $\eta = 0, 1, \dots$, from (8.47) and the recursion relation (8.48) we obtain (8.46). □

Corollary 8.3. *Let $T \in (\mathcal{E}' \cap L^1)(\mathbb{R}^1)$, let $f \in C^s[-r(T), r(T)]$ for some $s \in \mathbb{N}$, and suppose that (8.45) holds. Assume that $\lambda \in \mathcal{Z}(\widehat{T})$, $|\lambda| > 1$, $\eta \in \{0, \dots, m(\lambda, T)\}$. Then*

$$\left|\left\langle T^{\lambda,\eta}, f\right\rangle\right| \leqslant \frac{\mathrm{e}^{2r(T)}}{(|\lambda|-1)^s} \|T\|_{L^1[-r(T),r(T)]} \left\| f^{(s)} \right\|_{L^1[-r(T),r(T)]}.$$

Proof. Applying Proposition 8.12 for the case $q = 0$, we get

$$\left|\left\langle T^{\lambda,\eta}, f\right\rangle\right| \leqslant \sum_{k=0}^{\eta} \binom{s+k-1}{k} \frac{|\langle T^{\lambda,\eta-k}, f^{(s)}\rangle|}{|\lambda|^{k+s}}. \tag{8.49}$$

Now the desired inequality follows from (8.49) and (8.43). □

The following includes the analogue of Corollary 8.3 for an arbitrary T satisfying (8.17).

Proposition 8.13. *Let $r(T) > 0$, let $f \in C^{s+d_T}[-r(T), r(T)]$ for some $s \in \mathbb{N}$, and assume that (8.45) holds. Let $\lambda \in \mathcal{Z}(\widehat{T})$, $|\lambda| > 1$, $\eta \in \{0, \dots, m(\lambda, T)\}$. Then*

$$\left|\left\langle T^{\lambda,\eta}, f\right\rangle\right| \leqslant \frac{\gamma_1}{(|\lambda|-1)^s} \sum_{j=s}^{s+d_T} \left\| f^{(j)} \right\|_{L^1[-r(T),r(T)]} \tag{8.50}$$

and

$$\left|\left\langle T_{\lambda,\eta}, f\right\rangle\right| \leqslant \frac{\gamma_2 \sigma^{\lambda,\eta}(\widehat{T})}{(|\lambda|-1)^s} \sum_{j=s}^{s+d_T} \left\| f^{(j)} \right\|_{L^1[-r(T),r(T)]}, \tag{8.51}$$

where the constants $\gamma_1, \gamma_2 > 0$ are independent of λ, η, s, f.

Proof. By the definition of d_T there exists $U \in (\mathcal{E}' \cap L^1)(\mathbb{R}^1)$ such that $\operatorname{supp} U \subset [-r(T), r(T)]$ and $p(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t})U = T$ for some polynomial p of degree d_T. Without loss of generality we suppose that $p(\lambda) \neq 0$. Then $\lambda \in \mathcal{Z}(\widehat{T}) \cap \mathcal{Z}(\widehat{U})$ and $m(\lambda, T) = m(\lambda, U)$. Equality (8.45) can be rewritten as

$$\left\langle T, f^{(\nu)}\right\rangle = \left\langle U, g^{(\nu)}\right\rangle = 0, \qquad \nu \in \{0, \dots, s-1\},$$

where $g = p(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t})f$. By Corollary 8.3,

$$\left|\left\langle T^{\lambda,\eta}, f\right\rangle\right| \leqslant \gamma_3(|\lambda|-1)^{-s}\left\|g^{(s)}\right\|_{L^1[-r(T),r(T)]},$$

where $\gamma_3 > 0$ is independent of λ, η, s, f. This yields (8.50). Together, (8.50) and Proposition 8.5(iii) brings us to (8.51). □

Proposition 8.14. *Let $r(T) > 0$, $f \in C^\infty[-r(T), r(T)]$, and suppose that (8.32) holds for all $\nu \in \mathbb{Z}_+$. Then there exist $\gamma_1 > 1$, $\gamma_2, \gamma_3, \gamma_4 > 0$ independent of f such that for all $\lambda \in \mathcal{Z}(\widehat{T})$, $|\lambda| > \gamma_1$, $\eta \in \{0, \dots, m(\lambda, T)\}$, $s \in \mathbb{N}$,*

$$\left|\left\langle T_{\lambda,\eta}, f(-\cdot)\right\rangle\right| \leqslant \gamma_2 \frac{|\lambda|^{\gamma_3}\sigma_\lambda(\widehat{T})}{(|\lambda|-1)^s}\left(\left\|f^{(s)}\right\|_{L^1[-r(T),r(T)]} + \gamma_4^s\gamma_5\right),$$

where $\gamma_5 > 0$ is independent of λ, η, s.

Proof. Let $l \in \mathbb{N}$, let $\lambda_1, \dots, \lambda_l \in \mathcal{Z}(\widehat{T})$ be distinct numbers, and assume that $s_j \in \mathbb{N}$, $s_j \leqslant m(\lambda_j, T) + 1$, $j \in \{1, \dots, l\}$. Then there exists $Q \in \mathcal{E}'(\mathbb{R}^1)$ such that $\operatorname{supp} Q \subset [-r(T), r(T)]$ and (8.33) holds (see Theorem 6.3). Assuming l to be large enough, we can suppose that $Q \in L^1(\mathbb{R}^1)$. Define $g \in C^\infty[-r(T), r(T)]$ by formula (8.34). For $\lambda \in \mathcal{Z}(\widehat{T})\backslash\{\lambda_1, \dots, \lambda_l\}$, $\eta \in \{0, \dots, m(\lambda, T)\}$, and $s \in \mathbb{N}$, one derives from Propositions 8.10, 8.5(iii), and Corollary 8.3 that

$$\begin{aligned}\left|\left\langle T_{\lambda,\eta}, f(-\cdot)\right\rangle\right| &= \left|\left\langle Q_{\lambda,\eta}, g(-\cdot)\right\rangle\right| \\ &\leqslant \sum_{j=0}^{m(\lambda,Q)} \left|a_j^{\lambda,\eta}\big(\widehat{Q}\big)\right|\left|\left\langle Q^{\lambda,m(\lambda,Q)-j}, g(-\cdot)\right\rangle\right| \\ &\leqslant \gamma_6 \frac{\sigma_\lambda(\widehat{Q})}{(|\lambda|-1)^s}\left\|g^{(s)}\right\|_{L^1[-r(T),r(T)]},\end{aligned}$$

where $\gamma_6 > 0$ is independent of f, λ, η, s. To complete the proof it remains to apply Proposition 6.6(iv) and (8.34). □

A fundamental fact that will be used quite often is the following completeness result.

Theorem 8.2. *Let $T \in (\mathcal{E}' \cap L^1)(\mathbb{R}^1)$, $f \in L^1[-r(T), r(T)]$, and*

$$\left\langle T^{\lambda,\eta}, f\right\rangle = 0 \quad \text{for all } \lambda \in \mathcal{Z}\big(\widehat{T}\big), \eta \in \{0, \dots, m(\lambda, T)\}. \tag{8.52}$$

Then $f = 0$.

Proof. For brevity, we put $r(T) = r$. Consider the entire function

$$u(z) = \mathrm{e}^{-\mathrm{i}rz}\int_{-r}^{r} f(x)\int_{-r}^{x} T(t)\mathrm{e}^{\mathrm{i}z(x-t)}\,\mathrm{d}t\,\mathrm{d}x, \quad z \in \mathbb{C}. \tag{8.53}$$

Setting $f(x) = 0$ for $x \in \mathbb{R}^1\backslash[-r, r]$, one has

$$\begin{aligned} u(z) &= \mathrm{e}^{-\mathrm{i}rz}\int_{-r}^{r} f(-x)\int_{-r}^{-x} T(t)\mathrm{e}^{-\mathrm{i}z(x+t)}\,\mathrm{d}t\,\mathrm{d}x \\ &= \mathrm{e}^{-\mathrm{i}rz}\widehat{f}(-z)\widehat{T}(z) - \mathrm{e}^{-\mathrm{i}rz}\int_{-r}^{r} f(-x)\int_{-x}^{r} T(t)\mathrm{e}^{-\mathrm{i}z(x+t)}\,\mathrm{d}t\,\mathrm{d}x \\ &= \mathrm{e}^{-\mathrm{i}rz}\widehat{f}(-z)\widehat{T}(z) - \mathrm{e}^{-2\mathrm{i}rz}v(z), \end{aligned} \tag{8.54}$$

where

$$v(z) = \mathrm{e}^{\mathrm{i}rz}\int_{-r}^{r} f(-x)\int_{-r}^{x} T(-t)\mathrm{e}^{\mathrm{i}z(t-x)}\,\mathrm{d}t\,\mathrm{d}x. \tag{8.55}$$

Formulae (8.53) and (8.55) ensure that

$$|u(z)| + |v(z)| \leqslant c_1\mathrm{e}^{r|\mathrm{Im}\,z|}, \quad z \in \mathbb{C}, \tag{8.56}$$

where $c_1 > 0$ is independent of z. In addition, $u^{(\eta)}(\lambda) = 0$ for all λ, η because of (8.52) and (8.42). Hence, the functions

$$u_1(z) = \frac{u(z)}{\widehat{T}(z)} \quad \text{and} \quad u_2(z) = \frac{v(z)}{\widehat{T}(z)}$$

are entire. Relation (8.54) yields

$$u_1(z)\mathrm{e}^{\mathrm{i}rz} + u_2(z)\mathrm{e}^{-\mathrm{i}rz} = \widehat{f}(-z), \quad z \in \mathbb{C}. \tag{8.57}$$

According to Napalkov [159, Corollary 15.22], (8.56) and (8.18), for each $\varepsilon > 0$, there exists $c_2 = c_2(\varepsilon) > 0$ such that

$$|u_1(z)| + |u_2(z)| \leqslant c_2\mathrm{e}^{\varepsilon|z|} \quad \text{for all } z \in \mathbb{C}. \tag{8.58}$$

Next, let $\alpha \in (0, r)$ and $\beta \in (0, (r-\alpha)/2)$,

$$E_{1,\beta} = \{z \in \mathbb{C} : |z - r| \leqslant \beta\},$$

$$E_{2,\beta} = \{z \in \mathbb{C} : |z + r| \leqslant \beta\}.$$

Estimate (8.58) implies that there exist compactly supported measures μ_1 and μ_2 on $\mathbb{C}$ such that $\operatorname{supp}\mu_j \subset E_{j,\beta}$, $j \in \{1, 2\}$, and

$$\mu_1\big(\mathrm{e}^{-\mathrm{i}\zeta(\cdot - r)}\big) + \mu_2\big(\mathrm{e}^{-\mathrm{i}\zeta(\cdot + r)}\big) = \widehat{f}(-\zeta)$$

for all $\zeta \in \mathbb{C}$ (see Hörmander [126], Theorem 15.1.5, and (8.57)). Since ζ is arbitrary, this shows that

$$\mu_1\big(w(\cdot - r)\big) + \mu_2\big(w(\cdot + r)\big) = \int_{-r}^{r} f(-x)w(x)\,\mathrm{d}x$$

for each entire function $w : \mathbb{C} \to \mathbb{C}$. Thus,

$$\left|\int_{-r}^{r} f(-x)w(x)\,\mathrm{d}x\right| \leqslant c_3 \max_{z\in E_\beta} |w(z)|, \tag{8.59}$$

where $E_\beta = E_{1,\beta} \cup E_{2,\beta}$, and $c_3 > 0$ is independent of w. Assume now that

$$\varphi_k(z) = \frac{k}{\sqrt{\pi}}\mathrm{e}^{-(kz)^2}, \quad k \in \mathbb{N}, z \in \mathbb{C}. \tag{8.60}$$

Then $\varphi_k(z) \geqslant 0$ for $z \in \mathbb{R}^1$, and

$$\int_{\mathbb{R}^1} \varphi_k(t)\,\mathrm{d}t = 1.$$

Setting $w(z) = \varphi_k(z-\gamma)$, $\gamma \in (-\alpha,\alpha)$, we see from (8.59) and (8.60) that

$$(f * \varphi_k)(\gamma) \leqslant c_4 \exp\left(-\frac{k^2}{4}\left((r-\alpha)^2 - 4\beta^2\right)\right),$$

where $c_4 > 0$ is independent of k. This gives, on taking $k \to \infty$, that $f = 0$ on $(-\alpha,\alpha)$. As $\alpha \in (0,r)$ is arbitrary, we are done. □

For the next step, one omits the assumption that $T \in L^1(\mathbb{R}^1)$ in the previous theorem.

Corollary 8.4. *Let* $r(T) > 0$, $f \in C^{d_T}[-r(T), r(T)]$, *and suppose that* (8.52) *holds. Then* $f = 0$.

Proof. By the definition of d_T there exists a polynomial p of degree d_T such that $p(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t})U = T$ for some $U \in (\mathcal{E}' \cap L^1)(\mathbb{R}^1)$. Then $\mathcal{Z}(\widehat{U}) \subset \mathcal{Z}(\widehat{T})$ and $m(\lambda, U) \leqslant m(\lambda, T)$ for each $\lambda \in \mathcal{Z}(\widehat{U})$. In addition, $r(U) = r(T)$ and $\operatorname{supp} U \subset [-r(T), r(T)]$ because of Theorem 6.3. In view of Proposition 8.6 and (8.52), we find that

$$\langle T^{\lambda,\eta}, f\rangle = \left\langle U^{\lambda,\eta}, p\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right) f\right\rangle = 0$$

for all $\lambda \in \mathcal{Z}(\widehat{U})$ and $\eta \in \{0,\ldots,m(\lambda,U)\}$. Owing to Theorem 8.2,

$$p\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right) f = 0 \quad \text{on } [-r(T), r(T)], \tag{8.61}$$

and consequently $f \in C^\infty[-r(T), r(T)]$. Next, there exists a polynomial q such that $\mathcal{Z}(q) \cap \mathcal{Z}(p) = \varnothing$ and $q(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t})V = T$ for some $V \in (\mathcal{E}' \cap L^1)(\mathbb{R}^1)$. In the same way we obtain $q(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t})f = 0$. Combining this with (8.61), one concludes that $f = 0$. □

Corollary 8.5. *Let* $R > r(T)$, $f \in \mathcal{D}'(-R,R)$, *and let*

$$f * T^{\lambda, m(\lambda,T)} = 0 \quad \textit{in } \left(-R + r(T), R - r(T)\right) \quad \textit{for all } \lambda \in \mathcal{Z}\left(\widehat{T}\right). \tag{8.62}$$

Then $f = 0$.

Proof. First, assume that $r(T) > 0$. It follows by (8.62) and (8.28) that

$$f * T^{\lambda,\eta} = 0 \quad \text{for all } \lambda \in \mathcal{Z}\big(\widehat{T}\big), \eta \in \{0, \dots, m(\lambda, T)\}.$$

Now the desired result can easily be deduced from Corollary 8.4 with the help of the standard smoothing procedure (see Theorem 6.1(i)). For the case $r(T) = 0$, the required conclusion follows from (8.23) and Proposition 13.2 in Part III. □

Remark 8.1. Proposition 8.5(iii) and (6.5) show that Theorem 8.2 and Corollary 8.4 remain valid, provided that $T^{\lambda,\eta}$ are replaced by $T_{\lambda,\eta}$. In addition, Corollary 8.5 remains to be true, provided that (8.62) is replaced by the condition

$$f * T_{\lambda,\eta} = 0 \quad \text{for all } \lambda \in \mathcal{Z}\big(\widehat{T}\big), \eta \in \{0, \dots, m(\lambda, T)\}.$$

Corollary 8.6. *Let $T \in \mathcal{E}'_{\natural}(\mathbb{R}^1)$, $R > r(T)$, $f \in \mathcal{D}'(-R, R)$, and let*

$$f * T_{(\lambda)} = 0 \quad \text{in } \big(-R + r(T), R - r(T)\big) \quad \text{for all } \lambda \in \mathcal{Z}\big(\widehat{T}\big), \tag{8.63}$$

where $T_{(\lambda)} \in \mathcal{E}'_{\natural}(\mathbb{R}^1)$ is defined by

$$\widehat{T_{(\lambda)}}(z) = \begin{cases} \widehat{T}(z)(z^2 - \lambda^2)^{-m(\lambda,T)-1} & \text{if } \lambda \neq 0, \\ \widehat{T}(z) z^{-m(0,T)-1} & \text{if } \lambda = 0 \in \mathcal{Z}\big(\widehat{T}\big). \end{cases} \tag{8.64}$$

Then $f = 0$.

Proof. We see from (8.63), (8.64), and (8.23) that (8.62) holds. Hence, f must vanish because of Corollary 8.5. □

To go further, let $m \in \mathbb{N}$, $m \geqslant 2$, and let $T_1, \dots, T_m$ be nonzero distributions in the class $\mathcal{E}'(\mathbb{R}^1)$. For the rest of the section, we assume that

$$\mathcal{Z}\big(\widehat{T_l}\big) \neq 0 \quad \text{and} \quad \operatorname{supp} T_l \subset [-r(T_l), r(T_l)]$$

for all $l \in \{1, \dots, m\}$. Next, for $l, k \in \{1, \dots, m\}$, $\lambda_l \in \mathcal{Z}(\widehat{T_l})$, we define the distribution $T_{\lambda_1,\dots,\lambda_m,k} \in \mathcal{E}'(\mathbb{R}^1)$ by letting

$$\widehat{T_{\lambda_1,\dots,\lambda_m,k}} = \prod_{\substack{l=1,\\ l\neq k}}^{m} \widehat{(T_l)_{\lambda_l,0}}$$

(see Theorem 6.3).

Proposition 8.15. *Assume that $\bigcap_{l=1}^{m} \mathcal{Z}(\widehat{T_l}) = \varnothing$, $\lambda_l \in \mathcal{Z}(\widehat{T_l})$, and*

$$m\big(\lambda_l, \widehat{T_l}\big) = 0 \quad \text{for all } l \in \{1, \dots, m\}. \tag{8.65}$$

Let $c_1, \dots, c_m \in \mathbb{C}$,

$$\sum_{l=1}^{m} c_l = 0, \quad \textit{and} \quad \sum_{l=1}^{m} c_l \lambda_l = -1. \tag{8.66}$$

Then

$$(T_1)_{\lambda_1,0} * \cdots * (T_m)_{\lambda_m,0} = \sum_{l=1}^{m} \frac{c_l}{\widehat{T_l}'(\lambda_l)} T_l * T_{\lambda_1,\ldots,\lambda_m,l}. \tag{8.67}$$

Proof. According to (8.25) and (6.5),

$$\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_l\right)(T_l)_{\lambda_l,0} = \frac{T_l}{\widehat{T_l}'(\lambda_l)}, \quad l \in \{1, \ldots, m\}.$$

For $c_1, \ldots, c_m \in \mathbb{C}$, this yields

$$\sum_{l=1}^{m} \frac{c_l}{\widehat{T_l}'(\lambda_l)} T_l * T_{\lambda_1,\ldots,\lambda_m,l} = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{l=1}^{m} c_l (T_1)_{\lambda_1,0} * \cdots * (T_m)_{\lambda_m,0}\right) - \left(\sum_{l=1}^{m} \lambda_l c_l\right)(T_1)_{\lambda_1,0} * \cdots * (T_m)_{\lambda_m,0}.$$

The validity of (8.67) is therefore obvious from (8.66). □

To conclude we shall obtain an analog of (8.67) for the case $m = 2$ without assumption (8.65). For $\lambda_1 \in \mathcal{Z}(\widehat{T_1})$, $\lambda_2 \in \mathcal{Z}(\widehat{T_2})$, we set

$$\nu = \nu(\lambda_1, \lambda_2) = m(\lambda_1, T_1) + m(\lambda_2, T_2) + 2.$$

Proposition 8.16. *If $\mathcal{Z}(\widehat{T_1}) \cap \mathcal{Z}(\widehat{T_2}) = \varnothing$, then*

$$\begin{aligned}(T_1)_{\lambda_1,0} * (T_2)_{\lambda_2,0} = \frac{1}{(\lambda_2 - \lambda_1)^{2\nu}} &\Bigg(\sum_{k=0}^{\nu} (-1)^{\nu-k} \binom{2\nu}{\nu+k} \sum_{j=0}^{m(\lambda_1,T_1)} a_j^{\lambda_1,0}(\widehat{T_1}) \\ &\times \left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_1\right)^{k+j+m(\lambda_2,T_2)+1} \\ &\times \left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_2\right)^{\nu-k} \big(T_1 * (T_2)_{\lambda_2,0}\big) \\ &+ \sum_{k=1}^{\nu} (-1)^{\nu+k} \binom{2\nu}{\nu-k} \sum_{j=0}^{m(\lambda_2,T_2)} a_j^{\lambda_2,0}(\widehat{T_2}) \left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_1\right)^{\nu-k} \\ &\times \left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_2\right)^{k+j+m(\lambda_1,T_1)+1} \big(T_2 * (T_1)_{\lambda_1,0}\big)\Bigg)\end{aligned}$$

for all $\lambda_1 \in \mathcal{Z}(\widehat{T_1})$ and $\lambda_2 \in \mathcal{Z}(\widehat{T_2})$.

Proof. For $l \in \{1, 2\}$, we define $\mu_l \in \mathbb{Z}_+$ by letting $\mu_l = m(\lambda_2, T_2) + 1$ if $l = 1$ and $\mu_l = m(\lambda_1, T_1) + 1$ if $l = 2$. It follows from Proposition 8.5(ii) that

$$\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_l\right)^{\nu} (T_l)_{\lambda_l,0} = \sum_{j=0}^{m(\lambda_l, T_l)} a_j^{\lambda_l,0}(\widehat{T}_l)\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_l\right)^{j+\mu_l} T_l.$$

Hence,

$$\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_l\right)^{\nu} \big((T_1)_{\lambda_1,0} * (T_2)_{\lambda_2,0}\big) = \sum_{j=0}^{m(\lambda_l, T_l)} a_j^{\lambda_l,0}(\widehat{T}_l)\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_l\right)^{j+\mu_l} T_l * K_l,$$

where

$$K_1 = (T_2)_{\lambda_2,0} \quad \text{and} \quad K_2 = (T_1)_{\lambda_1,0}.$$

The required conclusion now follows from the fact that

$$\begin{aligned}(\lambda_2 - \lambda_1)^{2\nu} &(T_1)_{\lambda_1,0} * (T_2)_{\lambda_2,0} \\ &= \left(\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_1\right) - \left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_2\right)\right)^{2\nu} \big((T_1)_{\lambda_1,0} * (T_2)_{\lambda_2,0}\big).\end{aligned}$$

□

8.3 Expansions in Series of Exponentials

Let $T \in \mathcal{E}'(\mathbb{R}^1)$, $T \neq 0$. In the sequel we shall consider series of the form

$$\sum_{\lambda \in \mathcal{Z}(\widehat{T})} \sum_{\eta=0}^{m(\lambda,T)} c_{\lambda,\eta} \mathrm{e}^{\lambda,\eta}(t), \tag{8.68}$$

where $c_{\lambda,\eta} \in \mathbb{C}$, $\mathrm{e}^{\lambda,\eta}(t) = (\mathrm{i}t)^{\eta} \mathrm{e}^{\mathrm{i}\lambda t}$, $t \in (\alpha, \beta) \subset \mathbb{R}^1$. According to what has been said in Sect. 6.1, the convergence of series (8.68) in the space $\mathcal{D}'(\alpha, \beta)$ is understood as the convergence in $\mathcal{D}'(\alpha, \beta)$ of the sequence of its partial sums

$$S_N(t) = \sum_{\substack{\lambda \in \mathcal{Z}(\widehat{T}) \\ |\lambda| \leqslant N}} \sum_{\eta=0}^{m(\lambda,T)} c_{\lambda,\eta} \mathrm{e}^{\lambda,\eta}(t) \tag{8.69}$$

as $N \to +\infty$. The convergence of (8.68) in other spaces is defined likewise.

Lemma 8.2. *Let* $R > 0$, $\lambda \in \mathbb{C}$, $\eta, q \in \mathbb{Z}_+$, *and let*

$$A(R, \eta, q) = \begin{cases} R^{\eta} & \text{if } R \geqslant 1, \\ R^{\max\{0, \eta - q\}} & \text{if } R < 1. \end{cases}$$

Then

$$\left|\left(\mathrm{e}^{\lambda,\eta}(t)\right)^{(q)}\right| \leqslant A(R,\eta,q)(|\lambda|+\eta)^q \mathrm{e}^{-t\,\mathrm{Im}\,\lambda} \tag{8.70}$$

for each $t \in [-R, R]$.

Proof. The desired statement is obvious from the identity

$$\left(\mathrm{e}^{\lambda,\eta}(t)\right)^{(q)} = \sum_{j=0}^{\min\{q,\eta\}} \binom{q}{j} \frac{\eta!\mathrm{i}^{\eta+q-j}}{(\eta-j)!} \lambda^{q-j} t^{\eta-j} \mathrm{e}^{\mathrm{i}\lambda t}, \tag{8.71}$$

since $\eta!/(\eta-j)! \leqslant \eta^j$ for $\eta \geqslant 1$. □

Corollary 8.7. *Let* $R > 0$, $t \in [-R, R]$, $\lambda \in \mathcal{Z}(\widehat{T})$, $\eta \in \{0, \dots, m(\lambda, T)\}$, $q \in \mathbb{Z}_+$. *Then*

$$\sum_{\eta=0}^{m(\lambda,T)} \left|\left(\mathrm{e}^{\lambda,\eta}(t)\right)^{(q)}\right| \leqslant \gamma_1\gamma_2^q B(R,\lambda,q)(1+|\lambda|)^q \mathrm{e}^{-t\,\mathrm{Im}\,\lambda}, \tag{8.72}$$

where

$$B(R,\lambda,q) = \begin{cases} R^{m(\lambda,T)} & \text{if } R > 1, \\ m(\lambda,T)+1 & \text{if } R = 1, \\ \min\{q+1, m(\lambda,T)+1\} & \text{if } R < 1, \end{cases} \tag{8.73}$$

and the constants $\gamma_1, \gamma_2 > 0$ *depend only of* R, T.

Proof. Using (8.4), one has

$$|\lambda| + \eta \leqslant (1+|\lambda|)\left(1 + \frac{m(\lambda,T)}{1+|\lambda|}\right) \leqslant \gamma_3(1+|\lambda|),$$

where $\gamma_3 > 0$ depend only of T. This, together with (8.70), brings us to the required assertion. □

The following result gives very utilitarian sufficient conditions for the convergence of (8.68) in the spaces $\mathcal{D}'(\alpha,\beta)$ and $\mathcal{E}(\alpha,\beta)$.

Proposition 8.17.

(i) *Assume that*

$$\sup_{\lambda\in\mathcal{Z}(\widehat{T})} \frac{|\mathrm{Im}\,\lambda| + m(\lambda,T)}{\log(2+|\lambda|)} < +\infty \tag{8.74}$$

and

$$|c_{\lambda,\eta}| \leqslant (2+|\lambda|)^\gamma \quad \textit{for all } \lambda \in \mathcal{Z}\big(\widehat{T}\big), \eta \in \{0,\dots,m(\lambda,T)\}, \tag{8.75}$$

where $\gamma > 0$ *is independent of* λ, η. *Then series* (8.68) *converges in* $\mathcal{D}'(\mathbb{R}^1)$.

(ii) *Let* $R > 0$, *and let*

$$\sum_{\lambda \in \mathcal{Z}(\widehat{T})} \sum_{\eta=0}^{m(\lambda,T)} |c_{\lambda,\eta}|(|\lambda| + \eta)^q R^{\max\{0,\eta-q\}} e^{R|\operatorname{Im}\lambda|} < +\infty \tag{8.76}$$

for some $q \in \mathbb{Z}_+$. *Then series* (8.68) *converges in* $C^q[-R, R]$. *The same is true, provided that* (8.76) *is replaced by*

$$\sum_{\lambda \in \mathcal{Z}(\widehat{T})} \Big(\max_{0 \leqslant \eta \leqslant m(\lambda,T)} |c_{\lambda,\eta}|\Big) B(R, \lambda, q)(1 + |\lambda|)^q e^{R|\operatorname{Im}\lambda|} < +\infty,$$

where $B(R, \lambda, q)$ *is defined by* (8.73). *In particular, if* (8.74) *holds and*

$$\max_{0 \leqslant \eta \leqslant m(\lambda,T)} |c_{\lambda,\eta}| = O\big(|\lambda|^{-\gamma}\big) \quad \textit{as } \lambda \to \infty \tag{8.77}$$

for each fixed $\gamma > 0$, *then series* (8.68) *converges in* $\mathcal{E}(\mathbb{R}^1)$.

Proof. To prove (i), first observe that

$$\langle S_N, \varphi\rangle = \sum_{\substack{\lambda \in \mathcal{Z}(\widehat{T}), \\ |\lambda| \leqslant N}} \sum_{\eta=0}^{m(\lambda,T)} c_{\lambda,\eta} \widehat{\varphi}^{(\eta)}(\lambda) \tag{8.78}$$

for each $\varphi \in \mathcal{D}(\mathbb{R}^1)$ (see (8.69)). By Theorem 6.3, for each $\alpha > 0$, there exist $\gamma_1, \gamma_2 > 0$ such that

$$\big|\widehat{\varphi}(\zeta)\big| \leqslant \gamma_1 (2 + |\zeta|)^{-\alpha} e^{\gamma_2 |\operatorname{Im}\zeta|} \quad \text{for all } \zeta \in \mathbb{C}.$$

Applying (6.24) with $g = \widehat{\varphi}$, $z = \lambda \in \mathcal{Z}(\widehat{T})$, $s = \eta \leqslant m(\lambda, T)$, $r = \log(2 + |\lambda|)$, we see from (8.74) that

$$\max_{0 \leqslant \eta \leqslant m(\lambda,T)} \big|\widehat{\varphi}^{(\eta)}(\lambda)\big| = O\big(|\lambda|^{-\beta}\big) \quad \text{as } \lambda \to \infty$$

for each fixed $\beta > 0$. Using now (8.75), (8.78), and (8.5), we arrive at (i).

Assertion (ii) is an easy consequence of Lemma 8.2 and Corollary 8.7. □

Remark 8.2. It can be easily shown that condition (8.74) in Proposition 8.17(i) cannot be omitted. We note also that for $T \in \mathfrak{N}(\mathbb{R}^1)$, conditions (8.75) and (8.77) in Proposition 8.17 are necessary (see V.V. Volchkov [225], Part III, Lemma 1.5).

Proposition 8.18. *Assume that* (8.74) *holds and let*

$$\max_{0 \leqslant \eta \leqslant m(\lambda,T)} |c_{\lambda,\eta}| \leqslant \frac{M_q}{(2 + |\lambda|)^q}, \quad q = 1, 2, \ldots, \tag{8.79}$$

where the constants $M_q > 0$ *are independent of* λ *and satisfy*

$$\sum_{\nu=1}^{\infty} \frac{1}{\inf\limits_{q \geqslant \nu} M_q^{1/q}} = +\infty. \tag{8.80}$$

Then series (8.68) converges in $\mathcal{E}(\mathbb{R}^1)$ to $f \in \mathrm{QA}(\mathbb{R}^1)$.

Proof. Let $R > 0$ and $q \in \mathbb{Z}_+$. According to (8.70),

$$\left|\left(\mathrm{e}^{\lambda,\eta}(t)\right)^{(q)}\right| \leqslant (2 + |\lambda|)^{q+\gamma_1}, \quad t \in [-R, R],$$

where $\gamma_1 > 0$ is independent of λ, η, q, t. Let $s \in \mathbb{N}$, $s \geqslant \gamma_1 + 2$. Then we have

$$\sum_{\lambda \in \mathcal{Z}(\widehat{T})} \sum_{\eta=0}^{m(\lambda,T)} \left|c_{\lambda,\eta}\left(\mathrm{e}^{\lambda,\eta}(t)\right)^{(q)}\right| \leqslant \gamma_2 M_{q+s}, \quad q \in \mathbb{N},\ t \in [-R, R],$$

where $\gamma_2 > 0$ is independent of q, t (see (8.79) and (8.5)). Now the desired statement follows from Lemma 8.1(i). □

Proposition 8.19. *Let $\alpha > 0$ and suppose that*

$$|\mathrm{Im}\,\lambda| + m(\lambda, T) = o\left(|\lambda|^{1/\alpha}\right) \quad \text{as } \lambda \to \infty \tag{8.81}$$

and

$$|c_{\lambda,\eta}| \leqslant \gamma_1 \exp\left(-\gamma_2 |\lambda|^{1/\alpha}\right), \tag{8.82}$$

where the constants $\gamma_1, \gamma_2 > 0$ are independent of λ, η. Then series (8.68) converges in $\mathcal{E}(\mathbb{R}^1)$ to $f \in G^\alpha(\mathbb{R}^1)$.

Proof. Let $R > 0$, $t \in [-R, R]$, and $q \in \mathbb{N}$. Estimates (8.72), (8.81) and (8.82) imply that

$$\sum_{\lambda \in \mathcal{Z}(\widehat{T})} \sum_{\eta=0}^{m(\lambda,T)} \left|c_{\lambda,\eta}\left(\mathrm{e}^{\lambda,\eta}(t)\right)^{(q)}\right| \leqslant \gamma_3^q \sum_{\lambda \in \mathcal{Z}(\widehat{T})} (1 + |\lambda|)^{-2+q} \exp\left(-\gamma_4 |\lambda|^{1/\alpha}\right), \tag{8.83}$$

where $\gamma_3, \gamma_4 > 0$ depend only of $R, T, \alpha, \gamma_1, \gamma_2$. It is easy to see that

$$(1 + |\lambda|)^q \exp\left(-\gamma_4 |\lambda|^{1/\alpha}\right) \leqslant \gamma_5^q q^{\alpha q}$$

for some $\gamma_5 > 0$ independent of λ, q. This, together with (8.83) and (8.5), concludes the proof. □

For the rest of the section, we assume that

$$r(T) > 0 \quad \text{and} \quad \mathrm{supp}\, T \subset [-r(T), r(T)].$$

For $\lambda \in \mathcal{Z}(\widehat{T})$ and $q \in \mathbb{Z}_+$, let us define $B(r(T), \lambda, q)$ by formula (8.73).

Theorem 8.3. *Let $s \in \mathbb{N}$, $q \in \mathbb{Z}_+$, $q \geqslant d_T$, and let*

$$\sum_{\lambda \in \mathcal{Z}(\widehat{T})} \frac{\sigma_\lambda(\widehat{T})}{(1+|\lambda|)^{s-q}} B\big(r(T), \lambda, q\big) e^{r(T)|\operatorname{Im}\lambda|} < +\infty. \tag{8.84}$$

Suppose that $f \in C^{s+d_T}[-r(T), r(T)]$ and

$$\big\langle T, f^{(\nu)}(-\cdot)\big\rangle = 0 \tag{8.85}$$

for each $\nu \in \{0, \dots, s-1\}$. Then

$$f = \sum_{\lambda \in \mathcal{Z}(\widehat{T})} \sum_{\eta=0}^{m(\lambda,T)} c_{\lambda,\eta} e^{\lambda,\eta}, \tag{8.86}$$

where $c_{\lambda,\eta} = \langle T_{\lambda,\eta}, f(-\cdot)\rangle$, and the series converges in $C^q[-r(T), r(T)]$.

Proof. Owing to Proposition 8.13,

$$|c_{\lambda,\eta}| \leqslant \gamma(1+|\lambda|)^{-s}\sigma_\lambda\big(\widehat{T}\big),$$

where the constant $\gamma > 0$ is independent of λ, η. Then it follows by (8.84) and Proposition 8.17(ii) that the series in the right-hand part of (8.86) converges in $C^q[-r(T), r(T)]$. Denoting its sum by g, we obtain $c_{\lambda,\eta} = \langle T_{\lambda,\eta}, g(-\cdot)\rangle$ (see (8.24), (8.3), Proposition 8.5(iii), and Corollary 8.1). Therefore, $\langle T_{\lambda,\eta}, f(-\cdot) - g(-\cdot)\rangle = 0$ for all $\lambda \in \mathcal{Z}(\widehat{T})$ and $\eta \in \{0, \dots, m(\lambda, T)\}$. On account of Corollary 8.4, this finishes the proof. □

Corollary 8.8. *Assume that (8.74) is fulfilled and*

$$\sigma_\lambda\big(\widehat{T}\big) \leqslant (2+|\lambda|)^{\gamma_1}, \tag{8.87}$$

where $\gamma_1 > 0$ is independent of λ. Then the following statements are equivalent.

(i) *$f \in C^\infty[-r(T), r(T)]$, and (8.85) is satisfied for each $\nu \in \mathbb{Z}_+$.*
(ii) *Relation (8.86) holds with $c_{\lambda,\eta} = \langle T_{\lambda,\eta}, f(-\cdot)\rangle$, and the series in (8.86) converges to f in $\mathcal{E}[-r(T), r(T)]$.*

The proof follows directly from Theorem 8.3.

Corollary 8.9. *Assume that (8.74) and (8.87) are satisfied. Then the following assertions are equivalent.*

(i) *$f \in \mathrm{QA}[-r(T), r(T)]$, and (8.85) is satisfied for each $\nu \in \mathbb{Z}_+$.*
(ii) *Conditions (8.79) and (8.80) are fulfilled with $c_{\lambda,\eta} = \langle T_{\lambda,\eta}, f(-\cdot)\rangle$, and the series in (8.86) converges to f in $\mathcal{E}[-r(T), r(T)]$.*

Proof. The implication (i)→(ii) is a consequence of Corollary 8.8, Proposition 8.14, and Lemma 8.1 (i). The implication (ii)→(i) is obvious from Proposition 8.18. □

Remark 8.3. Proposition 6.6(iii) shows that Corollaries 8.8 and 8.9 remain valid, provided that (8.87) is replaced by

$$\frac{n_\lambda!}{|\widehat{T}^{(n_\lambda)}(\lambda)|}\big(1+\gamma\big(\lambda,\widehat{T}\big)\big)^{n_\lambda-1}\leqslant(2+|\lambda|)^{\gamma_1},$$

where $n_\lambda=n_\lambda(\widehat{T})$, the number $\gamma(\lambda,\widehat{T})$ is defined by (6.16), and the constant $\gamma_1>0$ is independent of λ.

Next, for $f\in C^\infty[-r(T),r(T)]$ and $\lambda\in\mathcal{Z}(\widehat{T})$, we define

$$\mu(\lambda,f)=\begin{cases}\inf\limits_{s\in\mathbb{N}}(|\lambda|-1)^{-s}\sum\limits_{j=s}^{s+d_T}\big\|f^{(j)}\big\|_{L^1[-r(T),r(T)]} & \text{if } |\lambda|>1,\\ 0 & \text{if } |\lambda|\leqslant 1.\end{cases}$$

Theorem 8.4. *Let $f\in C^\infty[-r(T),r(T)]$, and let (8.85) hold for all $\nu\in\mathbb{Z}_+$. Suppose that*

$$\sum_{\lambda\in\mathcal{Z}(\widehat{T})}\sigma_\lambda\big(\widehat{T}\big)B(r(T),\lambda,q)(1+|\lambda|)^q\mu(\lambda,f)e^{r(T)|\mathrm{Im}\,\lambda|}<+\infty$$

for some $q\in\mathbb{Z}_+$, $q\geqslant d_T$. Then relation (8.86) is satisfied with $c_{\lambda,\eta}=\langle T_{\lambda,\eta},f(-\cdot)\rangle$, and the series converges in $C^q[-r(T),r(T)]$.

Proof. There is no difficulty in modifying the proof of Theorem 8.3 using Proposition 8.13 and Corollary 8.4 to obtain the desired result. □

Corollary 8.10. *Let $\alpha>0$ and assume that $T\in\mathfrak{G}_\alpha(\mathbb{R}^1)$. Then the following statements are equivalent.*

(i) *$f\in G^\alpha[-r(T),r(T)]$, and (8.85) holds for each $\nu\in\mathbb{Z}_+$.*
(ii) *Condition (8.82) is satisfied with $c_{\lambda,\eta}=\langle T_{\lambda,\eta},f(-\cdot)\rangle$, and the series in (8.86) converges to f in $\mathcal{E}[-r(T),r(T)]$.*

Proof. First, assume that (i) holds. By the definition of $\mu(\lambda,f)$ one sees that

$$|\mu(\lambda,f)|\leqslant\gamma_3^\nu\nu^{\alpha\nu}(1+|\lambda|)^{-\nu},\quad \nu=1,2,\ldots,\tag{8.88}$$

where $\gamma_3>0$ is independent of λ,ν. Assuming $|\lambda|$ to be large enough and setting $\nu=[e^{-1}\gamma_3^{-1/\alpha}(1+|\lambda|)^{1/\alpha}]$, one concludes from (8.88) that

$$|\mu(\lambda,f)|\leqslant\gamma_4\exp\big(-\gamma_5|\lambda|^{1/\alpha}\big),$$

where $\gamma_4,\gamma_5>0$ are independent of λ. To complete the proof of the implication (i)→(ii) it remains to apply (8.4), Theorem 8.4, and Proposition 8.13.

The implication (ii)→(i) follows at once from Proposition 8.19. □

8.4 The Distribution ζ_T. Solution of the Lyubich Problem

By the definition of the class $\mathrm{Inv}_+(\mathbb{R}^1)$ one sees that for each $T \in \mathrm{Inv}_+(\mathbb{R}^1)$, there exists a unique distribution $\zeta_T^+ \in \mathcal{D}'(\mathbb{R}^1)$ such that

$$\operatorname{supp} \zeta_T^+ \subset [0, +\infty) \quad \text{and} \quad \mathrm{i}\zeta_T^+ * T = \delta_0. \tag{8.89}$$

Similarly, if $T \in \mathrm{Inv}_-(\mathbb{R}^1)$, then there is a unique $\zeta_T^- \in \mathcal{D}'(\mathbb{R}^1)$ such that

$$\operatorname{supp} \zeta_T^- \subset (-\infty, 0] \quad \text{and} \quad \mathrm{i}\zeta_T^- * T = -\delta_0. \tag{8.90}$$

Throughout this section we suppose that

$$T \in (\mathrm{Inv}_+ \cap \mathrm{Inv}_-)\big(\mathbb{R}^1\big) \quad \text{and} \quad \operatorname{supp} T \subset [-r(T), r(T)]. \tag{8.91}$$

Now define

$$\zeta_T = \zeta_T^+ + \zeta_T^-. \tag{8.92}$$

This distribution will play an important role later on in this book. The aim of this section is to establish some basic properties of ζ_T.

Proposition 8.20. *The following assertions hold.*

(i) $\zeta_T * T = 0$.
(ii) *If T is even* (*respectively odd*), *then ζ_T is odd* (*respectively even*).
(iii) *If $V \in \mathcal{E}'(\mathbb{R}^1)$ and $T = p(\frac{\mathrm{d}}{\mathrm{d}t})V$ for some polynomial p, then V belongs to* $(\mathrm{Inv}_+ \cap \mathrm{Inv}_-)(\mathbb{R}^1)$ *and $\zeta_V = p(\frac{\mathrm{d}}{\mathrm{d}t})\zeta_T$.*
(iv) *If $a, b \in \mathbb{R}^1$, $b - a > 2r(T)$, $u \in \mathcal{E}'(\mathbb{R}^1)$, and $\zeta_T * u = 0$ in (a, b), then $u = T * v$ for some $v \in \mathcal{E}'(\mathbb{R}^1)$.*
(v) *If $r(T) > 0$, then $\zeta_T = 0$ in $(-r(T), r(T))$. In addition, $\pm r(T) \in \operatorname{supp} \zeta_T$.*

Proof. Part (i) is clear from (8.89), (8.90), and (8.92). To prove (ii), first suppose that T is even. Then by the definition of ζ_T^+ and ζ_T^-, one concludes that $\zeta_T^+(-\cdot) = -\zeta_T^-(\cdot)$. This implies that ζ_T is odd because of (8.92). The case where T is odd can be treated by a similar way.

Regarding (iii), we set $\zeta_V^+ = p(\frac{\mathrm{d}}{\mathrm{d}t})\zeta_T^+$ and $\zeta_V^- = p(\frac{\mathrm{d}}{\mathrm{d}t})\zeta_T^-$. Then $\operatorname{supp} \zeta_V^+ \subset [0, +\infty)$ and $\operatorname{supp} \zeta_V^- \subset (-\infty, 0]$. Moreover, by assumption on V and (8.89), (8.90), it follows that

$$\mathrm{i}\zeta_V^+ * V = \delta_0, \qquad \mathrm{i}\zeta_V^- * V = -\delta_0.$$

In view of (8.92), this verifies (iii).

As for (iv), it is sufficient to show that the function $\widehat{u}/\widehat{T}$ is entire (see Theorem 6.4). Hence, we may confine our attention to the case where $\mathcal{Z}(\widehat{T}) \neq \varnothing$. Due to Theorem 6.2 and (8.92),

$$\mathrm{i}\zeta_T^+ * u = -\mathrm{i}\zeta_T^- * u \quad \text{in } \mathbb{R}^1.$$

This equality combined with (8.89) and (8.41) yields

$$e_+^{\lambda,\eta} * u = e_+^{\lambda,\eta} * T * i\zeta_T^+ * u = \frac{1}{\eta!} T^{\lambda,\eta} * \zeta_T^- * u \tag{8.93}$$

for all $\lambda \in \mathcal{Z}(\widehat{T})$ and $\eta \in \{0,\dots,m(\lambda,T)\}$. Relation (8.93) delivers $e_+^{\lambda,\eta} * u \in \mathcal{E}'(\mathbb{R}^1)$ since $\zeta_T^- = 0$ on $(0,+\infty)$. Therefore, $\widehat{u}^{(\eta)}(\lambda) = 0$, proving (iv).

Concerning (v), Theorem 6.2, together with (8.89)–(8.91), shows that $\zeta_T^+ = 0$ in $(-\infty, r(T))$ and $\zeta_T^- = 0$ in $(-r(T), +\infty)$. The same argument leads to the conclusion that $r(T) \in \operatorname{supp} \zeta_T^+$ and $-r(T) \in \operatorname{supp} \zeta_T^-$. According to (8.92), this ends the proof of (v). □

A fundamental fact that will be used quite often is the following.

Theorem 8.5. *Let $T \in \mathfrak{M}(\mathbb{R}^1)$. Then $T \in (\mathrm{Inv}_+ \cap \mathrm{Inv}_-)(\mathbb{R}^1)$ and*

$$\zeta_T = \sum_{\lambda \in \mathcal{Z}(\widehat{T})} \sum_{\eta=0}^{m(\lambda,T)} a_{m(\lambda,T)}^{\lambda,\eta}(\widehat{T}) e^{\lambda,\eta},$$

where the series converges in $\mathcal{D}'(\mathbb{R}^1)$. In particular, $\mathfrak{M}(\mathbb{R}^1) \cap \mathcal{D}(\mathbb{R}^1) = \varnothing$.

The proof is based on the following lemmas, which are of independent interest.

Lemma 8.3. *Let $U \in \mathfrak{M}(\mathbb{R}^1)$, $\operatorname{supp} U \subset [-r(U), r(U)]$, and let*

$$\sum_{\lambda \in \mathcal{Z}(\widehat{U})} \sigma_\lambda(\widehat{U}) e^{\varepsilon |\mathrm{Im}\,\lambda|} < +\infty \tag{8.94}$$

for some $\varepsilon > 0$. Then $U \in (\mathrm{Inv}_+ \cap \mathrm{Inv}_-)(\mathbb{R}^1)$,

$$\zeta_U^+ = \sum_{\lambda \in \mathcal{Z}(\widehat{U})} \sum_{\eta=0}^{m(\lambda,U)} a_{m(\lambda,U)}^{\lambda,\eta}(\widehat{U}) e_+^{\lambda,\eta}, \tag{8.95}$$

and

$$\zeta_U^- = \sum_{\lambda \in \mathcal{Z}(\widehat{U})} \sum_{\eta=0}^{m(\lambda,U)} a_{m(\lambda,U)}^{\lambda,\eta}(\widehat{U}) e_-^{\lambda,\eta}, \tag{8.96}$$

where $e_-^{\lambda,\eta} = e^{\lambda,\eta} - e_+^{\lambda,\eta}$, and the series in (8.95) and (8.96) converge in $\mathcal{D}'(\mathbb{R}^1)$.

Proof. It follows by Proposition 8.17(i) that the series in (8.95) and (8.96) converge in $\mathcal{D}'(\mathbb{R}^1 \backslash \{0\})$. Moreover, (8.94) and (6.9) imply that these series converge uniformly on $[-\varepsilon, \varepsilon]$. Denoting its sums by S^+ and S^-, we need only to show that $S^+ = \zeta_U^+$ and $S^- = \zeta_U^-$. Relations (8.95) and (8.96) yield

$$\operatorname{supp} S^+ \subset [0, +\infty), \qquad \operatorname{supp} S^- \subset (-\infty, 0].$$

Next, (8.41), (6.9), and Proposition 8.5(iii) allow us to write

$$\mathrm{i}S^+ * U = -\mathrm{i}S^- * U = \sum_{\lambda \in \mathcal{Z}(\widehat{U})} U_{\lambda,0}.$$

Our lemma is thereby established because of (8.94) and Proposition 8.9. □

Lemma 8.4. *Let $T \in \mathcal{E}'(\mathbb{R}^1)$, $T \neq 0$, and let $U = p(\frac{\mathrm{d}}{\mathrm{d}t})T$ for some polynomial p. Assume that $\lambda \in \mathcal{Z}(\widehat{U})$. Then*

$$p\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \sum_{\eta=0}^{m(\lambda,U)} a^{\lambda,\eta}_{m(\lambda,U)}\big(\widehat{U}\big)\mathrm{e}^{\lambda,\eta} = \begin{cases} \displaystyle\sum_{\eta=0}^{m(\lambda,T)} a^{\lambda,\eta}_{m(\lambda,T)}\big(\widehat{T}\big)\mathrm{e}^{\lambda,\eta} & \text{if } \lambda \in \mathcal{Z}\big(\widehat{T}\big), \\ 0 & \text{if } \lambda \notin \mathcal{Z}\big(\widehat{T}\big). \end{cases}$$

Proof. It is enough to consider the case where p is a linear polynomial. In this case the required formula follows from (6.33), (6.9), and Proposition 6.5. □

Proof of Theorem 8.5. Owing to Proposition 8.3, there exists a polynomial p such that the distribution $U = p(\frac{\mathrm{d}}{\mathrm{d}t})T$ satisfies (8.94) and $U \in \mathfrak{M}(\mathbb{R}^1)$. By Lemma 8.3,

$$\zeta_U = \sum_{\lambda \in \mathcal{Z}(\widehat{U})} \sum_{\eta=0}^{m(\lambda,U)} a^{\lambda,\eta}_{m(\lambda,U)}\big(\widehat{U}\big)\mathrm{e}^{\lambda,\eta},$$

where the series converges in $\mathcal{D}'(\mathbb{R}^1)$. To complete the proof it remains to apply Proposition 8.20(iii) and Lemma 8.4. □

We would like to continue our consideration with the following result which will be useful in Sect. 13.2. Here and below we set $\{\alpha\} = \alpha - [\alpha]$, where $\alpha \in \mathbb{R}^1$ and $[\alpha]$ is the integer part of α.

Theorem 8.6. *Let $k \in \mathbb{Z}_+$, $\alpha < 1$, $p \geqslant 1$, $p(1 - \{\alpha\}) < 1$, and let*

$$m = m(k,\alpha) = \begin{cases} k - 2 + \alpha & \text{if } \alpha \in \mathbb{Z}, \\ k - 1 + [\alpha] & \text{if } \alpha \notin \mathbb{Z},\ k + \alpha > 1, \\ k + [\alpha] & \text{if } \alpha \notin \mathbb{Z},\ k + \alpha < 1. \end{cases}$$

Assume that $r > 0$ and $T = V^{(k)}$ where

$$V(t) = \begin{cases} \left(r^2 - t^2\right)^{-\alpha} & \text{if } t \in (-r, r), \\ 0 & \text{if } |t| \geqslant r. \end{cases}$$

Then $T \in \mathfrak{N}(\mathbb{R}^1)$ and $\zeta_T \in L^{p,\mathrm{loc}}_m(\mathbb{R}^1)$.

Proof. By (7.7),

$$\widehat{T}(z) = c_1 z^k \mathbf{I}_{1/2-\alpha}(rz), \quad z \in \mathbb{C}, \tag{8.97}$$

where $c_1 \in \mathbb{C}\backslash\{0\}$ is independent of z. Hence, all the zeros of $\widehat{T}$ are real, and the set of zeros is symmetric with respect to $z = 0$. In addition, all the zeros of the function

$\widehat{T}(z)z^{-k}$ are simple (see Sect. 7.1). Thus,

$$\mathcal{Z}(\widehat{T})\backslash\{0\} = \{\lambda_1, \lambda_2, \dots\} \cup \{-\lambda_1, -\lambda_2, \dots\},$$

where $\{\lambda_n\}_{n=1}^{\infty}$ is the sequence of all positive zeros of $\widehat{T}$ numbered in the ascending order. It is known that

$$\lambda_n = \frac{\pi}{r}\left(n - \frac{\alpha}{2} + q_{\alpha,r}\right) + O\left(\frac{1}{n}\right) \quad \text{as } n \to \infty, \tag{8.98}$$

where $q_{\alpha,r} \in \mathbb{Z}$ is independent of n (see (7.9)). Going now to the asymptotic formulae for the Bessel functions (see (7.8)), we find from (7.3), (8.97), and (8.98) that

$$\widehat{T}'(\lambda_n) = c_2\lambda_n^{k+1}\mathbf{I}_{3/2-\alpha}(r\lambda_n) = c_3(-1)^n\lambda_n^{k+\alpha-1} + O\left(n^{k+\alpha-2}\right), \tag{8.99}$$

where $c_2, c_3 \in \mathbb{C}\backslash\{0\}$, and the constant in O does not depend on n. Therefore, $T \in \mathfrak{N}(\mathbb{R}^1)$, and by Theorem 8.5

$$\zeta_T(t) = u_1(t) + u_2(t) + (-1)^{k+1}u_2(-t), \quad t \in \mathbb{R}^1, \tag{8.100}$$

where

$$u_1(t) = \begin{cases} \sum\limits_{\eta=0}^{k-1} a_{k-1}^{0,\eta}(\widehat{T})(\mathrm{i}t)^{\eta} & \text{if } k \geqslant 1, \\ 0 & \text{if } k = 0, \end{cases} \tag{8.101}$$

and

$$u_2(t) = \frac{1}{c_2}\sum_{n=1}^{\infty} \frac{\lambda_n^{-1-k}e^{\mathrm{i}\lambda_n t}}{\mathbf{I}_{3/2-\alpha}(r\lambda_n)}. \tag{8.102}$$

According to (8.99) and Proposition 8.17(i), the series in (8.102) converges in $\mathcal{D}'(\mathbb{R}^1)$. Define $\beta = \beta(\alpha)$ by letting $\beta = \{\alpha\}$ if $\alpha \notin \mathbb{Z}$ and $\beta = 1$ if $\alpha \in \mathbb{Z}$. In the case of $k + \alpha > 1$ we have

$$u_2^{(m)}(t) = c_4 \exp\left(\mathrm{i}\frac{\pi t}{r}\left(q_{\alpha,r} - \frac{\alpha}{2}\right)\right)\sum_{n=1}^{\infty}\frac{e^{\mathrm{i}\pi n t(1+1/r)}}{n^{\beta}} + u_3(t),$$

where $c_4 \in \mathbb{C}\backslash\{0\}$ and $u_3 \in C(\mathbb{R}^1)$ (see (8.98), (8.99), (8.102)). This, together with (8.100), (8.101), and Edwards [60, Sect. 7.3.5(ii)], tells us that $\zeta_T^{(m)} \in L^{p,\mathrm{loc}}(\mathbb{R}^1)$, which disposes of the case $k + \alpha > 1$.

Next, let $k + \alpha \leqslant 1$ and

$$u_5(t) = u_4(t) + (-1)^{k+m+1}u_4(-t),$$

where

$$u_4(t) = \frac{\mathrm{i}^m}{c_2}\sum_{n=1}^{\infty}\frac{\lambda_n^{m-1-k}e^{\mathrm{i}\lambda_n t}}{\mathbf{I}_{3/2-\alpha}(r\lambda_n)}, \quad t \in \mathbb{R}^1. \tag{8.103}$$

The series in (8.103) converges in $\mathcal{D}'(\mathbb{R}^1)$, and $u_5^{(-m)} = \zeta_T - u_1$ because of (8.100). Using now Edwards [60, Sect. 7.3.5 (ii)], (8.98), and (8.99), we infer as in the case $k + \alpha > 1$ that $u_5 \in L^{p,\mathrm{loc}}(\mathbb{R}^1)$. Hence the theorem. □

A function $f \in C^\infty(\mathbb{R}^1)$ is said to be *mean w-periodic* if there exists $\mu \in \mathcal{E}'(\mathbb{R}^1)$ that satisfies the following conditions: (a) $\operatorname{supp}\mu \subset [0, w]$; (b) $0, w \in \operatorname{supp}\mu$; and (c) $f * \mu = 0$. Each w-periodic function $f \in C^\infty(\mathbb{R}^1)$ is obviously a mean w-periodic function.

The mean w-periodic functions are similar to the ordinary w-periodic functions in the following sense: if a mean w-periodic function is equal to zero everywhere on $[0, w]$, then it vanishes in $\mathbb{R}^1$ (see Sect. 13.2 below). We can say more about the w-periodic functions:

$$\text{if } E \subset \mathbb{R}^1,\ [0, w) \subset \big\{x \in \mathbb{R}^1 : x = y - [y/w]w \text{ for some } y \in E\big\}, \tag{8.104}$$

and f is w-periodic, then

$$f|_E = 0 \quad \Longrightarrow \quad f = 0. \tag{8.105}$$

Lyubich [148] has posed the following problem. Is the implication (8.105) valid for a mean w-periodic function f if E satisfies (8.104)? The following result shows that this problem has a negative solution.

Theorem 8.7. *Let $r > 0$, $a \geqslant 2r$, and*

$$\lambda_q = \frac{2\pi}{a}\left[\frac{aq}{2r}\right], \quad q = 1, 2, \ldots. \tag{8.106}$$

Then there exists even $T \in \mathfrak{N}(\mathbb{R}^1)$ with the following properties:

(1) $\widehat{T}(z) = \prod_{q=1}^{\infty}\left(1 - \frac{z^2}{\lambda_q^2}\right)$, $z \in \mathbb{C}$;
(2) $r(T) = r$;
(3) ζ_T *is a-periodical. In particular $\zeta_T = 0$ on the set* $\bigcup_{m\in\mathbb{Z}}(am - r, am + r)$.

Proof. From (8.106) one sees that $\lambda_{q+1} - \lambda_q \geqslant 2\pi/a$ and $|\lambda_q - \pi q/r| \leqslant 2\pi/a$ for all q. According to [225, Part V, Lemma 3.2], there exists $T \in \mathcal{E}'_\natural(\mathbb{R}^1)$ satisfying (1) such that $\operatorname{supp} T \subset [-r, r]$. Due to Theorem 6.3 and Proposition 6.1(ii), (iii), $r(T) = r$. Next, it is not difficult to adapt the argument in the proof of Lemma 3.2 in [225, Part V] to show that $|\widehat{T}'(\lambda_q)| > q^{-\gamma}$ for some $\gamma > 0$ independent of q. So $T \in \mathfrak{N}_\natural(\mathbb{R}^1)$ and $T \in \operatorname{Inv}(\mathbb{R}^1)$ in view of Theorem 8.5. In addition, for each $m \in \mathbb{Z}$, the function $(e^{iamz} - 1)/\widehat{T}(z)$ is entire. This means that $\delta_0(\cdot + am) - \delta_0(\cdot) = (T * \psi)(\cdot)$ for some $\psi \in \mathcal{E}'(\mathbb{R}^1)$ (see (6.35) and Theorem 6.4). Convolving with ζ_T and appealing to Proposition 8.20(i) and Theorem 8.5, we conclude that T satisfies (3). This proves the theorem. □

Chapter 9
Multidimensional Euclidean Case

In this chapter we proceed immediately to the study of transmutation operators. Harmonic analysis on $\mathrm{SO}(n)/\mathrm{SO}(n-1)$ is the starting point for the theory to be developed here.

We have seen in Part I that the space $L^2(\mathbb{S}^{n-1})$ is the orthogonal direct sum of the spaces $\mathcal{H}_1^{n,k}$, $k \in \mathbb{Z}_+$. By a passage to polar coordinates this enables us to expand an arbitrary function $f(x) \in L^2(\mathbb{R}^n)$ into a series analogous to (4.11), where the coefficients $f_{k,j}$ depend on $|x|$. The restriction of the Fourier transform to each of the summands can be identified with a classical Hankel transform. A generalization of these arguments to distributions leads to the definition of the spaces $\mathcal{E}'_{k,j}(\mathbb{R}^n)$ and the transform $\mathcal{F}_j^k$ on $\mathcal{E}'_{k,j}(\mathbb{R}^n)$ (see Sects. 9.1–9.3). In Sect. 9.3 we formulate a Paley–Wiener-type theorem for $\mathcal{F}_j^k$, which is of course a special case of the general fact for the Fourier transform. Thereby the mapping $\Lambda : \mathcal{E}'_\natural(\mathbb{R}^n) \to \mathcal{E}'_\natural(\mathbb{R}^1)$ given by

$$\mathcal{F}_1^0(T) = \widehat{\Lambda(T)}$$

is a bijection, and

$$\inf\left\{r \geqslant 0 : \operatorname{supp} T \subset \mathring{B}_r\right\} = \inf\left\{r \geqslant 0 : \operatorname{supp} \Lambda(T) \subset [-r, r]\right\}.$$

We use Λ to introduce some important classes of distributions for the study of mean periodic functions in $\mathbb{R}^n$. In Sect. 9.4, by means of the inversion formula for the transform $\mathcal{F}_j^k$, we define the injective operator $\mathfrak{A}_{k,j} : \mathcal{E}'_{k,j}(\mathbb{R}^n) \to \mathcal{D}'_\natural(\mathbb{R}^1)$ which is a fundamental tool to reduce n-dimensional convolution equations to one-dimensional ones. The key observation is the generalized transmutation property

$$\mathfrak{A}_{k,j}(f * T) = \mathfrak{A}_{k,j}(f) * \Lambda(T).$$

We indicate the basic properties of $\mathfrak{A}_{k,j}$ and compute the converse operator $\mathfrak{B}_{k,j}$. It turns out that $\mathfrak{B}_{0,1}$ coincides with the dual Abel transform. The results of Sect. 9.4 play an essential role in many of the proofs later. In particular, we apply them in Sect. 9.5 to study Bessel-type expansions.

V.V. Volchkov, V.V. Volchkov, *Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group,* Springer Monographs in Mathematics,

9.1 Introductory Results

Throughout Chap. 9 we shall consider a real Euclidean space $\mathbb{R}^n$ of dimension $n \geqslant 2$ with inner product $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ and Euclidean norm $|\cdot|$. For $0 \leqslant R \leqslant +\infty$ and $x \in \mathbb{R}^n$, we set

$$B_R(x) = \big\{y \in \mathbb{R}^n : |x-y| < R\big\}, \qquad B_R = B_R(0).$$

If $R < +\infty$, then $\mathring{B}_R(x)$ and $\mathring{B}_R$ denote the sets $\{y \in \mathbb{R}^n : |x-y| \leqslant R\}$ and $\mathring{B}_R(0)$, respectively. Also let

$$S_R = \big\{x \in \mathbb{R}^n : |x| = R\big\}.$$

For $0 \leqslant r \leqslant R \leqslant +\infty$, $r < +\infty$, let

$$B_{r,R} = B_R \backslash \mathring{B}_r \quad \text{and} \quad \mathring{B}_{r,R} = \mathring{B}_R \backslash B_r.$$

Denote by ϱ and $\sigma = \{\sigma_1, \dots, \sigma_n\}$ the polar coordinates in $\mathbb{R}^n$ (for each $x \in \mathbb{R}^n$, we set $\varrho = |x|$, and if $x \in \mathbb{R}^n\backslash\{0\}$, then we put $\sigma = x/\varrho$). Let ω_{n-1} be the surface area of the sphere $\mathbb{S}^{n-1} = S_1$, that is,

$$\omega_{n-1} = 2\pi^{n/2}\big/\Gamma(n/2).$$

Following Sect. 4.1, we write $\mathcal{H}_1^{n,k}$ for the space of all homogeneous complex-valued harmonic polynomials on $\mathbb{R}^n$ of degree k. Let

$$\mathcal{H}_1^{n,k}\big(\mathbb{S}^{n-1}\big) = \mathcal{H}_1^{n,k}\big|_{\mathbb{S}^{n-1}},$$

and let $d(n,k)$ be the dimension (over $\mathbb{C}$) of $\mathcal{H}_1^{n,k}(\mathbb{S}^{n-1})$. A simple calculation shows that $d(n,0) = 1$, $d(n,1) = n$, and

$$d(n,k) = \binom{n+k-1}{k} - \binom{n+k-3}{k-2} \quad \text{if } k \geqslant 2$$

(see Stein and Weiss [203], Chap. 4). We note that $\mathcal{H}_1^{n,k}(\mathbb{S}^{n-1})$ and $\mathcal{H}_1^{n,k}$ have the same dimension, since the map $h \to h|_{\mathbb{S}^{n-1}}$ is a vector space isomorphism of $\mathcal{H}_1^{n,k}$ onto $\mathcal{H}_1^{n,k}(\mathbb{S}^{n-1})$.

We define $Y_1^0(\sigma) = 1/\sqrt{\omega_{n-1}}$ for all $\sigma \in \mathbb{S}^{n-1}$. Next, for $k \geqslant 1$, let $\{Y_j^k\}$, $j \in \{1, \dots, d(n,k)\}$, be a fixed orthonormal basis in the space $\mathcal{H}_1^{n,k}(\mathbb{S}^{n-1})$ regarded as a subspace of $L^2(\mathbb{S}^{n-1})$. We set $Y_j^k(x) = \varrho^k Y_j^k(\sigma)$ for $x = \varrho\sigma \in \mathbb{R}^n\backslash\{0\}$, $k \in \mathbb{Z}_+$. Using this relation, let us extend Y_j^k to a polynomial on $\mathbb{C}^n$.

Let $T_1^n(\tau)$, $\tau \in \mathrm{O}(n)$, be the quasi-regular representation of the group $\mathrm{O}(n)$ in $L^2(\mathbb{S}^{n-1})$. As we already know from Sect. 4.1, $T_1^n(\tau)$ is a direct sum of pairwise nonequivalent irreducible unitary representations $T_1^{n,k}(\tau)$ acting on the spaces

$\mathcal{H}_1^{n,k}(\mathbb{S}^{n-1})$. Let $\{t_{i,j}^k(\tau)\}$ denote a representation matrix of $T_1^{n,k}(\tau)$, that is,

$$Y_j^k\left(\tau^{-1}\sigma\right) = \sum_{i=1}^{d(n,k)} t_{i,j}^k(\tau) Y_i^k(\sigma), \quad \sigma \in \mathbb{S}^{n-1}. \tag{9.1}$$

This relation yields

$$t_{i,j}^k(\tau) = \int_{\mathbb{S}^{n-1}} Y_j^k\left(\tau^{-1}\sigma\right)\overline{Y_i^k(\sigma)}\, \mathrm{d}\omega(\sigma), \quad \tau \in \mathrm{O}(n),$$

whence $t_{i,j}^k$ is real-analytic on $\mathrm{O}(n)$, and

$$t_{i,j}^k\left(\tau^{-1}\right) = \overline{t_{j,i}^k(\tau)}. \tag{9.2}$$

Let $\mathcal{O}$ be a nonempty open subset of $\mathbb{R}^n$ such that

$$\tau\mathcal{O} = \mathcal{O} \quad \text{for each } \tau \in \mathrm{O}(n).$$

We associate with each function $f \in L^{1,\mathrm{loc}}(\mathcal{O})$ its Fourier series into spherical harmonics

$$f(x) \sim \sum_{k=0}^{\infty} \sum_{j=1}^{d(n,k)} f_{k,j}(\varrho) Y_j^k(\sigma), \quad x \in \mathcal{O}, \tag{9.3}$$

where

$$f_{k,j}(\varrho) = \int_{\mathbb{S}^{n-1}} f(\varrho\sigma)\overline{Y_j^k(\sigma)}\, \mathrm{d}\omega(\sigma).$$

By the Fubini theorem the functions $f_{k,j}$ are well defined for almost all $\varrho \in \{r > 0 : S_r \subset \mathcal{O}\}$. Moreover, the functions

$$f^{k,j,i}(x) = f_{k,j}(\varrho) Y_i^k(\sigma) \tag{9.4}$$

are in $L^{1,\mathrm{loc}}(\mathcal{O})$. Owing to (1.80),

$$f^{k,j,i}(x) = d(n,k) \int_{\mathrm{O}(n)} f\left(\tau^{-1}x\right)\overline{t_{i,j}^k(\tau)}\, \mathrm{d}\tau, \tag{9.5}$$

where $\mathrm{d}\tau$ is the normalized Haar measure on the group $\mathrm{O}(n)$. Combining this with (1.77), one obtains

$$\int_E \left|f^{k,j,i}(x)\right| \mathrm{d}x \leqslant \sqrt{d(n,k)} \int_E |f(x)|\, \mathrm{d}x \tag{9.6}$$

for each nonempty $\mathrm{O}(n)$-invariant compact set $E \subset \mathcal{O}$. Next, for each $\psi \in \mathcal{D}(\mathcal{O})$, it follows from (9.2) and (9.5) that

$$\begin{aligned}\int_{\mathcal{O}} f^{k,j,i}(x)\psi(x)\, \mathrm{d}x &= d(n,k) \int_{\mathcal{O}} f(x) \int_{\mathrm{O}(n)} \psi\left(\tau^{-1}x\right) t_{j,i}^k(\tau)\, \mathrm{d}\tau\, \mathrm{d}x \\ &= \int_{\mathcal{O}} f(x)\overline{(\overline{\psi})_{k,j}(\varrho)}\; \overline{Y_i^k(\sigma)}\, \mathrm{d}x. \end{aligned} \tag{9.7}$$

According to (9.7), we now extend the map $f \to f^{k,j,i}$ and expansion (9.3) to distributions $f \in \mathcal{D}'(\mathcal{O})$ as follows:

$$\langle f^{k,j,i}, \psi\rangle = \left\langle f, d(n,k)\int_{\mathrm{O}(n)} \psi\big(\tau^{-1}x\big)t^k_{j,i}(\tau)\,\mathrm{d}\tau\right\rangle$$
$$= \big\langle f, \overline{(\overline{\psi})_{k,j}(\varrho)}\ \overline{Y^k_i(\sigma)}\big\rangle, \quad \psi \in \mathcal{D}(\mathcal{O}), \tag{9.8}$$

and

$$f \sim \sum_{k=0}^{\infty}\sum_{j=1}^{d(n,k)} f^{k,j}, \quad \text{where } f^{k,j} = f^{k,j,j}. \tag{9.9}$$

The following result summarizes basic properties of the distributions $f^{k,j,i}$.

Proposition 9.1.

(i) *The mapping* $f \to f^{k,j,i}$ *is a continuous operator from* $\mathcal{D}'(\mathcal{O})$ *into* $\mathcal{D}'(\mathcal{O})$.
(ii) ord $f^{k,j,i} \leqslant$ ord f.
(iii) *If* $f \in C^m(\mathcal{O})$ *for some* $m \in \mathbb{Z}_+$, *then* $f^{k,j,i} \in C^m(\mathcal{O})$.
(iv) *Let* u *be a radial distribution in* $\mathcal{E}'(\mathbb{R}^n)$ *such that the set* $\mathcal{U} = \{x \in \mathbb{R}^n : x - \operatorname{supp} u \subset \mathcal{O}\}$ *is nonempty. Then*

$$(f * u)^{k,j,i} = f^{k,j,i} * u \quad \text{in } \mathcal{U}. \tag{9.10}$$

In particular, $(p(\Delta)f)^{k,j,i} = p(\Delta)f^{k,j,i}$ *for each polynomial* p.
(v) *If* $x \in \operatorname{supp} f^{k,j,i}$, *then* $\tau x \in \operatorname{supp} f^{k,j,i}$ *for all* $\tau \in \mathrm{O}(n)$.
(vi) *Series* (9.9) *converges to* f *in the space* $\mathcal{D}'(\mathcal{O})$ *(respectively* $\mathcal{E}(\mathcal{O})$*) for* $f \in \mathcal{D}'(\mathcal{O})$ *(respectively* $\mathcal{E}(\mathcal{O})$*).*

Proof. Parts (i)–(iii) are clear from the definition of $f^{k,j,i}$ and (9.5). Turning to (iv), let $\psi \in \mathcal{D}(\mathcal{U})$. Then

$$\langle (f*u)^{k,j,i}, \psi\rangle = \left\langle f(x), \left\langle u(y), d(n,k)\int_{\mathrm{O}(n)} \psi\big(\tau^{-1}x + \tau^{-1}y\big)t^k_{j,i}(\tau)\,\mathrm{d}\tau\right\rangle\right\rangle$$
$$= \left\langle f(x), d(n,k)\int_{\mathrm{O}(n)} \big\langle u(y), \psi\big(\tau^{-1}x + y\big)\big\rangle t^k_{j,i}(\tau)\,\mathrm{d}\tau\right\rangle$$
$$= \langle f^{k,j,i} * u, \psi\rangle,$$

proving (9.10). In order to complete the proof of (iv), it is enough to substitute $u = p(\Delta)\delta_0$ into (9.10).

Part (v) is obvious from (9.4) if $f \in L^{1,\mathrm{loc}}(\mathcal{O})$. The general case reduces to this one by means of the standard smoothing trick (see Theorem 6.1) and (iv).

For the proof of (vi), we refer the reader to Helgason [122], Chap. 5, Sect. 3.1, where a generalization of this statement is contained. □

For an arbitrary set $\mathfrak{W}(\mathcal{O}) \subset \mathcal{D}'(\mathcal{O})$, we shall write $\mathfrak{W}_{k,j}(\mathcal{O})$ for the set

$$\{f \in \mathfrak{W}(\mathcal{O}) : f = f^{k,j}\}.$$

Observe that $\mathfrak{W}_{0,1}(\mathcal{O}) = \mathfrak{W}_{\natural}(\mathcal{O})$, where, as usual, $\mathfrak{W}_{\natural}(\mathcal{O})$ is the set of all radial distributions in $\mathfrak{W}(\mathcal{O})$.

We end this section by quoting some relations concerning the action of the operator Δ on $C^2_{k,j}(\mathcal{O})$. For a nonempty open set $E \subset (0, +\infty)$ and $\alpha \in \mathbb{R}^1$, let us regard the differential operator $D(\alpha)$ defined on $C^1(E)$ by the formula

$$\big(D(\alpha)\varphi\big)(\varrho) = \varrho^{\alpha}\frac{\mathrm{d}}{\mathrm{d}\varrho}\left(\varrho^{-\alpha}\varphi(\varrho)\right), \quad \varphi \in C^1(E). \tag{9.11}$$

It can be verified that for each $f \in C^2_{k,j}(\mathcal{O})$,

$$\begin{aligned}(\Delta f)(x) &= \big(D(1-n-k)D(k)f_{k,j}\big)(\varrho)Y^k_j(\sigma)\\ &= \big(D(k-1)D(2-n-k)f_{k,j}\big)(\varrho)Y^k_j(\sigma)\\ &= \left(f''_{k,j}(\varrho) + \frac{n-1}{\varrho}f'_{k,j}(\varrho) - \frac{k(n+k-2)}{\varrho^2}f_{k,j}(\varrho)\right)Y^k_j(\sigma)\end{aligned} \tag{9.12}$$

(see V.V. Volchkov [225], Part I, formula (5.20)).

9.2 Spherical Functions and Their Generalizations

Let $k \in \mathbb{Z}_+$, $j \in \{1, \dots, d(n,k)\}$, $\eta \in \mathbb{Z}_+$. For $\lambda \in \mathbb{C}$, we define

$$\Phi_{\lambda,\eta,k,j}(x) = \Phi_{\lambda,\eta,k}(\varrho)Y^k_j(\sigma), \quad x \in \mathbb{R}^n\backslash\{0\}, \tag{9.13}$$

where

$$\Phi_{\lambda,\eta,k}(\varrho) = \tau_n\varrho^k\left(\frac{\partial}{\partial z}\right)^{\varkappa}\left(\mathbf{I}_{\frac{n}{2}+k-1}(z\varrho)\right)\Big|_{z=\lambda}, \tag{9.14}$$

$$\tau_n = 2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)\sqrt{\omega_{n-1}} \quad \text{and} \quad \varkappa = \begin{cases}\eta & \text{if } \lambda \neq 0,\\ 2\eta & \text{if } \lambda = 0.\end{cases} \tag{9.15}$$

It is easy to see from (9.13)–(9.15) that the function $\Phi_{\lambda,\eta,k,j}$ admits continuous extension to 0. As usual, we assume that $\Phi_{\lambda,\eta,k,j}$ is defined at 0 by continuity. Then $\Phi_{\lambda,\eta,k,j} \in \mathrm{RA}(\mathbb{R}^n)$ and

$$\Phi_{\lambda,\eta,k,j}(0) = \delta_{0,\eta}\delta_{0,k}, \tag{9.16}$$

(see (7.1)). In addition, relations (9.14) and (7.1) yield

$$\Phi_{0,\eta,k,j}(x) = \frac{(-1)^{\eta}(2\eta)!\Gamma(n/2)\sqrt{\omega_{n-1}}}{\eta!\Gamma(\frac{n}{2}+k+\eta)2^{2\eta+k}}|x|^{2\eta}Y^k_j(x), \quad x \in \mathbb{R}^n. \tag{9.17}$$

The function $\Phi_{\lambda,0,0,1}$ is called the *spherical function* on $\mathbb{R}^n$.

Next, for $\lambda \in \mathbb{C}\backslash(-\infty; 0)$, we set

$$\Psi_{\lambda,\eta,k,j}(x) = \Psi_{\lambda,\eta,k}(\varrho)Y_j^k(\sigma), \quad x \in \mathbb{R}^n\backslash\{0\}, \tag{9.18}$$

where

$$\Psi_{\lambda,\eta,k}(\varrho) = \tau_n \varrho^k \left(\frac{\partial}{\partial z}\right)^\eta \left(\mathbf{N}_{\frac{n}{2}+k-1}(z\varrho)\right)\Big|_{z=\lambda} \quad \text{if } \lambda \neq 0 \tag{9.19}$$

and

$$\Psi_{0,\eta,k}(\varrho) = \begin{cases} \varrho^{2\eta-n-k+2} & \text{if either } n \text{ is odd or } 2\eta < 2k+n-2, \\ \varrho^{2\eta-n-k+2}\log\varrho & \text{otherwise.} \end{cases} \tag{9.20}$$

The aim of this section is to consider basic properties of $\Phi_{\lambda,\eta,k,j}$ and $\Psi_{\lambda,\eta,k,j}$ that we shall frequently use later.

Proposition 9.2.

(i) *If* $\lambda \in \mathbb{C}$, *then*

$$D(k)\Phi_{\lambda,0,k} = -\lambda^2\Phi_{\lambda,0,k+1}$$

and

$$D(1-k-n)\Phi_{\lambda,0,k+1} = \Phi_{\lambda,0,k}$$

(*see* (9.11)). *Moreover, for the functions* $\Psi_{\lambda,0,k}$, *the same equalities are true, provided that* $\lambda \in \mathbb{C}\backslash(-\infty, 0]$.

(ii) *If* $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, *then*

$$\Phi_{\lambda,0,k}(\varrho)\frac{\mathrm{d}}{\mathrm{d}\varrho}\Psi_{\lambda,0,k}(\varrho) - \Psi_{\lambda,0,k}(\varrho)\frac{\mathrm{d}}{\mathrm{d}\varrho}\Phi_{\lambda,0,k}(\varrho) = \frac{2}{\pi}\tau_n^2\lambda^{2-2k-n}\varrho^{1-n}.$$

Proof. Part (i) follows at once from (7.3). To prove (ii) it is enough to combine (9.14), (9.19), and (7.6). □

Theorem 9.1.

(i) *If* $q \in \mathbb{Z}_+$ *and* $q \leqslant \eta$, *then*

$$\Delta^q\Phi_{0,\eta,k,j} = \begin{cases} (-1)^q(-2\eta)_{2q}\Phi_{0,\eta-q,k,j} & \text{if } q \leqslant \eta, \\ 0 & \text{if } q > \eta. \end{cases} \tag{9.21}$$

(ii) *If* $q \in \mathbb{Z}_+$ *and* $\lambda \in \mathbb{C}\backslash\{0\}$, *then*

$$\Delta^q\Phi_{\lambda,\eta,k,j} = (-1)^q \sum_{\nu=\max\{0,\eta-2q\}}^{\eta} \binom{\eta}{\nu}\frac{(2q)!\lambda^{2q-\eta+\nu}}{(2q-\eta+\nu)!}\Phi_{\lambda,\nu,k,j} \tag{9.22}$$

and

$$\left(\Delta+\lambda^2\right)^q \Phi_{\lambda,\eta,k,j} = \sum_{\nu=\max\{0,\eta-2q\}}^{\eta} \frac{\eta!2^{2q-\eta+\nu}(-q)_{2q-\eta+\nu}}{\nu!(2q-\eta+\nu)!}(-1)^{q+\nu-\eta}\lambda^{2q-\eta+\nu}\Phi_{\lambda,\nu,k,j}. \quad (9.23)$$

In addition, for the functions $\Psi_{\lambda,\eta,k,j}$, the same relations hold in $\mathbb{R}^n\setminus\{0\}$, provided that $\lambda\in\mathbb{C}\setminus(-\infty,0]$.

(iii) *If $\lambda\in\mathbb{C}\setminus(-\infty,0)$, then $\Psi_{\lambda,0,0,1}\in L^{1,\mathrm{loc}}(\mathbb{R}^n)$ and*

$$\left(\Delta+\lambda^2\right)\Psi_{\lambda,0,0,1} = \begin{cases} 2^n\Gamma(n/2)\pi^{\frac{n}{2}-1}\lambda^{2-n}\delta_0 & \text{if } \lambda\in\mathbb{C}\setminus(-\infty,0],\\ (2-n)\sqrt{\omega_{n-1}}\delta_0 & \text{if } \lambda=0 \text{ and } n>2,\\ \sqrt{2\pi}\delta_0 & \text{if } \lambda=0,\ n=2.\end{cases}$$

The proof starts with the following lemma.

Lemma 9.1. *Let $q,m\in\mathbb{Z}_+$, $q\geqslant m$. Then*

$$\sum_{l=m}^{q}(-1)^l\binom{q}{l}\binom{2l}{2m-1} = (-1)^{q+1}\frac{2^{2q-2m+1}}{(2q-2m+1)!}(-q)_{2q-2m+1} \quad (9.24)$$

and

$$\sum_{l=m}^{q}(-1)^l\binom{q}{l}\binom{2l}{2m} = (-1)^{q}\frac{2^{2(q-m)}}{(2q-2m)!}(-q)_{2q-2m}, \quad (9.25)$$

where $\binom{2l}{2m-1}$ is set to be equal to zero for $m=0$, $l\in\mathbb{Z}_+$.

Proof. To verify (9.24) we set

$$b_{l,m}=\sum_{\nu=m}^{l}(-1)^\nu\binom{l}{\nu}a_{\nu,m},\quad l\in\mathbb{Z}_+, l\geqslant m,$$

where

$$a_{\nu,m}=(-1)^{\nu+1}\frac{2^{2\nu-2m+1}}{(2\nu-2m+1)!}(-\nu)_{2\nu-2m+1}.$$

Then

$$a_{q,m}=\sum_{l=m}^{q}(-1)^l\binom{q}{l}b_{l,m}$$

(see Riordan [183], Sect. 2.1). Hence, it is enough to show that

$$b_{l,m}=\binom{2l}{2m-1}. \quad (9.26)$$

For the case where $l \leqslant 2m-1$, one has

$$\begin{aligned} b_{l,m} &= \sum_{\nu=m}^{l} 2^{2(\nu-m)+1}\binom{l}{\nu}\binom{\nu}{2(\nu-m)+1} \\ &= \sum_{p=0}^{l-m} 2^{2p+1}\binom{l}{m+p}\binom{m+p}{2p+1} \\ &= \sum_{p=0}^{l-m} 4^{p}\binom{l}{m+p}\binom{m+p}{2p}\frac{2m-2p}{2p+1} \\ &= \sum_{p=0}^{l-m} 4^{p}\binom{l}{m+p}\binom{m+p}{2p}\frac{2m+1}{2p+1} - \sum_{p=0}^{l-m} 4^{p}\binom{l}{m+p}\binom{m+p}{2p} \\ &= \sum_{p=0}^{l-m} 4^{p}\binom{l}{l-m+p}\binom{l-m+p}{2p}\frac{2m+1}{2p+1} \\ &\quad - \sum_{p=0}^{l-m} 4^{p}\binom{l}{l-m+p}\binom{l-m+p}{2p}. \end{aligned}$$

Utilizing the equalities

$$\binom{2l+1}{2m} = \sum_{\nu=0}^{m} 4^{\nu}\frac{2m+1}{2\nu+1}\binom{m+\nu}{2\nu}\binom{l}{m+\nu} \tag{9.27}$$

and

$$\binom{2l}{2m} = \sum_{\nu=0}^{m} 4^{\nu}\binom{m+\nu}{2\nu}\binom{l}{m+\nu} \tag{9.28}$$

(see Riordan [183], Chap. 1, Sect. 1.6, Exercise 16), we arrive at (9.26) for $l \leqslant 2m-1$.

Suppose now that $l \geqslant 2m$. Then

$$\begin{aligned} b_{l,m} &= \sum_{\nu=m}^{2m-1} (-1)^{\nu}\binom{l}{\nu}a_{\nu,m} \\ &= \sum_{\nu=m}^{2m-1} 2^{2(\nu-m)+1}\binom{l}{\nu}\binom{\nu}{2(\nu-m)+1} \\ &= \sum_{p=0}^{m-1} 2^{2p+1}\binom{l}{m+p}\binom{m+p}{2p+1} \\ &= \sum_{p=0}^{m-1} 4^{p}\binom{l}{m+p}\binom{m+p}{2p}\frac{2m+1}{2p+1} - \sum_{p=0}^{m-1} 4^{p}\binom{l}{m+p}\binom{m+p}{2p}. \end{aligned}$$

Combining this with (9.27) and (9.28), we get

$$b_{l,m} = \binom{2l+1}{2m} - \binom{2l}{2m} = \binom{2l}{2m-1},$$

so (9.24) is established. The proof of (9.25) is quite similar. □

Proof of Theorem 9.1. One deduces from (9.12) and Proposition 9.2(i) that

$$\Delta \Phi_{\lambda,0,k,j} = -\lambda^2 \Phi_{\lambda,0,k,j} \quad \text{in } \mathbb{R}^n \tag{9.29}$$

for each $\lambda \in \mathbb{C}$. In addition, the same equality in which $\Phi_{\lambda,0,k,j}$ is replaced by $\Psi_{\lambda,0,k,j}$ is true in $\mathbb{R}^n \backslash \{0\}$ when $\lambda \in \mathbb{C}\backslash(-\infty, 0]$. For any $\lambda \in \mathbb{C}$, formulae (9.29), (9.13), and (9.14) imply

$$\Delta^q \Phi_{\lambda,\eta,k,j} = \left(\frac{\partial}{\partial z}\right)^{\varkappa} \left((-z^2)^q \Phi_{z,0,k,j}\right)\Big|_{z=\lambda},$$

where $\varkappa$ is defined by (9.15). Now a straightforward calculation reveals that (9.21) and (9.22) are valid. As for (9.23), one infers from (9.22) that

$$\begin{aligned}
&(\Delta + \lambda^2)^q \Phi_{\lambda,\eta,k,j} \\
&= \sum_{l=0}^{q} (-1)^l \binom{q}{l} \sum_{\nu=\max\{0,\eta-2l\}} \binom{\eta}{\nu} \frac{(2l)!}{(2l-\eta+\nu)!} \lambda^{2q-\eta+\nu} \Phi_{\lambda,\nu,k,j} \\
&= \sum_{\nu=\max\{0,\eta-2q\}}^{\eta} \frac{\eta!}{\nu!} \lambda^{2q-\eta+\nu} \Phi_{\lambda,\nu,k,j} \sum_{l \geqslant (\eta-\nu)/2}^{q} (-1)^l \binom{q}{l} \binom{2l}{\eta-\nu}.
\end{aligned}$$

Applying (9.24) and (9.25) with $m = [(\eta - \nu + 1)/2]$, we see that two expressions in (9.23) actually coincide. The analogues of (9.22) and (9.23) for the functions $\Psi_{\lambda,\eta,k,j}$ in $\mathbb{R}^n\backslash\{0\}$ can be obtained in the same way. Finally, the proof of (iii) can be found in Treves [214, Sect. 5.3]. Hence the theorem. □

Corollary 9.1. *For each* $\lambda \in \mathbb{C}$*, one has*

$$\left(\Delta + \lambda^2\right)^{\eta+1} \Phi_{\lambda,\eta,k,j} = 0 \quad \text{in } \mathbb{R}^n. \tag{9.30}$$

Furthermore, if $\lambda \in \mathbb{C}\backslash(-\infty, 0)$*, then*

$$\left(\Delta + \lambda^2\right)^{\eta+1} \Psi_{\lambda,\eta,k,j} = 0 \quad \text{in } \mathbb{R}^n\backslash\{0\}. \tag{9.31}$$

Proof. For $\lambda = 0$, relations (9.30) and (9.31) are consequences of (9.17), (9.20), (9.18), and (9.12). In the remaining cases Theorem 9.1(ii) is applicable. □

As another application of Theorem 9.1, we deduce a result on the linear independence of the functions $\Phi_{\lambda,\eta,k,j}$ and $\Psi_{\lambda,\eta,k,j}$.

Proposition 9.3. *Let* $\{\lambda_1, \dots, \lambda_m\}$ *be a set of complex numbers such that* $\lambda_l \notin (-\infty, 0)$ *for each* $l \in \{1, \dots, m\}$ *and the numbers* $\lambda_1^2, \dots, \lambda_m^2$ *are distinct. Assume that* $\mathcal{O}$ *is a nonempty open subset of* $\mathbb{R}^n$ *and let* $\nu \in \mathbb{Z}_+$. *Then the following assertions hold.*

(i) *If* $0 \notin \mathcal{O}$ *and*

$$\sum_{l=1}^{m}\sum_{\eta=0}^{\nu} \alpha_{l,\eta}\Phi_{\lambda_l,\eta,k,j} + \beta_{l,\eta}\Psi_{\lambda_l,\eta,k,j} = 0 \quad \text{in } \mathcal{O} \tag{9.32}$$

for some $\alpha_{l,\eta}, \beta_{l,\eta} \in \mathbb{C}$, *then* $\alpha_{l,\eta} = \beta_{l,\eta} = 0$ *for all* l, η.

(ii) *If* $0 \in \mathcal{O}$ *and there exists* $f \in C^\infty(\mathcal{O})$ *such that*

$$f = \sum_{l=1}^{m}\sum_{\eta=0}^{\nu} \gamma_{l,\eta}\Psi_{\lambda_l,\eta,k,j} \quad \text{in } \mathcal{O}\backslash\{0\} \tag{9.33}$$

for some $\gamma_{l,\eta} \in \mathbb{C}$, *then* $\gamma_{l,\eta} = 0$ *for all* l, η.

Proof. For the case where $m = 1$ and $\lambda_1 = 0$, the required statements are obvious from (9.17), (9.18), and (9.20). Let $m \geqslant 2$, $s \in \{1, \dots, m\}$, and suppose that $\lambda_s \neq 0$. Write

$$p_{s,\mu}(z) = \left(\lambda_s^2 + z\right)^{\mu} \prod_{\substack{1\leqslant l\leqslant m\\ l\neq s}} \left(\lambda_l^2 + z\right)^{\nu+1}, \quad \mu \in \{0, \dots, \nu\},\ z \in \mathbb{C}.$$

To show (i), first assume that $|\alpha_{s,\mu}| + |\beta_{s,\mu}| \neq 0$ for some μ and that μ is the largest number with this property. Applying $p_{s,\mu}(\Delta)$ to (9.32), we get by Theorem 9.1 (ii),

$$(\alpha_{s,\mu}\Phi_{\lambda_s,0,k,j} + \beta_{s,\mu}\Psi_{\lambda_s,0,k,j})(-2\lambda_s)^{\mu}\mu! \prod_{\substack{1\leqslant l\leqslant m\\ l\neq s}} \left(\lambda_l^2 - \lambda_s^2\right)^{\nu+1} = 0 \quad \text{in } \mathcal{O}.$$

Going now to the definition of $\Phi_{\lambda,0,k,j}$ and $\Psi_{\lambda,0,k,j}$, we have a contradiction, whence $\alpha_{s,\eta} = \beta_{s,\eta} = 0$ for all $\eta \in \{0, \dots, \nu\}$. Therefore, (i) is proved if $0 \notin \{\lambda_1, \dots, \lambda_m\}$, and for otherwise the argument in the earlier case of $m = 1$ and $\lambda_1 = 0$ is applicable.

Let us prove (ii). As before, it is enough to verify that $\gamma_{s,\eta} = 0$ for all $\eta \in \{0, \dots, \nu\}$. Proceeding by contradiction, we suppose that $\mu \in \{0, \dots, \nu\}$ is the largest number such that $\gamma_{s,\mu} \neq 0$. Applying $p_{s,\mu}(\Delta)$ to (9.33), one obtains

$$\gamma_{s,\mu}\Psi_{\lambda_s,0,k,j} = c p_{s,\mu}(\Delta) f \quad \text{in } \mathcal{O}\backslash\{0\},$$

where $c \in \mathbb{C}$ (see Theorem 9.1(ii)). However, this is impossible since $0 \in \mathcal{O}$ and $f \in C^\infty(\mathcal{O})$. The proof is now complete. □

For the rest of the section, we assume that $\alpha \in \{-1, 1\}$ and, for brevity, write

$$H_{\lambda,\eta,\alpha} = \Phi_{\lambda,\eta,k,j} + \mathrm{i}\alpha\Psi_{\lambda,\eta,k,j}. \tag{9.34}$$

The behavior of $H_{\lambda,\eta,\alpha}(x)$ for $\lambda|x| \to \infty$ with $\operatorname{Re}\lambda \geqslant 0$ is given in the following proposition.

Proposition 9.4. *Let* $\lambda \in \mathbb{C}\backslash\{0\}$, $|\arg\lambda| \leqslant \pi/2$, $\theta > 2$. *Suppose that* $x \in \mathbb{R}^n$ *and* $|x| > (\theta\eta + 1)/|\lambda|$. *Then*

$$H_{\lambda,\eta,\alpha}(x) = \sqrt{\frac{2}{\pi}}\frac{\tau_n(\mathrm{i}\alpha)^\eta Y_j^k(\sigma)}{\lambda^{\frac{n-1}{2}+k}\varrho^{\frac{n-1}{2}-\eta}}\Bigg(\exp\Big(\mathrm{i}\alpha\Big(\lambda\varrho - \frac{\pi}{4}(n+2k-1)\Big)\Big) + O\big((1+\eta)(|\lambda|\varrho)^{-1}\mathrm{e}^{-\alpha\varrho\operatorname{Im}\lambda}\big)\Bigg),$$

where the constant in O depends only on n, k, and θ.

Proof. According to (9.14), (9.19), and (9.34),

$$H_{\lambda,\eta,\alpha}(x) = \tau_n\varrho^{k+\eta}Y_j^k(\sigma)g_{\eta,\alpha}(\lambda\varrho),$$

where

$$g_{\eta,\alpha}(z) = \left(\frac{\partial}{\partial z}\right)^\eta\big(\mathbf{I}_{\frac{n}{2}+k-1}(z) + \mathrm{i}\alpha\mathbf{N}_{\frac{n}{2}+k-1}(z)\big). \tag{9.35}$$

The required statement now follows by Proposition 7.1. □

In the sequel an important part is played by the following often used fact.

Proposition 9.5. *Let* $0 < r < R$, $q \in \mathbb{Z}_+$, $\theta > 2$, $\lambda \in \mathbb{C}\backslash\{0\}$, $|\arg\lambda| \leqslant \pi/2$, *and let* $|\lambda| > (\eta+1)\theta/r$. *Then we have the following.*

(i) $\|\Phi_{\lambda,\eta,k,j}\|_{C^q(\dot{B}_{r,R})} + \|\Psi_{\lambda,\eta,k,j}\|_{C^q(\dot{B}_{r,R})} \leqslant \gamma_1|\lambda|^{q-\frac{n-1}{2}-k}R^\eta\mathrm{e}^{R|\operatorname{Im}\lambda|}$, *where the constant* $\gamma_1 > 0$ *is independent of* λ, η.

(ii) $\|\Delta^q\Phi_{\lambda,\eta,k,j}\|_{C(\dot{B}_{r,R})} + \|\Delta^q\Psi_{\lambda,\eta,k,j}\|_{C(\dot{B}_{r,R})} \leqslant \gamma_2|\lambda|^{2q-\frac{n-1}{2}-k}R^\eta\exp\big(R|\operatorname{Im}\lambda| + \frac{2q\eta}{|\lambda|}\big)$, *where* $\gamma_2 > 0$ *is independent of* λ, η, q.

Proof. For (i), let $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_+^n$, and let $m = \beta_1 + \cdots + \beta_n \leqslant q$. We can write

$$\left(\frac{\partial}{\partial x}\right)^\beta H_{\lambda,\eta,\alpha}(x) = \sum_{\mu=0}^m f_{\mu,\alpha,\beta}(x)\left(\frac{\partial}{\partial z}\right)^\eta\big(z^\mu g_{\mu,\alpha}(z\varrho)\big)\Big|_{z=\lambda}, \quad x \in \dot{B}_{r,R}, \tag{9.36}$$

where $f_{\mu,\alpha,\beta} \in C^\infty(\mathbb{R}^n\backslash\{0\})$, and $g_{\mu,\alpha}$ is defined by (9.35). Since

$$\left(\frac{\partial}{\partial z}\right)^\eta\big(z^\mu g_{\mu,\alpha}(z\varrho)\big) = \sum_{l=\max\{0,\eta-\mu\}}^{\eta}\binom{\eta}{l}g_{\mu+l,\alpha}(z\varrho)\frac{\mu!\varrho^l}{(\mu-\eta+l)!}z^{\mu-\eta+l},$$

one has the estimate

$$\left|\left(\frac{\partial}{\partial z}\right)^{\eta}\left(z^{\mu} g_{\mu,\alpha}(z\varrho)\right)\Big|_{z=\lambda}\right| \leqslant \max_{0\leqslant\nu\leqslant\eta+q} |g_{\nu,\alpha}(\lambda\varrho)| \sum_{l=0}^{\eta}\binom{\eta}{l}(|\lambda|\varrho)^{l}\left(\frac{q}{|\lambda|}\right)^{\eta-l}.$$

Now (9.36), Proposition 7.1, and the inequality

$$\left(\varrho+\frac{q}{|\lambda|}\right)^{\eta} \leqslant \varrho^{\eta}\exp\left(\frac{q\eta}{\varrho|\lambda|}\right)$$

lead to (i).

Regarding (ii),

$$\Delta^{q} H_{\lambda,\eta,\alpha} = (-1)^{q} \sum_{\nu=\max\{0,\eta-2q\}}^{\eta}\binom{\eta}{\nu}\frac{(2q)!\lambda^{2q-\eta+\nu}}{(2q-\eta+\nu)!}H_{\lambda,\nu,\alpha}$$

because of Theorem 9.1(ii). Therefore,

$$\left|\Delta^{q} H_{\lambda,\eta,\alpha}(x)\right| \leqslant \max_{0\leqslant\mu\leqslant\eta}\left|H_{\lambda,\mu,\alpha}(x)\right| \sum_{m=0}^{\eta}\binom{\eta}{m}(2q)^{m}|\lambda|^{2q-m}, \quad x\in \dot{B}_{r,R},$$

which, together with (i), brings us to (ii). □

We complement Proposition 9.5(ii) by the following result.

Proposition 9.6. *Let* $0<r<R$, $q\in\mathbb{N}$, $\lambda\in\mathbb{C}\backslash\{0\}$. *Then the following statements are valid.*

(i) *If* $x\in\dot{B}_{R}$, *then*

$$\Delta^{q}\Phi_{\lambda,\eta,k,j}(x) = (-1)^{q}(2q)^{\eta}\lambda^{2q-\eta}\left(\Phi_{\lambda,0,k,j}(x)+O\left(q^{-1}\right)\right),$$

where the constant in O *is independent of* q, x.

(ii) *If* $x\in\dot{B}_{r,R}$ *and* $\lambda\in\mathbb{C}\backslash(-\infty,0]$, *then*

$$\Delta^{q}\Psi_{\lambda,\eta,k,j}(x) = (-1)^{q}(2q)^{\eta}\lambda^{2q-\eta}\left(\Psi_{\lambda,0,k,j}(x)+O\left(q^{-1}\right)\right),$$

where the constant in O *does not depend of* q *and* x.

Proof. This is immediate from Theorem 9.1(ii). □

To conclude this section we should mention an upper estimate for the function $(\frac{\partial}{\partial x})^{\beta}\Phi_{\lambda,\eta,k,j}(x)$, where $\beta\in\mathbb{Z}_{+}^{n}$, $x\in\mathbb{R}^{n}$.

Proposition 9.7. *Let* $x\in\mathbb{R}^{n}$, $\lambda\in\mathbb{C}\backslash\{0\}$, $\beta=(\beta_1,\ldots,\beta_n)\in\mathbb{Z}_{+}^{n}$. *Then*

$$\left|\left(\frac{\partial}{\partial x}\right)^{\beta}\Phi_{\lambda,\eta,k,j}(x)\right| \leqslant \gamma e^{\varrho|\operatorname{Im}\lambda|}A_{\lambda,\eta}(\varrho,\beta_1+\cdots+\beta_n), \tag{9.37}$$

where

$$A_{\lambda,\eta}(\varrho,\nu)=\begin{cases}|\lambda|^{\nu-k}\left(\varrho+\dfrac{\nu-k}{|\lambda|}\right)^{\eta} & \text{if } \nu\geqslant k,\\[2ex] |\lambda|^{\nu-k}\left(\varrho+\dfrac{\eta+k-\nu}{|\lambda|}\right)^{\eta} & \text{if } \nu<k,\end{cases}$$

and the constant $\gamma>0$ *depends only on* n *and* k.

Proof. Putting $\nu=\beta_1+\cdots+\beta_n$, we see from (9.13)–(9.15) and (5.2) that

$$\left(\frac{\partial}{\partial x}\right)^{\beta}\Phi_{\lambda,\eta,k,j}(x)=\gamma_1\left(\frac{\partial}{\partial z}\right)^{\eta}\left(\int_{\mathbb{S}^{n-1}}\mathrm{i}^{\nu}\xi^{\beta}z^{\nu-k}\mathrm{e}^{\mathrm{i}z\langle x,\xi\rangle_{\mathbb{R}}}Y_j^k(\xi)\,\mathrm{d}\omega(\xi)\right)\Big|_{z=\lambda},$$

where γ_1 depends only on n,k. This yields

$$\left|\left(\frac{\partial}{\partial x}\right)^{\beta}\Phi_{\lambda,\eta,k,j}(x)\right|\leqslant|\gamma_1|\int_{\mathbb{S}^{n-1}}\left|\left(\frac{\partial}{\partial z}\right)^{\eta}\left(z^{\nu-k}\mathrm{e}^{\mathrm{i}z\langle x,\xi\rangle_{\mathbb{R}}}\right)\Big|_{z=\lambda}Y_j^k(\xi)\right|\mathrm{d}\omega(\xi). \tag{9.38}$$

However,

$$\left(\frac{\partial}{\partial z}\right)^{\eta}\left(z^{\nu-k}\mathrm{e}^{\mathrm{i}z\langle x,\xi\rangle_{\mathbb{R}}}\right)=\mathrm{e}^{\mathrm{i}z\langle x,\xi\rangle_{\mathbb{R}}}\sum_{m=0}^{\eta}\binom{\eta}{m}(\mathrm{i}\langle x,\xi\rangle_{\mathbb{R}})^{\eta-m}z^{\nu-k-m}a_m, \tag{9.39}$$

where $a_0=1$ and $a_m=\prod_{l=0}^{m-1}(\nu-k-l)$ if $m\geqslant 1$. The validity of (9.37) is now obvious from (9.38) and (9.39). □

9.3 Hankel-Like Integral Transforms

Let $k\in\mathbb{Z}_+$, $j\in\{1,\dots,d(n,k)\}$. For $f\in\mathcal{E}'_{k,j}(\mathbb{R}^n)$, we define

$$\mathcal{F}_j^k(f)(z)=\left\langle f,\overline{\Phi_{\overline{z},0,k,j}}\right\rangle,\quad z\in\mathbb{C}. \tag{9.40}$$

Formulae (9.40) and (9.13)–(9.15) show that $\mathcal{F}_j^k(f)$ is an even entire function of variable z. It follows by (9.40) and (5.2) that

$$\mathcal{F}_j^k(f)\left(\sqrt{\zeta_1^2+\cdots+\zeta_n^2}\right)Y_j^k(\zeta)=\frac{\tau_n\mathrm{i}^k}{(2\pi)^{n/2}}\widehat{f}(\zeta) \tag{9.41}$$

for all $f\in\mathcal{E}'_{k,j}(\mathbb{R}^n)$, $\zeta=(\zeta_1,\dots,\zeta_n)\in\mathbb{C}^n$, where τ_n is defined by (9.15). If $f\in\mathcal{E}'_{\natural}(\mathbb{R}^n)$, then the function

$$\widetilde{f}(z)=\mathcal{F}_1^0(f)(z),\quad z\in\mathbb{C}, \tag{9.42}$$

is called the *spherical transform* of f.

There is a close connection between $\mathcal{F}_j^k$ and the well-known Hankel transform. More precisely, for $f \in (\mathcal{E}'_{k,j} \cap C)(\mathbb{R}^n)$, it follows from (9.40) and (9.13)–(9.15) that

$$\mathcal{F}_j^k(f)(z) = \tau_n \int_0^\infty f_{k,j}(\varrho)\varrho^{n-1+k}\mathbf{I}_{\frac{n}{2}+k-1}(z\varrho)\,\mathrm{d}\varrho, \quad z \in \mathbb{C}.$$

The integral in the right-hand part of this equality is the Hankel transform of the function $\varrho^{1-k} f_{k,j}(\varrho)$ (see Koornwinder [138], formula (5.4)).

We now establish some basic properties of $\mathcal{F}_j^k$.

Proposition 9.8. *If $f \in \mathcal{E}'_{k,j}(\mathbb{R}^n)$ and $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$, then*

$$\mathcal{F}_j^k(f * T)(z) = \widetilde{T}(z)\mathcal{F}_j^k(f)(z), \quad z \in \mathbb{C}. \tag{9.43}$$

In particular,

$$\mathcal{F}_j^k\big(p(\Delta)T\big)(z) = p\big(-z^2\big)\mathcal{F}_j^k(f)(z) \tag{9.44}$$

for each polynomial p.

Proof. First, assume that $z \neq 0$. Then there is $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$ such that $Y_j^k(\zeta) \neq 0$ and $\sqrt{\zeta_1^2 + \cdots + \zeta_n^2} = z$. Appealing to (9.41), (9.42), and (6.35), we arrive at (9.43) and (9.44). For the remaining value $z = 0$, the required equalities are obtained by continuity. □

The following result contains the Paley–Wiener theorem and the inversion formula for the transform $\mathcal{F}_j^k$.

Theorem 9.2.

(i) *If $f \in \mathcal{E}'_{k,j}(\mathbb{R}^n)$, then*

$$\big|\mathcal{F}_j^k(f)(z)\big| \leqslant \gamma_1(1+|z|)^{\gamma_2}\mathrm{e}^{r(f)|\mathrm{Im}\,z|}, \quad z \in \mathbb{C}, \tag{9.45}$$

where $\gamma_2 = \mathrm{ord}\, f - k$, and $\gamma_1 > 0$ is independent of z. Moreover, if $f \in (\mathcal{E}'_{k,j} \cap C^m)(\mathbb{R}^n)$ for some $m \in \mathbb{Z}_+$, then (9.45) holds with $\gamma_2 = -m-k$.

(ii) *Let w be an even entire function and suppose that*

$$|w(z)| \leqslant \gamma_1(1+|z|)^{\gamma_2}\mathrm{e}^{R|\mathrm{Im}\,z|}, \quad z \in \mathbb{C}, \tag{9.46}$$

where $\gamma_1 > 0$, $\gamma_2 \in \mathbb{R}^1$, and $R \geqslant 0$ are independent of z. Then there exists a unique $f \in \mathcal{E}'_{k,j}(\mathbb{R}^n)$ such that $\mathcal{F}_j^k(f) = w$. In addition, $r(f) \leqslant R$ and $\mathrm{ord}\, f \leqslant \max\{0, 1+\gamma_2+k+n/2\}$. Next, if $\gamma_2 = -(k+n+1+l)$ for some $l \in \mathbb{Z}_+$, then $f \in (\mathcal{E}'_{k,j} \cap C^l)(\mathbb{R}^n)$ and

$$f(x) = \frac{1}{\tau_n^2}\int_0^\infty \lambda^{n+2k-1}\Phi_{\lambda,0,k,j}(x)\mathcal{F}_j^k(f)(\lambda)\,\mathrm{d}\lambda, \quad x \in \mathbb{R}^n. \tag{9.47}$$

Proof. Part (i) is clear from (9.41), (6.34), and Theorem 6.3. Turning to (ii), let $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$. Then

$$\left| \operatorname{Im} \sqrt{\zeta_1^2 + \cdots + \zeta_n^2} \right|^2 \leqslant \sum_{m=1}^{n} (\operatorname{Im} \zeta_m)^2 \tag{9.48}$$

because of the Schwarz inequality. By (9.46), (9.48), and Theorem 6.3 there is a unique $f \in \mathcal{E}'(\mathbb{R}^n)$ such that $\widehat{f}(\zeta) = w(\sqrt{\zeta_1^2 + \cdots + \zeta_n^2}) Y_j^k(\zeta)$. Moreover, Theorem 6.3, (9.46), and (9.41) ensure us that $f \in \mathcal{E}'_{k,j}(\mathbb{R}^n)$, $r(f) \leqslant R$, $\operatorname{ord} f \leqslant \max\{0, 1 + \gamma_2 + k + n/2\}$, and $\mathcal{F}_j^k(f) = w$. In order to complete the proof one needs only Proposition 6.14 and (5.2). □

We are now in a position to prove the following statement needed in the sequel.

Proposition 9.9. *Let E be an infinite bounded subset of $\mathbb{C}$, and let $A(E, k, j)$ be the set of all finite linear combinations of the functions $\Phi_{\lambda,0,k,j}$ with $\lambda \in E$. Then, for each $R > 0$, the set $A(E, k, j)$ is dense in $C^\infty_{k,j}(B_R)$ with the topology induced by $C^\infty(B_R)$.*

Proof. Let $u \in \mathcal{E}'(B_R)$ and suppose that $\langle u, \Phi_{\lambda,0,k,j} \rangle = 0$ for each $\lambda \in E$. According to (9.40) and (9.8), we can write

$$\mathcal{F}_j^k\big((\overline{u})^{k,j}\big)(z) = 0, \quad z \in \mathbb{C},$$

since $\mathcal{F}_j^k((\overline{u})^{k,j})$ is an entire function. So $(\overline{u})^{k,j} = 0$, giving $< u, f >= 0$ for each $f \in C^\infty_{k,j}(B_R)$ by (9.8). Thus, every $u \in \mathcal{E}'(B_R)$ which is orthogonal to $A(E, k, j)$ is also orthogonal to $C^\infty_{k,j}(B_R)$, and we conclude by the Hahn–Banach theorem. □

Next, let

$$\operatorname{conj}\big(\mathcal{E}'_{k,j}(\mathbb{R}^n)\big) = \big\{ f \in \mathcal{E}'(\mathbb{R}^n) : \overline{f} \in \mathcal{E}'_{k,j}(\mathbb{R}^n) \big\},$$

and let $T \in \operatorname{conj}(\mathcal{E}'_{k,j}(\mathbb{R}^n))$. Thanks to Theorems 6.3 and 9.2(i), there exists a unique distribution $\Lambda^{k,j}(T) \in \mathcal{E}'_\natural(\mathbb{R}^1)$ such that

$$\widehat{\Lambda^{k,j}(T)}(z) = \overline{\mathcal{F}_j^k(\overline{T})(\overline{z})} = \langle T, \Phi_{z,0,k,j} \rangle, \quad z \in \mathbb{C}. \tag{9.49}$$

Moreover, the transform $\Lambda^{k,j} : T \to \Lambda^{k,j}(T)$ sets up a bijection between the spaces $\operatorname{conj}(\mathcal{E}'_{k,j}(\mathbb{R}^n))$ and $\mathcal{E}'_\natural(\mathbb{R}^1)$. For future use, we note that

$$\operatorname{ord} \Lambda^{k,j}(T) \leqslant \max\{0, \operatorname{ord} T - k + 1\} \quad \text{and} \quad r\big(\Lambda^{k,j}(T)\big) = r(T) \tag{9.50}$$

(see Theorems 6.3 and 9.2(i)). In the sequel the mapping $\Lambda^{0,1} : \mathcal{E}'_\natural(\mathbb{R}^n) \to \mathcal{E}'_\natural(\mathbb{R}^1)$ will be denoted by Λ.

If $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$ and $r(T) > 0$, then $r(\Lambda(T)) > 0$, and hence the set $\mathcal{Z}(\widehat{\Lambda(T)})$ is infinite (see Proposition 6.1 (iv) and Corollary 6.2). This ensures us that there is a polynomial of degree $1 + \operatorname{ord} T$ such that

$$p\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right)u = \Lambda(T) \quad \text{for some } u \in \mathcal{E}'\big(\mathbb{R}^1\big).$$

This yields

$$\widehat{u}(z) = \frac{\widehat{\Lambda(T)}(z)}{p(z)} = \frac{\widetilde{T}(z)}{p(z)}, \quad z \in \mathbb{C}.$$

Using now Theorem 9.2(i), we see that $\widehat{u} \in L^2(\mathbb{R}^1)$. Thus, $u \in L^2(\mathbb{R}^1)$, and

$$d_{\Lambda(T)} \leqslant 1 + \operatorname{ord} T \quad \text{if } T \in \mathcal{E}'_\natural\big(\mathbb{R}^n\big), r(T) > 0. \tag{9.51}$$

Let $\alpha > 0$, and let $\mathfrak{W}(\mathbb{R}^1)$ denote one of the classes $\mathfrak{M}(\mathbb{R}^1)$, $\mathfrak{N}(\mathbb{R}^1)$, $\mathfrak{E}(\mathbb{R}^1)$, $\mathfrak{G}_\alpha(\mathbb{R}^1)$, $\mathrm{Inv}_+(\mathbb{R}^1)$, $\mathfrak{R}(\mathbb{R}^1)$ (see Sect. 8.1). We define the multidimensional analogs of these classes by

$$\mathfrak{W}(\mathbb{R}^n) = \big\{T \in \mathcal{E}'_\natural\big(\mathbb{R}^n\big) : \Lambda(T) \in \mathfrak{W}\big(\mathbb{R}^1\big)\big\}. \tag{9.52}$$

Next, let $\mathfrak{I}(\mathbb{R}^n)$ (respectively $\mathfrak{I}_\alpha(\mathbb{R}^n)$) be the set of all nonzero distributions $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$ with the following property: $T \in \mathfrak{I}(\mathbb{R}^n)$ (respectively $T \in \mathfrak{I}_\alpha(\mathbb{R}^n)$) if and only if either the set $\mathcal{Z}(\widetilde{T})$ is not infinite or

$$n_\lambda\big(\widetilde{T}\big)\big(1 + |\mathrm{Im}\,\lambda| + \log n_\lambda\big(\widetilde{T}\big)\big) + \log\big(1 + \sigma_\lambda\big(\widetilde{T}\big)\big) = O(\log|\lambda|)$$

as $\lambda \to \infty$, $\lambda \in \mathcal{Z}(\widetilde{T})$ (respectively

$$n_\lambda\big(\widetilde{T}\big)\big(1 + |\mathrm{Im}\,\lambda| + \log n_\lambda\big(\widetilde{T}\big)\big) + \log\big(1 + \sigma_\lambda\big(\widetilde{T}\big)\big) = o\big(|\lambda|^{1/\alpha}\big)$$

as $\lambda \to \infty$, $\lambda \in \mathcal{Z}(\widetilde{T})$).

9.4 Transmutation Operators Induced by the Converse Hankel Transform. Connection with the Dual Abel Transform

In this section we define an operator which allows one to reduce a number of problems concerning convolution equations in $\mathbb{R}^n$, $n \geqslant 2$, to the one-dimensional case.

Let $k \in \mathbb{Z}_+$, $j \in \{1, \dots, d(n,k)\}$. For $f \in \mathcal{E}'_{k,j}(\mathbb{R}^n)$ and $\psi \in \mathcal{D}(\mathbb{R}^1)$, we put

$$\langle \mathfrak{A}_{k,j}(f), \psi\rangle = \frac{1}{\tau_n^2}\int_0^\infty \lambda^{n+2k-1}\mathcal{F}_j^k(f)(\lambda)\int_{\mathbb{R}^1}\psi(t)\cos(\lambda t)\,\mathrm{d}t\,\mathrm{d}\lambda. \tag{9.53}$$

It is not hard to make sure that $\mathfrak{A}_{k,j}(f) \in \mathcal{D}'_\natural(\mathbb{R}^1)$. Consider the main properties of the mapping $\mathfrak{A}_{k,j} : f \to \mathfrak{A}_{k,j}(f)$.

Lemma 9.2.

(i) *For* $f \in \mathcal{E}'_{k,j}(\mathbb{R}^n)$ *and* $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$*, we have*

$$\mathfrak{A}_{k,j}(f * T) = \mathfrak{A}_{k,j}(f) * \Lambda(T). \tag{9.54}$$

(ii) *Suppose that* $f \in (\mathcal{E}'_{k,j} \cap C^{N+n+k+2})(\mathbb{R}^n)$ *for some* $N \in \mathbb{Z}_+$*. Then* $\mathfrak{A}_{k,j}(f) \in C^N_\natural(\mathbb{R}^1)$ *and*

$$f_{k,j}(\varrho) = \frac{2^{1-k}\Gamma(n/2)\sqrt{\omega_{n-1}}}{\sqrt{\pi}\Gamma(\frac{n-1}{2}+k)} \int_0^\varrho \mathfrak{A}_{k,j}(f)(t) \frac{(\varrho^2-t^2)^{\frac{n-3}{2}+k}}{\varrho^{n+k-2}}\, \mathrm{d}t. \tag{9.55}$$

(iii) *Let* $f \in \mathcal{E}'_{k,j}(\mathbb{R}^n)$*,* $r \in (0, +\infty]$*. Then* $f = 0$ *in* B_r *if and only if* $\mathfrak{A}_{k,j}(f) = 0$ *on* $(-r, r)$*.*

Proof. Relation (9.54) can be easily derived with the aid of (9.53), (9.49), and (9.43). Assume now that the conditions of part (ii) are satisfied. By Theorem 9.2(i),

$$\lambda^{n+2k-1}\mathcal{F}^k_j(f)(\lambda) = O\big(\lambda^{-N-3}\big) \quad \text{as } \lambda \to +\infty.$$

Hence, $\mathfrak{A}_{k,j}(f) \in C^N_\natural(\mathbb{R}^1)$ and

$$\mathfrak{A}_{k,j}(f)(t) = \frac{1}{\tau_n^2}\int_0^\infty \lambda^{n+2k-1}\mathcal{F}^k_j(f)(\lambda)\cos(\lambda t)\, \mathrm{d}\lambda. \tag{9.56}$$

Using (9.56), (9.47), and (7.7), we obtain (9.55). Part (iii) follows from (9.55) by the standard approximation argument (see (9.54), (9.50), and Theorem 6.1). □

Equality (9.56) shows that the mapping $\mathfrak{A}_{k,j}$ is closely connected with the inversion formula for the Hankel transform (see (9.47)).

Part (iii) of Lemma 9.2 makes it possible to extend $\mathfrak{A}_{k,j}$ to the space $\mathcal{D}'_{k,j}(B_R)$, $R \in (0, +\infty]$. We shall do this by the formula

$$\langle \mathfrak{A}_{k,j}(f), \psi\rangle = \langle \mathfrak{A}_{k,j}(f\eta), \psi\rangle, \qquad f \in \mathcal{D}'_{k,j}(B_R), \psi \in \mathcal{D}(-R, R), \tag{9.57}$$

where $\eta \in \mathcal{D}_\natural(B_R)$ is selected so that $\eta = 1$ in $B_{r_0(\psi)+\varepsilon}$ for some $\varepsilon \in (0, R - r_0(\psi))$. Then $\mathfrak{A}_{k,j}(f) \in \mathcal{D}'_\natural(-R, R)$ and

$$\mathfrak{A}_{k,j}(f|_{B_r}) = \mathfrak{A}_{k,j}(f)|_{(-r,r)}$$

for all $r \in (0, R]$.

Theorem 9.3. *For* $R \in (0, +\infty]$*,* $N \in \mathbb{Z}_+$*, and* $\nu = N + n + k + 2$*, the following are true.*

(i) *If* $f \in \mathcal{D}'_{k,j}(B_R)$*,* $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$*, and* $r(T) < R$*, then* (9.54) *is valid on* $(r(T) - R, R - r(T))$*. In particular, we have the transmutation property*

$$\mathfrak{A}_{k,j}\big(\Delta^N f\big) = \mathfrak{A}_{k,j}(f)^{(2N)}. \tag{9.58}$$

(ii) *Let* $f \in \mathcal{D}'_{k,j}(B_R)$*,* $r \in (0, R]$*. Then* $f = 0$ *in* B_r *if and only if* $\mathfrak{A}_{k,j}(f) = 0$ *on* $(-r, r)$*.*

(iii) *When* $f \in C^\nu_{k,j}(B_R)$*, then* $\mathfrak{A}_{k,j}(f) \in C^N_\natural(-R, R)$*, and* (9.55) *holds for* $\varrho \in (0, R)$*. In addition,*

$$\mathfrak{A}_{k,j}(f)^{(2\mu)}(0) = \frac{2^k (n/2)_k}{\sqrt{\omega_{n-1}}} \lim_{x\to 0} \frac{\Delta^\mu(f)(x)(Y^k_j(x/|x|))^{-1}}{|x|^k} \tag{9.59}$$

if $\mu \in \big\{0, \dots, [N/2]\big\}$*.*

(iv) *The mapping* $\mathfrak{A}_{k,j}$ *is continuous from* $\mathcal{D}'_{k,j}(B_R)$ *into* $\mathcal{D}'_\natural(-R, R)$ *and from* $C^\nu_{k,j}(B_R)$ *into* $C^N_\natural(-R, R)$*.*

(v) *If* $f \in \mathcal{D}'_{k,j}(B_R)$*, then* $\operatorname{ord} \mathfrak{A}_{k,j}(f) \leqslant n + k + 2 + \operatorname{ord} f$*.*

(vi) *Let* $f \in C^\nu_{k,j}(B_R)$ *have all derivatives of order* $\leqslant \nu$ *vanishing at* 0*. Then*

$$\mathfrak{A}_{k,j}(f)^{(s)}(0) = 0, \quad s = 0, \dots, N.$$

(vii) *For* $\lambda \in \mathbb{C}$ *and* $\mu \in \mathbb{Z}_+$*, we have*

$$\mathfrak{A}_{k,j}(\Phi_{\lambda,\mu,k,j}) = u_{\lambda,\mu}, \quad \textit{where}$$
$$u_{\lambda,\mu}(t) = \begin{cases} \dfrac{1}{2}\big(\mathrm{e}^{\lambda,\mu}(t) + \mathrm{e}^{\lambda,\mu}(-t)\big) & \textit{if } \lambda \neq 0, \\ (-1)^\mu t^{2\mu} & \textit{if } \lambda = 0. \end{cases} \tag{9.60}$$

(viii) *Suppose that* $T \in \operatorname{conj}(\mathcal{E}'_{k,j}(\mathbb{R}^n))$*,* $r(T) < R$ *and* $f \in C^s_{k,j}(B_R)$*, where* $s = \max\{n + k + 2, \operatorname{ord} T + n + 3\}$*. Then*

$$\langle T, f\rangle = \big\langle \Lambda^{k,j}(T), \mathfrak{A}_{k,j}(f)\big\rangle. \tag{9.61}$$

Proof. The definition of $\mathfrak{A}_{k,j}$ on $\mathcal{D}'_{k,j}(B_R)$ and Lemma 9.2 imply (i), (ii), and the first assertion in part (iii). To prove (9.59) it suffices to use (ii), (9.56), (9.47), and (9.58). In (iv) assume first that $\{f_q\}_{q=1}^\infty \in \mathcal{D}'_{k,j}(B_R)$ and $f_q \to 0$ in $\mathcal{D}'(B_R)$ as $q \to +\infty$. Take $\psi \in \mathcal{D}(-R, R)$ and choose $\eta \in \mathcal{D}_\natural(B_R)$ such that $\eta = 1$ in $B_{r_0(\psi)+\varepsilon}$ for some $\varepsilon \in (0, R - r_0(\psi))$. For $\lambda > 0$, set

$$\eta_\lambda(x) = \overline{\Phi_{\lambda,0,k,j}(x)}\eta(x).$$

Then the integral

$$\frac{1}{\tau_n^2}\int_0^\infty \lambda^{n+2k-1}\int_{-R}^{R} \psi(t)\cos(\lambda t)\,\mathrm{d}t\,\eta_\lambda(x)\,\mathrm{d}\lambda$$

converges in $\mathcal{D}(B_R)$ (see (9.37)). Denoting its value by $\varphi(x)$, we have

$$\langle \mathfrak{A}_{k,j}(f_q), \psi\rangle = \frac{1}{\tau_n^2}\int_0^\infty \lambda^{n+2k-1}\int_{-R}^R \psi(t)\cos(\lambda t)\,\mathrm{d}t\langle f_q, \eta_\lambda\rangle\,\mathrm{d}\lambda = \langle f_q, \varphi\rangle,$$

whence $\mathfrak{A}_{k,j}(f_q) \to 0$ in $\mathcal{D}'(-R, R)$. Now let $\{f_q\}_{q=1}^\infty \in C_{k,j}^\nu(B_R)$ and $f_q \to 0$ in $C^\nu(B_R)$. Fix $r \in (0, R)$. Again pick up $\eta \in \mathcal{D}_\natural(B_R)$ such that $\eta = 1$ in $B_{r+\varepsilon}$ for some $\varepsilon \in (0, R - r)$. Using (9.56), (9.30), (9.37), and (1.44), we get

$$\|\mathfrak{A}_{k,j}(\eta f_q)\|_{C^N[-r,r]} \leqslant c\|\eta f_q\|_{C^\nu(E)}, \tag{9.62}$$

where $E = \operatorname{supp}\eta$, and the constant $c > 0$ does not depend on q. Because of (9.62) and (ii),

$$\lim_{q\to+\infty}\|\mathfrak{A}_{k,j}(f_q)\|_{C^N[-r,r]} = 0.$$

Thus, $\mathfrak{A}_{k,j}(f_q) \to 0$ in $C^N(-R, R)$. Thereby (iv) is established. Moreover, the proof of (iv) shows that (v) also holds. Part (vi) follows from (ii) and (iv) by [122, Chap. 2, Lemma 1.3]. Next, owing to (9.55) and (7.7),

$$\int_0^\varrho \big(\mathfrak{A}_{k,j}(\Phi_{\lambda,0,k,j})(t) - \cos(\lambda t)\big)\big(\varrho^2 - t^2\big)^{\frac{n-3}{2}+k}\,\mathrm{d}t = 0$$

for all $\varrho \in (0, R)$. Therefore (see [225, Part I, Lemma 8.1]),

$$\mathfrak{A}_{k,j}(\Phi_{\lambda,0,k,j})(t) = \cos(\lambda t), \quad t \in (-R, R). \tag{9.63}$$

Differentiating (9.63) with respect to λ, we deduce (vii). Finally, by (9.63) and (9.49),

$$\langle T, \Phi_{\lambda,0,k,j}\rangle = \big\langle \Lambda^{k,j}(T), \mathfrak{A}_{k,j}(\Phi_{\lambda,0,k,j})\big\rangle, \quad \lambda \in \mathbb{C}.$$

On account of the arbitrariness of λ, the previous equality, part (iv), and (9.50) give (9.61) (see Proposition 9.9). □

Remark 9.1. Let $r \in (0, +\infty)$ and $f \in C_{k,j}^\nu(\mathring{B}_r)$, where ν is given in Theorem 9.3. Clearly, there exists $f_1 \in C_{k,j}^\nu(\mathbb{R}^n)$ for which $f_1|_{\mathring{B}_r} = f$. In addition, if $f_2 \in C_{k,j}^\nu(\mathbb{R}^n)$ and $f_2|_{\mathring{B}_r} = f$, then $\mathfrak{A}_{k,j}(f_1) = \mathfrak{A}_{k,j}(f_2)$ on $[-r, r]$ because of Theorem 9.3(ii), (iii). So, $\mathfrak{A}_{k,j}$ is well defined as a mapping from $C_{k,j}^\nu(\mathring{B}_r)$ into $C_\natural^N[-r, r]$ by $\mathfrak{A}_{k,j}(f) = \mathfrak{A}_{k,j}(f_1)|_{[-r,r]}$.

Theorem 9.4. *Let $r \in (0, +\infty)$, $k \in \mathbb{Z}_+$, and $j \in \{1, \dots, d(n, k)\}$. Then there is a constant $c > 0$ such that*

$$\int_{-r}^r \big|\mathfrak{A}_{k,j}(f)^{(M)}(t)\big|\,\mathrm{d}t \leqslant c\sum_{i=0}^{[(n+k)/2]+1}\int_{B_r}\big|\Delta^{[(M+1)/2]+i}(f)(x)\big|\,\mathrm{d}x \tag{9.64}$$

for all $M \in \mathbb{Z}_+$ and $f \in C_{k,j}^\mu(\mathring{B}_r)$, where $\mu = 2 + \max\{M + k + n, 2[(M + 1)/2] + 2[(n + k)/2]\}$.

Proof. Since $C^{\infty}_{k,j}$ is dense in $C^{\mu}_{k,j}$, it is enough to consider the case $f \in C^{\infty}_{k,j}$. First, assume that $M = 2N$, $N \in \mathbb{Z}_+$. For brevity, we set $h = (\mathfrak{A}_{k,j}(f))^{(2N)}$. Define

$$g(t) = \begin{cases} e^{-i \arg h(t)} & \text{if } t \in [-r, r] \text{ and } h(t) \neq 0, \\ 1 & \text{if } t \in [-r, r] \text{ and } h(t) = 0, \\ 0 & \text{if } |t| > r. \end{cases}$$

Then

$$|\widehat{g}(\lambda)| \leqslant 2r e^{r|\operatorname{Im}\lambda|}, \quad \lambda \in \mathbb{C}, \tag{9.65}$$

and

$$\int_{-r}^{r} |h(t)|\, dt = \int_{-r}^{r} h(t) g(t)\, dt. \tag{9.66}$$

Next, $g = \Lambda^{k,j}(T)$ for some $T \in \operatorname{conj}(\mathcal{E}'_{k,j}(\mathbb{R}^n))$ with $r(T) = r$. We conclude from (9.65), (9.49), and Theorem 9.2(ii) that $\operatorname{ord} T \leqslant 1+k+\dfrac{n}{2}$. Therefore, by (9.66) and Theorem 9.3(i), (viii),

$$\int_{-r}^{r} |h(t)|\, dt = \langle T, \Delta^N f\rangle. \tag{9.67}$$

Let

$$\alpha_i = \frac{\omega_{n-1}\lambda_i^{n+2k-2}\widehat{g}(\lambda_i)}{2^{n-1}\pi^n r^2 J^2_{n/2+k}(\lambda_i r)}, \quad i = 1, \dots, [(n+k)/2]+1,$$

where $\lambda_1, \lambda_2, \dots$ is the sequence of all positive zeroes of $J_{n/2+k-1}(r\lambda)$ arranged in ascending order of magnitude (see Sect. 7.1). In view of (9.65) and (7.10), the function

$$w(x) = \begin{cases} \displaystyle\sum_{i=1}^{[(n+k)/2]+1} \alpha_i \overline{\Phi_{\lambda_i,0,k,j}(x)} & \text{if } x \in B_r, \\ 0 & \text{if } x \in \mathbb{R}^n \backslash B_r, \end{cases} \tag{9.68}$$

possesses the following properties:

(a) the inequality

$$|w(x)| \leqslant c_1, \quad x \in \mathbb{R}^n, \tag{9.69}$$

holds, where the constant c_1 depends only on n, k, and r;

(b) $\mathcal{F}^k_j(\overline{T})(\lambda_i) = \mathcal{F}^k_j(\overline{w})(\lambda_i)$, $i = 1, \dots, [(n+k)/2]+1$.

Introduce the even entire function

$$u(\lambda) = \frac{\mathcal{F}^k_j(\overline{T})(\lambda) - \mathcal{F}^k_j(\overline{w})(\lambda)}{P(-\lambda^2)},$$

where

$$P\big(-\lambda^2\big) = \prod_{i=1}^{[(n+k)/2]+1} \big(\lambda_i^2 - \lambda^2\big).$$

Taking into account (9.65), (9.49), (9.69), and (9.37) and using the maximum-modulus principle, we obtain

$$|u(\lambda)| \leqslant c_2 \frac{e^{r|\operatorname{Im}\lambda|}}{(1+|\lambda|)^{2[(n+k)/2]+2}}, \quad \lambda \in \mathbb{C}, \tag{9.70}$$

where the constant c_2 depends only on n, k, r. Due to Theorem 9.2(ii), (9.70), (9.47), (9.44), and Proposition 9.7, there exists $T_1 \in C_{k,j}(\mathbb{R}^n)$ such that $\operatorname{supp} T_1 \subset \dot{B}_r$, $T - w = \overline{P(\Delta)T_1}$, and

$$|T_1(x)| \leqslant c_3, \quad x \in \mathbb{R}^n, \tag{9.71}$$

where the constant c_3 depends only on n, k, r. In terms of T_1 we can write (9.67) in the form

$$\int_{-r}^{r} |h(t)|\, dt = \overline{\langle T_1, P(\Delta)\Delta^N \overline{f}\rangle} + \langle w, \Delta^N f\rangle.$$

The previous part and estimates (9.69) and (9.71) yield (9.64) for $M = 2N$.

Now let $M = 2N - 1$, $N \in \mathbb{N}$. Since

$$\mathfrak{A}_{k,j}(f)^{(2N-1)}(t) = \int_0^t \mathfrak{A}_{k,j}(f)^{(2N)}(u)\, du,$$

we have

$$\int_{-r}^{r} \left|\mathfrak{A}_{k,j}(f)^{(2N-1)}(t)\right| dt \leqslant r \int_{-r}^{r} \left|\mathfrak{A}_{k,j}(f)^{(2N)}(t)\right| dt.$$

Hence, by the above we obtain (9.64) in the case under consideration. Thus, the theorem is proved. □

We see from assertion (ii) of Theorem 9.3 that the mapping $\mathfrak{A}_{k,j}$ is injective. Our further purpose is to find the converse operator $\mathfrak{A}_{k,j}^{-1}$.

If $F \in \mathcal{E}'_{\natural}(\mathbb{R}^1)$, set

$$\begin{aligned}\langle \mathfrak{B}_{k,j}(F), w\rangle &= \frac{1}{\pi}\int_0^\infty \widehat{F}(\lambda)\mathcal{F}_j^k\big(\overline{(w)_{k,j}(\varrho)}Y_j^k(\sigma)\big)(\lambda)\, d\lambda \\ &= \frac{1}{\pi}\int_0^\infty \widehat{F}(\lambda)\langle w, \Phi_{\lambda,0,k,j}\rangle\, d\lambda, \quad w \in \mathcal{D}\big(\mathbb{R}^n\big).\end{aligned} \tag{9.72}$$

Using (9.30), Proposition 9.7, and Theorem 6.3, we infer that $\mathfrak{B}_{k,j}(F) \in \mathcal{D}'_{k,j}(\mathbb{R}^n)$ and the mapping $\mathfrak{B}_{k,j} : \mathcal{E}'_{\natural}(\mathbb{R}^1) \to \mathcal{D}'_{k,j}(\mathbb{R}^n)$ is continuous.

Lemma 9.3.

(i) *For $F \in \mathcal{E}'_{\natural}(\mathbb{R}^1)$ and $T \in \mathcal{E}'_{\natural}(\mathbb{R}^n)$, one has*

$$\mathfrak{B}_{k,j}(F) * T = \mathfrak{B}_{k,j}\big(F * \Lambda(T)\big). \tag{9.73}$$

(ii) *Assume that* $F \in (\mathcal{E}'_\natural \cap C^s)(\mathbb{R}^1)$ *with some* $s \geqslant 2$. *Then* $\mathfrak{B}_{k,j}(F) \in C^{s+k-2}_{k,j}(\mathbb{R}^1)$ *and*

$$\mathfrak{B}_{k,j}(F)(x) = \frac{2^{1-k}\Gamma(n/2)\sqrt{\omega_{n-1}}}{\sqrt{\pi}\Gamma((n-1+2k)/2)} \int_0^\varrho F(t) \frac{(\varrho^2 - t^2)^{\frac{n-3}{2}+k}}{\varrho^{n+k-2}} \, \mathrm{d}t Y^k_j(\sigma). \tag{9.74}$$

(iii) *Let* $F \in \mathcal{E}'_\natural(\mathbb{R}^1)$ *and* $r \in (0, +\infty]$. *Then* $F = 0$ *on* $(-r, r)$ *if and only if* $\mathfrak{B}_{k,j}(F) = 0$ *in* B_r.

Proof. Take $w \in \mathcal{D}(\mathbb{R}^n)$ arbitrarily. Taking (9.49), (9.30), and Proposition 9.8 into account, we get

$$\begin{aligned}\langle \mathfrak{B}_{k,j}(F) * T, w\rangle &= \langle \mathfrak{B}_{k,j}(F), w * T\rangle \\ &= \frac{1}{\pi}\int_0^\infty \widehat{F}(\lambda)\widehat{\Lambda(T)}(\lambda)\langle w, \Phi_{\lambda,0,k,j}\rangle \, \mathrm{d}\lambda \\ &= \big\langle \mathfrak{B}_{k,j}\big(F * \Lambda(T)\big), w\big\rangle,\end{aligned}$$

which is the formula (9.73). To verify (ii) observe that for $F \in (\mathcal{E}'_\natural \cap C^s)(\mathbb{R}^1)$, $s \geqslant 2$,

$$\mathfrak{B}_{k,j}(F)(x) = \frac{1}{\pi}\int_0^\infty \widehat{F}(\lambda)\Phi_{\lambda,0,k,j}(x) \, \mathrm{d}\lambda, \quad x \in \mathbb{R}^n \tag{9.75}$$

(see (9.72) and Proposition 9.7). Now applying (7.7) and the inversion formula for the Fourier-cosine transform, we arrive at (ii). Part (iii) is immediate from (ii) by regularization (see (9.50) and (9.73)). □

Thanks to Lemma 9.3(iii), the mapping $\mathfrak{B}_{k,j}$ can be extended to the space $\mathcal{D}'_\natural(-R, R)$, $R \in (0, +\infty]$, by

$$\langle \mathfrak{B}_{k,j}(F), w\rangle = \langle \mathfrak{B}_{k,j}(F\eta), w\rangle, \quad F \in \mathcal{D}'_\natural(-R, R), w \in \mathcal{D}(B_R), \tag{9.76}$$

where $\eta \in \mathcal{D}_\natural(-R, R)$ and $\eta = 1$ on $(-r_0(w) - \varepsilon, r_0(w) + \varepsilon)$ with some $\varepsilon \in (0, R - r_0(w))$. It is easy to show that $\mathfrak{B}_{k,j}(F) \in \mathcal{D}'_{k,j}(B_R)$ and

$$\mathfrak{B}_{k,j}(F|_{(-r,r)}) = \mathfrak{B}_{k,j}(F)|_{B_r}$$

when $r \in (0, R]$.

Theorem 9.5. *For* $R \in (0, +\infty]$, $s \in \{2, 3, \ldots\}$, *the following are true.*

(i) *Let* $F \in \mathcal{D}'_\natural(-R, R)$, $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$, *and* $r(T) < R$. *Then* (9.73) *holds in* $B_{R-r(T)}$. *In particular,*

$$\Delta^N\big(\mathfrak{B}_{k,j}(F)\big) = \mathfrak{B}_{k,j}\big(F^{(2N)}\big) \tag{9.77}$$

for all $N \in \mathbb{Z}_+$.

(ii) *Let* $F \in \mathcal{D}'_\natural(-R, R)$, $r \in (0, R]$. *Then* $F = 0$ *on* $(-r, r)$ *if and only if* $\mathfrak{B}_{k,j}(F) = 0$ *in* B_r.

(iii) *If* $F \in C^s_\natural(-R, R)$, *then* $\mathfrak{B}_{k,j}(F) \in C^{s+k-2}_{k,j}(B_R)$, *and* (9.74) *is valid in* $B_R \backslash \{0\}$. *In this case*

$$\mathfrak{B}_{0,1}(F)(x) = \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} F(\langle x, \eta \rangle_{\mathbb{R}})\, d\omega(\eta). \tag{9.78}$$

Furthermore,

$$\Delta^N\big(\mathfrak{B}_{0,1}(F)\big)(0) = F^{(2N)}(0)$$

when $N \in \{0, \dots, [s/2] - 1\}$.

(iv) *The map* $\mathfrak{B}_{k,j}$ *is continuous from* $\mathcal{D}'_\natural(-R, R)$ *into* $\mathcal{D}'_{k,j}(B_R)$ *and from* $C^s_\natural(-R, R)$ *into* $C^{s+k-2}_{k,j}(B_R)$.

(v) *If* $F \in \mathcal{D}'_\natural(-R, R)$, *then* $\operatorname{ord} \mathfrak{B}_{k,j}(F) \leqslant \max\{0, \operatorname{ord} F - k + 3\}$.

(vi) *Let* $F \in C^s_\natural(-R, R)$ *and* $F^{(\nu)}(0) = 0$ *for* $\nu = 0, \dots, s$. *Then* $\mathfrak{B}_{k,j}(F)$ *has all derivatives of order* $\leqslant s + k - 2$ *vanishing at* 0.

(vii) *For* $F \in \mathcal{D}'_\natural(-R, R)$, *we have* $\mathfrak{A}_{k,j}(\mathfrak{B}_{k,j}(F)) = F$.

(viii) *Assume that* $T \in \operatorname{conj}(\mathcal{E}'_{k,j}(\mathbb{R}^n))$, $m = \max\{2, \operatorname{ord} T - k + 2\}$, $r(T) < R$, *and* $F \in C^m_\natural(-R, R)$. *Then*

$$\langle T, \mathfrak{B}_{k,j}(F) \rangle = \big\langle \Lambda^{k,j}(T), F \big\rangle. \tag{9.79}$$

Proof. The argument of Theorem 9.3 is applicable with minor modification. In the first place we use Lemma 9.3, (9.75), and the inversion formula for the Fourier-cosine transform instead of Lemma 9.2, (9.56), and (9.47), respectively. Next, relation (9.78) is an immediate consequence of (9.74) and the Funk–Hecke theorem (see [225, Part I, Theorem 5.1]). Part (vii) follows from (9.75) and (9.60) by regularization. The rest of the proof now duplicates Theorem 9.3. □

Corollary 9.2. *For each* $R \in (0, +\infty]$, *the transform* $\mathfrak{A}_{k,j}$ *sets up a homeomorphism between:*

(i) $\mathcal{D}'_{k,j}(B_R)$ *and* $\mathcal{D}'_\natural(-R, R)$;
(ii) $C^\infty_{k,j}(B_R)$ *and* $C^\infty_\natural(-R, R)$.

Moreover,

$$\mathfrak{A}^{-1}_{k,j} = \mathfrak{B}_{k,j}. \tag{9.80}$$

The proof is obvious from the above theorems.

Remark 9.2. Relation (9.78) shows that the mapping $\mathfrak{B}_{0,1}$ coincides with the dual Abel transform (see Koornwinder [139], formula (5.7)).

We now consider an application of the transform $\mathfrak{A}_{0,1}$ in the theory of positive definite functions.

Let E be a vector space over the field of real numbers. The function $f : E \to \mathbb{C}$ is said to be *positive definite* on E, written $f \in \Phi(E)$, if for any family $\{x_k\}_{k=1}^m$ of $x_k \in E$ for all k and for any number set $\{\xi_k\}_{k=1}^m$, $\xi_k \in \mathbb{C}$ for all k,

$$\sum_{k,l=1}^{m} f(x_k - x_l)\xi_k\overline{\xi}_l \geqslant 0.$$

The following result allows us to reduce the investigation of the class $(C_\natural \cap \Phi)(\mathbb{R}^n)$ to the investigation of $(C_\natural \cap \Phi)(\mathbb{R}^1)$.

Proposition 9.10. *In order that* $f \in (C_\natural \cap \Phi)(\mathbb{R}^n)$, *it is necessary and sufficient that* $\mathfrak{A}_{0,1}(f) \in (C_\natural \cap \Phi)(\mathbb{R}^1)$.

Proof. Let $m \in \mathbb{N}$. By the Bochner–Khintchin theorem, $f \in (C_\natural \cap \Phi)(\mathbb{R}^m)$ if and only if there exists a finite positive measure μ_+ on $[0, +\infty)$ such that

$$F(x) = 2^{\frac{m}{2}-1}\Gamma\left(\frac{m}{2}\right)\int_0^\infty \mathbf{I}_{\frac{m}{2}-1}(\lambda|x|)\,\mathrm{d}\mu_+(\lambda), \quad x \in \mathbb{R}^m$$

(see Trigub and Belinsky [216], Chap. 6, Sect. 6.3). If $m = n$, this equality is equivalent to

$$\mathfrak{A}_{0,1}(f)(t) = \int_0^\infty \cos(\lambda t)\,\mathrm{d}\mu_+(\lambda), \quad t \in \mathbb{R}^1$$

(see Corollary 9.2(i) and (9.60)). According to what has been said above, this concludes the proof. □

Remark 9.3. Assume that $F \in C_\natural^s[-r,r]$, $s \geqslant 2$, $r \in (0,+\infty)$. Take $F_1 \in C_\natural^s(\mathbb{R}^1)$ such that $F_1|_{[-r,r]} = F$. According to Theorem 9.5(ii), (iii), $\mathfrak{B}_{k,j}$ is well defined as a mapping from $C_\natural^s[-r,r]$ into $C_{k,j}^{s+k-2}(\dot{B}_r)$ by $\mathfrak{B}_{k,j}(F) = \mathfrak{B}_{k,j}(F_1)|_{\dot{B}_r}$.

We present an analog of inequality (9.64) for the operator $\mathfrak{B}_{k,j}$.

Theorem 9.6. *Let* $r \in (0,+\infty)$. *Then there exists a constant* $c > 0$ *such that for all* $N \in \mathbb{Z}_+$ *and* $F \in C_\natural^{2N+2}[-r,r]$,

$$\int_{B_r} \left|\Delta^N\left(\mathfrak{B}_{k,j}(F)\right)(x)\right|\mathrm{d}x \leqslant c\int_{-r}^{r}\left(\left|F^{(2N)}(t)\right| + \left|F^{(2N+2)}(t)\right|\right)\mathrm{d}t. \tag{9.81}$$

Proof. Denote by H the function $\Delta^N(\mathfrak{B}_{k,j}(F))$. Put

$$T(x) = \left(\int_{\mathbb{S}^{n-1}} G(\varrho\xi)Y_j^k(\xi)\,\mathrm{d}\omega(\xi)\right)\overline{Y_j^k(\sigma)},$$

where

$$G(x) = \begin{cases} \mathrm{e}^{-\mathrm{i}\arg H(x)} & \text{if } x \in \dot{B}_r \text{ and } H(x) \neq 0, \\ 1 & \text{if } x \in \dot{B}_r \text{ and } H(x) = 0, \\ 0 & \text{if } |x| > r. \end{cases}$$

Passing to polar coordinates, we have

$$\int_{B_r} |H(x)|\,\mathrm{d}x = \int_{B_r} H(x)G(x)\,\mathrm{d}x = \int_{B_r} H(x)T(x)\,\mathrm{d}x. \tag{9.82}$$

By (9.82), (9.77), and (9.79),

$$\int_{B_r} |H(x)|\,\mathrm{d}x = \left\langle \Lambda^{k,j}(T) - W, F^{(2N)} \right\rangle + \left\langle W, F^{(2N)} \right\rangle \tag{9.83}$$

with

$$W(t) = \begin{cases} \dfrac{1}{r}\overline{\mathcal{F}_j^k(\overline{T})}\left(\dfrac{\pi}{r}\right)\cos\left(\dfrac{\pi}{r}t\right) & \text{if } t \in [-r, r], \\ 0 & \text{if } t \in \mathbb{R}^1 \backslash [-r, r]. \end{cases}$$

Bearing in mind that $(\widehat{\Lambda^{k,j}(T)} - \widehat{W})(\pm\pi/r) = 0$, we deduce (9.81) from (9.83) in the same way as in the case of the operator $\mathfrak{A}_{k,j}$. □

Corollary 9.3. *Let $R \in (0, +\infty]$, $f \in \mathcal{D}'_{k,j}(B_R)$, $\alpha > 0$. Then $f \in (\mathcal{D}'_{k,j} \cap G^\alpha)(B_R)$ if and only if $\mathfrak{A}_{k,j}(f) \in (\mathcal{D}'_\natural \cap G^\alpha)(-R, R)$.*

The proof follows from Theorems 9.6 and 9.4.

We consider now the mapping $\mathcal{A}_j^k : \mathcal{D}'_{k,j}(B_R) \to \mathcal{D}'_\natural(B_R)$, $R \in (0, +\infty]$, defined by

$$\mathcal{A}_j^k = \mathfrak{A}_{0,1}^{-1}\mathfrak{A}_{k,j}. \tag{9.84}$$

In accordance with (9.60),

$$\mathcal{A}_j^k(\Phi_{\lambda,\mu,k,j}) = \Phi_{\lambda,\mu,0,1}. \tag{9.85}$$

Next, under the assumptions of Theorem 9.3(i),

$$\mathcal{A}_j^k(f * T) = \mathcal{A}_j^k(f) * T \tag{9.86}$$

in $B_{R-r(T)}$ (see (9.54), (9.73), and (9.80)).

Lemma 9.4. *Let $f \in C_{k,j}^{n+k+4}(B_R)$, $R \in (0, +\infty]$. Then*

$$\mathcal{A}_j^k(f) = \begin{cases} f & \text{if } k = 0, \\ \dfrac{1}{\sqrt{\omega_{n-1}}} D(1-n)\cdots D(2-k-n)(f_{k,j}) & \text{if } k \geqslant 1, \end{cases}$$

where $D(\cdot)$ is given by (9.11).

Proof. We can assume, without loss of generality, that $f \in (\mathcal{E}'_{k,j} \cap C^{n+k+4})(B_R)$ (see Theorems 9.3(ii) and 9.5(ii)). Represent f in the form (9.47) and apply to (9.47) the operator $\mathcal{A}_j^k$. Taking (9.85) and Proposition 9.2(i) into account, we obtain the desired relation. □

Corollary 9.4. *Let $f \in \mathcal{D}'_{k,j}(B_R)$, $R \in (0, +\infty]$. Suppose that $f = 0$ in $B_{r_1,r_2} = \{x \in \mathbb{R}^n : r_1 < |x| < r_2\}$ for some $r_1 \in \mathbb{R}^1$, $r_2 \in (0, R]$. Then $\mathcal{A}_j^k(f) = 0$ in B_{r_1,r_2}.*

The proof follows from Lemma 9.4 by the standard smoothing trick.

We extend now the map $\mathcal{A}_j^k$ to the space $\mathcal{D}'_{k,j}(\mathcal{O})$, where $\mathcal{O}$ is a nonempty open $\mathrm{O}(n)$-invariant subset of $\mathbb{R}^n$. By analogy with (9.57) and (9.76) we put

$$\langle \mathcal{A}_j^k(f), w\rangle = \langle \mathcal{A}_j^k(f\eta), w\rangle, \quad f \in \mathcal{D}'_{k,j}(\mathcal{O}), w \in \mathcal{D}(\mathcal{O}),$$

where $\eta \in \mathcal{D}_\natural(\mathcal{O})$ and $\eta = 1$ in some open set $\mathcal{O}_1 \subset \mathcal{O}$ such that $\operatorname{supp} w \subset \mathcal{O}_1$. Then $\mathcal{A}_j^k(f) \in \mathcal{D}'_\natural(\mathcal{O})$ and

$$\mathcal{A}_j^k(f|_{\mathcal{U}}) = \mathcal{A}_j^k(f)|_{\mathcal{U}} \tag{9.87}$$

for any nonempty open $\mathrm{O}(n)$-invariant subset $\mathcal{U} \subset \mathcal{O}$ (see Corollary 9.4).

The main properties of the mapping $\mathcal{A}_j^k$ can be deduced from the corresponding ones of $\mathfrak{A}_{k,j}$ in view of (9.84). They are collected in the following:

Theorem 9.7.

(i) *The mapping $\mathcal{A}_j^k$ is continuous from $\mathcal{D}'_{k,j}(\mathcal{O})$ into $\mathcal{D}'_\natural(\mathcal{O})$ and from $C_{k,j}^{N+n+k+4}(\mathcal{O})$ into $C_\natural^N(\mathcal{O})$, $N \in \mathbb{Z}_+$.*

(ii) *Let $f \in \mathcal{D}'_{k,j}(\mathcal{O})$ and $B_r \subset \mathcal{O}$ for some $r \in (0, +\infty]$. Then $f = 0$ in B_r if and only if $\mathcal{A}_j^k(f) = 0$ in B_r.*

(iii) *For $f \in \mathcal{D}'_{k,j}(\mathcal{O})$, one has*

$$\operatorname{supp} \mathcal{A}_j^k(f) \subset \operatorname{supp} f.$$

(iv) *If $f \in \mathcal{D}'_{k,j}(\mathcal{O})$, $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$, and the set $\mathcal{O}^T = \{x \in \mathbb{R}^n : x - \operatorname{supp} T \subset \mathcal{O}\}$ is nonempty, then (9.86) is valid in $\mathcal{O}^T$. In particular,*

$$\mathcal{A}_j^k\big(p(\Delta)f\big) = p(\Delta)\big(\mathcal{A}_j^k(f)\big)$$

for every polynomial p.

(v) *Let $0 \notin \mathcal{O}$. Then*

$$\mathcal{A}_j^k(\Psi_{\lambda,\mu,k,j}) = \Psi_{\lambda,\mu,0,1}$$

for $\lambda \neq 0$.

The proof follows from (9.87), Lemma 9.4, and Theorems 9.3 and 9.5.

Note also that (9.64) and (9.81) imply the inequality

$$\int_{B_r} \left|\Delta^N\big(\mathcal{A}_j^k(f)\big)(x)\right| \mathrm{d}x \leqslant c \sum_{i=0}^{[\frac{n+k}{2}]+2} \int_{B_r} \left|\Delta^{N+i}(f)(x)\right| \mathrm{d}x,$$

valid for $r \in (0, +\infty)$, $N \in \mathbb{Z}_+$, and $f \in C_{k,j}^{2N+n+k+4}(\dot{B}_r)$ with some constant $c > 0$ not depending on N and f.

9.5 Bessel-Type Decompositions for Some Classes of Functions with Generalized Boundary Conditions

Assume that $\nu > -1$, $f \in C^{\infty}[0,1]$, and let

$$f^{(l)}(0) = f^{(l)}(1) = 0 \quad \text{for all } l \in \mathbb{Z}_+. \tag{9.88}$$

Bearing (7.10) in mind, we assign to f its Fourier–Bessel series

$$f(t) = \sum_{m=1}^{\infty} c_m J_\nu(\lambda_m t), \qquad t \in [0,1], \tag{9.89}$$

where

$$c_m = \frac{2}{J^2_{\nu+1}(\lambda_m)} \int_0^1 t J_\nu(\lambda_m t) f(t)\,\mathrm{d}t,$$

and $\{\lambda_m\}_{m=1}^{\infty}$ is the sequence of all positive zeros of J_ν numbered in the ascending order. It can be shown that the series in (9.89) converges to f in $C^{\infty}[0,1]$ (see the proof of Theorem 7.1). For the case $2\nu \in \mathbb{Z}_+$, this statement admits various far-reaching generalizations. Some of them will be considered in this section. More precisely, we intend to show that under certain conditions a function can be expanded into a generalized spherical function series. First, we introduce and study some biorthogonal systems that will play an important role in the following.

Let $T \in \mathcal{E}'_{\natural}(\mathbb{R}^n)$, $T \neq 0$, and let

$$\mathcal{Z}_T = \left\{\lambda \in \mathcal{Z}\big(\widetilde{T}\big) : \operatorname{Re}\lambda \geqslant 0, \mathrm{i}\lambda \notin (0,+\infty)\right\}.$$

Throughout the section we suppose that $\mathcal{Z}_T \neq \varnothing$. Now define

$$n(\lambda,T) = \begin{cases} n_\lambda(\widetilde{T}) - 1 & \text{if } \lambda \in \mathcal{Z}_T\backslash\{0\}, \\ n_\lambda(\widetilde{T})/2 - 1 & \text{if } \lambda = 0 \in \mathcal{Z}_T. \end{cases}$$

Theorem 9.2(i) and Proposition 6.1(i), (iii) show that if $\mathcal{Z}_T$ is infinite, then

$$n(\lambda,T) = o(|\lambda|) \quad \text{as } \lambda \to +\infty, \lambda \in \mathcal{Z}_T,$$

and

$$\sum_{\lambda \in \mathcal{Z}_T} \frac{n(\lambda,T)}{(1+|\lambda|)^{1+\varepsilon}} < +\infty \quad \text{for each } \varepsilon > 0. \tag{9.90}$$

Let $\lambda \in \mathcal{Z}_T$ and $\eta \in \{0,\dots,n(\lambda,T)\}$. Formulae (6.21) and (6.22) and Theorem 9.2(i) show that

$$\left|b^{\lambda,\eta}\big(\widetilde{T},z\big)\right| \leqslant \gamma(1+|z|)^{\operatorname{ord} T-2}\mathrm{e}^{r(T)|\operatorname{Im} z|}, \quad z \in \mathbb{C}, \tag{9.91}$$

where $\gamma > 0$ is independent of z. Owing to Theorem 9.2(ii) and (9.91), for all $k \in \mathbb{Z}_+$ and $j \in \{1, \dots, d(n,k)\}$, there exists $T_{\lambda,\eta,k,j} \in \mathrm{conj}(\mathcal{E}'_{k,j}(\mathbb{R}^n))$ such that

$$r(T_{\lambda,\eta,k,j}) = r(T)$$

and

$$\mathcal{F}^k_j\big(\overline{T_{\lambda,\eta,k,j}}\big)(z) = \overline{b^{\lambda,\eta}\big(\widetilde{T}, \overline{z}\big)}, \quad z \in \mathbb{C}. \tag{9.92}$$

In addition,

$$\mathrm{ord}\ T_{\lambda,\eta,k,j} \leqslant \mathrm{ord}\ T + \frac{n}{2} + k - 1. \tag{9.93}$$

In the sequel, for brevity, we shall write $T_{\lambda,\eta,0,1} = T_{\lambda,\eta}$. Notice that $T_{\lambda,\eta} \in \mathcal{E}'_\natural(\mathbb{R}^n)$ and

$$\widetilde{T_{\lambda,\eta}}(z) = b^{\lambda,\eta}\big(\widetilde{T}, z\big), \quad z \in \mathbb{C}. \tag{9.94}$$

Next, by reasoning like that above one sees that there exists $T^{\lambda,\eta} \in \mathcal{E}'_\natural(\mathbb{R}^n)$ such that

$$r\big(T^{\lambda,\eta}\big) = r(T), \qquad \mathrm{ord}\ T^{\lambda,\eta} \leqslant \max\{0, n/2 - 1 - 2\eta + \mathrm{ord}\ T\}, \tag{9.95}$$

and

$$\widetilde{T^{\lambda,\eta}}(z)\big(z^2 - \lambda^2\big)^{\eta+1} = \widetilde{T}(z), \quad z \in \mathbb{C}. \tag{9.96}$$

Let us now establish basic properties of the distributions $T_{\lambda,\eta,k,j}$ and $T^{\lambda,\eta}$.

Proposition 9.11.

(i) *If* $\mu \in \mathcal{Z}_T$ *and* $\nu \in \{0, \dots, n(\mu, T)\}$, *then*

$$\langle T_{\lambda,\eta,k,j}, \Phi_{\mu,\nu,k,j}\rangle = \delta_{\lambda,\mu}\delta_{\eta,\nu}.$$

(ii) $T_{\lambda,\eta} = \sum_{p=0}^{n(\lambda,T)} b_p^{\lambda,\eta}(\widetilde{T})T^{\lambda,n(\lambda,T)-p}$.

(iii) *If* $T \in \mathfrak{R}(\mathbb{R}^n)$, *then*

$$\sum_{\lambda \in \mathcal{Z}_T} T_{\lambda,0} = \delta_0, \tag{9.97}$$

where the series converges unconditionally in the space $\mathcal{D}'(\mathbb{R}^n)$.

(iv) *Assume that* $T = (\Delta + c)Q$ *for some* $c \in \mathbb{C}$, $Q \in \mathcal{E}'_\natural(\mathbb{R}^n)$. *Then* $\mathcal{Z}_Q \subset \mathcal{Z}_T$ *and* $T_{\lambda,0} = Q_{\lambda,0} - b^{\lambda,0}_{n(\lambda,T)}(\widetilde{T})T$ *for all* $\lambda \in \mathcal{Z}_Q$.

Proof. Part (i) is a consequence of (9.92) and (6.23). Utilizing (9.94), (9.96), (6.21), and (6.22), we arrive at (ii). Next, to prove (iii) let $f \in \mathcal{D}(\mathbb{R}^n)$. Applying (6.34) and Proposition 6.14, one obtains

$$\langle T_{\lambda,0}, f\rangle = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \widehat{f}(x)\widetilde{T_{\lambda,0}}(|x|)\,\mathrm{d}x.$$

Now (6.34) and (8.12) imply that the series in (9.97) converges unconditionally in $\mathcal{D}'(\mathbb{R}^n)$ to some distribution $g \in \mathcal{E}'_\natural(\mathbb{R}^n)$. In addition,

$$\widetilde{g}(z) = \sum_{\lambda \in \mathcal{Z}_T} b^{\lambda,0}\big(\widetilde{T}, z\big), \quad z \in \mathbb{C},$$

and the function $u = (\widetilde{g} - 1)/\widetilde{T}$ is entire (see (6.23), (8.12), and Proposition 6.6(v)). The rest of the proof of (iii) follows analogously to that of Proposition 8.9. Finally, part (iv) is a consequence of (9.94), (6.21), (6.22), and Proposition 6.9. □

Proposition 9.12. *Let $l \in \mathbb{Z}_+$ and assume that D is an arbitrary differential operator in $\mathbb{R}^n$ of order at most l. Then the following results are true.*

(i) *If $T \in (\mathcal{E}'_\natural \cap C^m)(\mathbb{R}^n)$ where $m = l + k + n - 1$, then $T_{\lambda,\eta,k,j} \in (\mathcal{E}' \cap C^l)(\mathbb{R}^n)$ and*

$$|(DT_{\lambda,\eta,k,j})(x)| \leqslant \gamma_1 \sigma^{\lambda,\eta}\big(\widetilde{T}\big), \quad x \in \mathbb{R}^n,$$

where $\gamma_1 > 0$ is independent of λ, η, x.

(ii) *If $T \in (\mathcal{E}'_\natural \cap C^{l+n-1})(\mathbb{R}^n)$, then $T^{\lambda,\eta} \in (\mathcal{E}'_\natural \cap C^l)(\mathbb{R}^n)$ and*

$$\big|D\big(T^{\lambda,\eta}\big)(x)\big| \leqslant \gamma_2, \quad x \in \mathbb{R}^n,$$

where $\gamma_2 > 0$ is independent of λ, η, x.

The proof follows at once from (9.92), (9.96), (9.37), Theorem 9.2, and Proposition 6.6(ii).

Proposition 9.13. *For each $\lambda \in \mathcal{Z}_T$, the following statements are valid.*

(i) $(\Delta + \lambda^2)T^{\lambda,0} = -T$.
In addition, if $n(\lambda, T) \geqslant 1$, then

$$\big(\Delta + \lambda^2\big)T^{\lambda,\eta+1} = -T^{\lambda,\eta} \quad \textit{for all } \eta \in \{0, \dots, n(\lambda, T) - 1\}.$$

(ii) $(\Delta + \lambda^2)T_{\lambda,n(\lambda,T)} = -b^{\lambda,n(\lambda,T)}_{n(\lambda,T)}(\widetilde{T})T$.

(iii) *If $\lambda \neq 0$ and $n(\lambda, T) \geqslant 1$, then*

$$\big(\Delta + \lambda^2\big)T_{\lambda,n(\lambda,T)-1} + 2\lambda n(\lambda, T)T_{\lambda,n(\lambda,T)} = -b^{\lambda,n(\lambda,T)-1}_{n(\lambda,T)}\big(\widetilde{T}\big)T.$$

(iv) *If $\lambda \neq 0$ and $n(\lambda, T) \geqslant 2$, then*

$$\big(\Delta + \lambda^2\big)T_{\lambda,\eta} + 2\lambda(\eta + 1)T_{\lambda,\eta+1} + (\eta + 2)(\eta + 1)T_{\lambda,\eta+2} = -b^{\lambda,\eta}_{n(\lambda,T)}\big(\widetilde{T}\big)T$$

for all $\eta \in \{0, \dots, n(\lambda, T) - 2\}$.

(v) *If $0 \in \mathcal{Z}_T$ and $n(0, T) \geqslant 1$, then*

$$\Delta T_{0,\eta} + (2\eta + 2)(2\eta + 1)T_{0,\eta+1} = -b^{0,\eta}_{n(0,T)}\big(\widetilde{T}\big)T$$

for all $\eta \in \{0, \dots, n(0, T) - 1\}$.

Proof. The desired statements can be obtained directly, by using (9.94), (9.96), and Proposition 6.8. □

Proposition 9.14. *Let* $r(T) > 0$*, and let* $m = n + k + 3 + \operatorname{ord} T$*. Then*

$$\langle T_{\lambda,\eta,k,j}, f\rangle = \big\langle T_{\lambda,\eta}, \mathcal{A}_j^k(f)\big\rangle \quad \text{if } f \in C_{k,j}^{m+[n/2]}\big(\mathring{B}_{r(T)}\big), \tag{9.98}$$

and

$$\langle T_{\lambda,\eta,k,j}, f\rangle = \begin{cases} 2\langle \Lambda(T)_{\lambda,\eta}, \mathfrak{A}_{k,j}(f)\rangle & \text{if } \lambda \in \mathcal{Z}_T\backslash\{0\}, \\ \langle \Lambda(T)_{0,2\eta}, \mathfrak{A}_{k,j}(f)\rangle & \text{if } \lambda = 0 \in \mathcal{Z}_T, \end{cases} \tag{9.99}$$

for $f \in C_{k,j}^{m-1}(\mathring{B}_{r(T)})$.

Proof. If $f \in \Phi_{z,0,k,j}$, $z \in \mathbb{C}$, then relations (9.98) and (9.99) are obvious from (9.94), (9.85), (9.60), and (6.19). The general case reduces to this one by appealing to Proposition 9.9 and Theorems 9.7(i) and 9.3(iv) (see also (9.93) and (9.50)). □

Relations (9.98) and (9.99) allow us to obtain a multidimensional analog of estimate (8.51).

Proposition 9.15. *Let* $s \in \mathbb{N}$*,* $r(T) > 0$*, and* $f \in C_{k,j}^{m}(\mathring{B}_{r(T)})$*, where* $m = 2n + 2s + k + 7 + \operatorname{ord} T$*. Assume that*

$$\big\langle T, \mathcal{A}_j^k\big(\Delta^\nu f\big)\big\rangle = 0 \tag{9.100}$$

for all $\nu \in \{0,\dots,s\}$*. Let* $\lambda \in \mathcal{Z}_T$*,* $|\lambda| > 1$*, and* $\eta \in \{0,\dots,n(\lambda,T)\}$*. Then*

$$|\langle T_{\lambda,\eta,k,j}, f\rangle| \leqslant \frac{\gamma\sigma^{\lambda,\eta}(\widetilde{T})}{(|\lambda|-1)^{2s}} \sum_{i=0}^{l} \big\|\Delta^{s+i} f\big\|_{L^1(B_{r(T)})},$$

where

$$l = 2 + [(n+k)/2] + [\operatorname{ord} T/2], \tag{9.101}$$

and the constant $\gamma > 0$ *is independent of* λ, η, s, f.

Proof. Owing to (9.100), (9.58), (9.84), and Theorem 9.3(viii),

$$\big\langle \Lambda(T), \big(\mathfrak{A}_{k,j}(f)\big)^{(p)}\big\rangle = 0 \tag{9.102}$$

for each even $p \in \{0,\dots,2s\}$. Moreover, since $\Lambda(T)$ and $\mathfrak{A}_{k,j}(f)$ are even, equality (9.102) remains valid for all $p \in \{0,\dots 2s\}$. Collecting together the results given by (9.99), (9.51), Theorem 9.3(iv), and Proposition 8.13, we deduce that

$$|\langle T_{\lambda,\eta,k,j}, f\rangle| \leqslant \frac{\gamma_1\sigma^{\lambda,\eta}(\widehat{\Lambda(T)})}{(|\lambda|-1)^{2s}} \sum_{p=2s}^{2s+1+\operatorname{ord} T} \big\|\big(\mathfrak{A}_{k,j}(f)\big)^{(p)}\big\|_{L^1[-r(T),r(T)]},$$

where $\gamma_1 > 0$ is independent of λ, η, s, f. Now the required result is immediate by Theorem 9.4. □

Our next task is to find analogs of Theorem 8.2 and its corollaries.

Theorem 9.8. *Let* $r(T) > 0$ *and* $f \in C^m_{k,j}(\mathring{B}_{r(T)})$, *where* $m = n + k + 3 + \operatorname{ord} T$. *Suppose that*

$$\langle T_{\lambda,\eta,k,j}, f\rangle = 0 \quad \textit{for all}\ \lambda \in \mathcal{Z}_T,\ \eta \in \{0, \dots, n(\lambda, T)\}. \tag{9.103}$$

Then $f = 0$.

Proof. In view of (9.103) and (9.99),

$$\langle \Lambda(T)_{\lambda,\eta}, \mathfrak{A}_{k,j}(f)\rangle = 0 \tag{9.104}$$

for all $\lambda \in \mathcal{Z}_T\backslash\{0\}$ and $\eta \in \{0, \dots, n(\lambda, T)\}$. Assume now that $0 \in \mathcal{Z}_T$. As before, we have

$$\langle \Lambda(T)_{0,\nu}, \mathfrak{A}_{k,j}(f)\rangle = 0$$

for each even $\nu \in \{0, \dots, 2n(0, T)\}$. Because $\mathfrak{A}_{k,j}(f)$ is even and $\Lambda(T)_{0,\mu}$ is odd for each odd $\mu \in \{0, \dots, m(0, \Lambda(T))\}$, one concludes that (9.104) holds for all $\lambda \in \mathcal{Z}(\widehat{\Lambda(T)})$ and $\eta \in \{0, \dots, m(\lambda, \Lambda(T))\}$. Corollary 8.4 and Remark 8.1 yield $\mathfrak{A}_{k,j}(f) = 0$ (see (9.50), (9.51), and Theorem 9.3(iv)). A final ingredient is Theorem 9.3(ii). □

Corollary 9.5. *Let* $r(T) > 0$, *let* $f \in C^m_{\natural}(\mathring{B}_{r(T)})$, *where* $m = n + 3 + \operatorname{ord} T$, *and let*

$$\left\langle T^{\lambda,\eta}, f\right\rangle = 0 \quad \textit{for all}\ \lambda \in \mathcal{Z}_T, \eta \in \{0, \dots, n(\lambda, T)\}. \tag{9.105}$$

Then $f = 0$.

Proof. Applying Theorem 9.8 with $k = 0$, $j = 1$, one sees from Proposition 9.11(ii) and (9.105) that f must vanish. □

Theorem 9.9. *Let* $R > r(T)$, $f \in \mathcal{D}'(B_R)$, *and let*

$$f * T^{\lambda,n(\lambda,T)} = 0 \quad \textit{for all}\ \lambda \in \mathcal{Z}_T. \tag{9.106}$$

Then $f = 0$. *The same is true if* (9.106) *is replaced by*

$$f * T_{\lambda,\eta} = 0 \quad \textit{for all}\ \lambda \in \mathcal{Z}_T, \eta \in \{0, \dots, n(\lambda, T)\}. \tag{9.107}$$

Proof. First, assume that $f \in \mathcal{D}'_{k,j}(B_R)$ for some $k \in \mathbb{Z}_+$ and $j \in \{1, \dots, d(n, k)\}$. Using (9.106) and Theorem 9.3(i), one has $\mathfrak{A}_{k,j}(f) * \Lambda(T^{\lambda,n(\lambda,T)}) = 0$ for all $\lambda \in \mathcal{Z}_T$. Bearing in mind that $\Lambda(T^{\lambda,n(\lambda,T)}) = (\Lambda(T))_{(\lambda)}$ (see (8.64)), we conclude from Corollary 8.6 that $\mathfrak{A}_{k,j}(f) = 0$. Thus, $f = 0$ in view of Theorem 9.3(ii). The general case reduces to this one by using Proposition 9.1(iv), (vi).

Suppose now that (9.107) is satisfied. Then (9.106) holds because of Proposition 9.11(ii), and the above argument shows that $f = 0$. □

Next, assume that $m \in \mathbb{N}$, $m \geqslant 2$, and $T_1, \dots, T_m$ are nonzero distributions in $\mathcal{E}'_{\natural}(\mathbb{R}^n)$ such that $\mathcal{Z}_{T_l} \neq \varnothing$ for all $l \in \{1, \dots, m\}$. For $l, p \in \{1, \dots, m\}$ and $\lambda_l \in \mathcal{Z}_{T_l}$, let us define the distribution $T_{\lambda_1,\dots,\lambda_m,p} \in \mathcal{E}'_{\natural}(\mathbb{R}^n)$ by the formula

$$\widetilde{T_{\lambda_1,\dots,\lambda_m,p}} = \prod_{\substack{l=1,\\ l\neq p}}^{m} \widetilde{(T_l)_{\lambda_l,0}}$$

(see Theorem 9.2).

Proposition 9.16. *Let $\bigcap_{l=1}^{m} \mathcal{Z}_{T_l} = \varnothing$ and suppose that*

$$\mathcal{Z}(w_l) \cap \mathcal{Z}(w_l') = \varnothing \quad \textit{for all } l \in \{1, \dots, m\}, \tag{9.108}$$

where $w_l(z) = \widetilde{T}_l(\sqrt{z})$, $z \in \mathbb{C}$. Let $c_1, \dots, c_m \in \mathbb{C}$, $\sum_{l=1}^{m} c_l = 0$, and $\sum_{l=1}^{m} c_l \lambda_l^2 = 1$. Then

$$(T_1)_{\lambda_1,0} * \cdots * (T_m)_{\lambda_m,0} = -\sum_{l=1}^{m} c_l b_0^{\lambda_l,0}\big(\widetilde{T}_l\big) T_l * T_{\lambda_1,\dots,\lambda_m,l}.$$

Proof. Once Proposition 9.13(ii) has been established, the proof of the desired equality is identical to that of Proposition 8.15. □

Consider now an analog of Proposition 9.16 for $m = 2$ without assumption (9.108). For $\lambda_1 \in \mathcal{Z}_{T_1}$ and $\lambda_2 \in \mathcal{Z}_{T_2}$, we put

$$\nu = \nu(\lambda_1, \lambda_2) = n(\lambda_1, T_1) + n(\lambda_2, T_2) + 2.$$

Proposition 9.17. *If $\mathcal{Z}_{T_1} \cap \mathcal{Z}_{T_2} = \varnothing$, then*

$$\begin{aligned}(T_1)_{\lambda_1,0} * (T_2)_{\lambda_2,0} = \big(\lambda_1^2 - \lambda_2^2\big)^{-2\nu} &\Bigg(\sum_{p=0}^{\nu} \binom{2\nu}{\nu + p} \sum_{q=0}^{n(\lambda_1,T_1)} b_q^{\lambda_1,0}\big(\widetilde{T}_1\big) \\ &\times \big(-\Delta - \lambda_1^2\big)^{q+p+n(\lambda_2,T_2)+1}\big(\Delta + \lambda_2^2\big)^{\nu-p}\big(T_1 * (T_2)_{\lambda_2,0}\big) \\ &+ \sum_{p=1}^{\nu} \binom{2\nu}{\nu - p} \sum_{q=0}^{n(\lambda_2,T_2)} b_q^{\lambda_2,0}\big(\widetilde{T}_2\big) \\ &\times \big(\Delta + \lambda_1^2\big)^{\nu-p}\big(-\Delta - \lambda_2^2\big)^{q+p+n(\lambda_1,T_1)+1}\big(T_2 * (T_1)_{\lambda_1,0}\big)\Bigg)\end{aligned}$$

for all $\lambda_1 \in \mathcal{Z}_{T_1}$, $\lambda_2 \in \mathcal{Z}_{T_2}$.

Proof. Propositions 9.11(ii) and 9.13(i) yield

$$\big(-\Delta - \lambda_l^2\big)^{n(\lambda_l,T_l)+1}(T_l)_{\lambda_l,0} = \sum_{q=0}^{n(\lambda_l,T_l)} b_q^{\lambda_l,0}\big(\widetilde{T}_l\big)\big(-\Delta - \lambda_l^2\big)^q T_l, \quad l \in \{1, 2\}.$$

The rest of the proof follows analogously to that of Proposition 8.16. □

For the rest of the section, we assume that $r(T) > 0$. Our next object is to show that under certain conditions a function can be expanded into a series of the form

$$\sum_{\lambda\in\mathcal{Z}_T}\sum_{\eta=0}^{n(\lambda,T)} c_{\lambda,\eta}\Phi_{\lambda,\eta,k,j}, \tag{9.109}$$

where $c_{\lambda,\eta}\in\mathbb{C}$, $k\in\mathbb{Z}_+$, $j\in\{1,\dots,d(n,k)\}$.

We shall begin with the following auxiliary results.

Proposition 9.18.

(i) *Assume that*

$$\sup_{\lambda\in\mathcal{Z}_T}\frac{|\operatorname{Im}\lambda|+n(\lambda,T)}{\log(2+|\lambda|)}<+\infty, \tag{9.110}$$

and let

$$|c_{\lambda,\eta}|\leqslant(2+|\lambda|)^{\gamma}\quad \text{for all } \lambda\in\mathcal{Z}_T,\eta\in\{0,\dots,n(\lambda,T)\}, \tag{9.111}$$

where $\gamma>0$ is independent of λ,η. Then series (9.109) *converges in $\mathcal{D}'(\mathbb{R}^n)$.*

(ii) *Let $R>0$, $q\in\mathbb{Z}_+$, $\lambda\in\mathcal{Z}_T\backslash\{0\}$, $\eta\in\{0,\dots,n(\lambda,T)\}$, and*

$$A_{\lambda,\eta}(R,q)=\begin{cases}|\lambda|^{q-k}\left(R+\dfrac{q-k}{|\lambda|}\right)^{\eta} & \text{if } q\geqslant k,\\ |\lambda|^{q-k}\left(R+\dfrac{\eta+k-q}{|\lambda|}\right)^{\eta} & \text{if } q<k.\end{cases} \tag{9.112}$$

Suppose that

$$\sum_{\lambda\in\mathcal{Z}_T\backslash\{0\}}\sum_{\eta=0}^{n(\lambda,T)}|c_{\lambda,\eta}|A_{\lambda,\eta}(R,\nu)e^{R\operatorname{Im}\lambda}<+\infty$$

for all $\nu\in\{0,\dots,q\}$. Then series (9.109) *converges in $C^q(\dot{B}_R)$. In particular, if* (9.110) *holds and*

$$\max_{0\leqslant\eta\leqslant n(\lambda,T)}|c_{\lambda,\eta}|=O\big(|\lambda|^{-\gamma}\big)\quad \text{as } \lambda\to\infty \tag{9.113}$$

for each fixed $\gamma>0$, then series (9.109) *converges in $\mathcal{E}(\mathbb{R}^n)$.*

Proof. For any $\varphi\in\mathcal{D}(\mathbb{R}^n)$, one has

$$\langle\Phi_{\lambda,0,k,j},\varphi\rangle=\overline{\mathcal{F}_j^k(\psi)(\overline{\lambda})}, \tag{9.114}$$

where $\psi=\overline{\varphi}^{k,j}$. Bearing (9.110) in mind, we see from Theorem 9.2(i), (6.24), and (9.114) that

$$\langle\Phi_{\lambda,\eta,k,j},\varphi\rangle=O\big(|\lambda|^{-\gamma}\big)\quad \text{as } \lambda\to\infty$$

for each fixed $\gamma>0$. This, together with (9.90) and (9.111), brings us to (i).

Assertion (ii) follows at once from Proposition 9.7. □

Remark 9.4. It can be shown that Proposition 9.18(i) is no longer valid without assumption (9.110). In addition, for a broad class of distribution T, conditions (9.111) and (9.113) in Proposition 9.18 are necessary (see V.V. Volchkov [225], Part III, Theorem 2.5).

Proposition 9.19. *Suppose that* (9.110) *holds and let*

$$\max_{0\leqslant\eta\leqslant n(\lambda,T)} |c_{\lambda,\eta}| \leqslant \frac{M_q}{(2+|\lambda|)^{2q}}, \quad q=1,2,\dots, \tag{9.115}$$

where the constants $M_q > 0$ are independent of λ, and

$$\sum_{\nu=1}^{\infty} \frac{1}{\inf_{q\geqslant\nu} M_q^{1/2q}} = +\infty. \tag{9.116}$$

Then series (9.109) *converges in $\mathcal{E}(\mathbb{R}^n)$ to $f \in \mathrm{QA}(\mathbb{R}^n)$.*

Proof. For $R > 0$ and $q \in \mathbb{Z}_+$, estimate (9.37) yields

$$\left|\Delta^q \Phi_{\lambda,\eta,k,j}(x)\right| \leqslant \gamma^q (2+|\lambda|)^{2q}, \quad x \in \mathring{B}_R,$$

where $\gamma > 0$ is independent of λ, η, and q. Using now (9.115), (9.116), (9.90), Lemma 8.1(i), and Proposition 9.18(ii), we arrive at the desired statement. □

Proposition 9.20. *Let $\alpha > 0$, let*

$$|\mathrm{Im}\,\lambda| + n(\lambda,T) = o\big(|\lambda|^{1/\alpha}\big) \quad \text{as } \lambda\to\infty, \tag{9.117}$$

and assume that

$$|c_{\lambda,\eta}| \leqslant \gamma_1 \exp\big(-\gamma_2|\lambda|^{1/\alpha}\big), \tag{9.118}$$

where the constants $\gamma_1, \gamma_2 > 0$ are independent of λ, η. Then series (9.109) *converges in $\mathcal{E}(\mathbb{R}^n)$ to $f \in G^\alpha(\mathbb{R}^n)$.*

Proof. Let $R > 0$, $x \in \mathring{B}_R$, $q \in \mathbb{N}$. Applying (9.117), (9.118), and (9.37), we obtain

$$\sum_{\lambda\in\mathcal{Z}_T} \sum_{\eta=0}^{n(\lambda,T)} \left|c_{\lambda,\eta}\Delta^q \Phi_{\lambda,\eta,k,j}(x)\right| \leqslant \gamma_3^q \sum_{\lambda\in\mathcal{Z}_T} (1+|\lambda|)^{-2+2q} \exp\big(-\gamma_4|\lambda|^{1/\alpha}\big), \tag{9.119}$$

where $\gamma_3, \gamma_4 > 0$ depend only of $R, T, \alpha, \gamma_1, \gamma_2$. Next,

$$(1+|\lambda|)^{2q} \exp\big(-\gamma_4|\lambda|^{1/\alpha}\big) \leqslant \gamma_5^q q^{2\alpha q}$$

for some $\gamma_5 > 0$ independent of λ and q. The required conclusion now follows from (9.119) and (9.90). □

In the sequel for numbers $\lambda \in \mathcal{Z}_T\backslash\{0\}$, $\eta \in \{0,\dots,n(\lambda,T)\}$, $q \in \mathbb{Z}_+$, we define $A_{\lambda,\eta}(r(T),q)$ by (9.112).

Theorem 9.10. *Let* $s, q \in \mathbb{N}$, $q \geqslant n + k + 3 + \operatorname{ord} T$, *and let* $f \in C^m_{k,j}(\mathring{B}_{r(T)})$, *where* $m = 2n + 2s + k + 7 + \operatorname{ord} T$. *Assume that*

$$\langle T, \mathcal{A}^k_j(\Delta^\nu f)\rangle = 0 \tag{9.120}$$

for each $\nu \in \{0, \dots, s\}$ *and that*

$$\sum_{\lambda\in\mathcal{Z}_T\setminus\{0\}} \sum_{\eta=0}^{n(\lambda,T)} \frac{\sigma^{\lambda,\eta}(\widetilde{T})}{(1+|\lambda|)^{2s}} A_{\lambda,\eta}(r(T),\mu) e^{r(T)|\operatorname{Im}\lambda|} < +\infty \tag{9.121}$$

for all $\mu \in \{0, \dots, q\}$. *Then*

$$f = \sum_{\lambda\in\mathcal{Z}_T} \sum_{\eta=0}^{n(\lambda,T)} c_{\lambda,\eta} \Phi_{\lambda,\eta,k,j}, \tag{9.122}$$

where $c_{\lambda,\eta} = \langle T_{\lambda,\eta,k,j}, f\rangle$, *and the series converges in* $C^q(\mathring{B}_{r(T)})$. *In addition, if* $|\lambda| > 1$ *and* $\eta \in \{0, \dots, n(\lambda, T)\}$, *then*

$$|c_{\lambda,\eta}| \leqslant \frac{\gamma\sigma^{\lambda,\eta}(\widetilde{T})}{(|\lambda|-1)^{2s}} \sum_{i=0}^{l} \left\|\Delta^{s+i} f\right\|_{L^1(B_{r(T)})}, \tag{9.123}$$

where l *is defined by* (9.101), *and the constant* $\gamma > 0$ *is independent of* λ, η, s, f.

Proof. Estimate (9.123) is a consequence of Proposition 9.15. Now property (9.121) and Proposition 9.18(ii) tell us that the series in the right-hand part of (9.122) converges in $C^q(\mathring{B}_{r(T)})$. Denoting its sum by g, we get $c_{\lambda,\eta} = \langle T_{\lambda,\eta,k,j}, g\rangle$ (see Proposition 9.11(i) and (9.93)). Hence, $\langle T_{\lambda,\eta,k,j}, f - g\rangle = 0$ for all $\lambda \in \mathcal{Z}_T$ *and* $\eta \in \{0, \dots, n(\lambda, T)\}$. To complete the proof it remains to apply Theorem 9.8. □

Corollary 9.6. *If* $T \in \mathfrak{M}(\mathbb{R}^n)$, *then the following items are equivalent.*

(i) $f \in C^\infty_{k,j}(\mathring{B}_{r(T)})$, *and* (9.120) *holds for all* $\nu \in \mathbb{Z}_+$.
(ii) *Equality* (9.122) *is true, where* $c_{\lambda,\eta} = \langle T_{\lambda,\eta,k,j}, f\rangle$, *and the series converges in* $\mathcal{E}(\mathring{B}_{r(T)})$.

The proof follows from (9.85), Theorem 9.1(i), (ii), and Theorem 9.10.

We note that assumption (9.120) in Theorem 9.10 is an analog of boundary conditions (9.88).

Corollary 9.7. *If* $T \in \mathfrak{M}(\mathbb{R}^n)$, *then the following assertions are equivalent.*

(i) $f \in (C^\infty_{k,j} \cap \mathrm{QA})(\mathring{B}_{r(T)})$, *and* (9.120) *is satisfied for each* $\nu \in \mathbb{Z}_+$.
(ii) *Conditions* (9.115) *and* (9.116) *are fulfilled with* $c_{\lambda,\eta} = \langle T_{\lambda,\eta,k,j}, f\rangle$, *and the series in* (9.122) *converges to* f *in* $\mathcal{E}(\mathring{B}_{r(T)})$.

Proof. The implication (ii)→(i) is a consequence of Proposition 9.19, (9.85), and Theorem 9.1(i), (ii). The proof that (i)→(ii) will be presented later (see Theorems 14.20, 14.17(iii), and 14.11 and (14.32)). □

To continue, for $f \in C^{\infty}_{k,j}(\mathring{B}_{r(T)})$, $\lambda \in \mathcal{Z}_T$, and $\eta \in \{0, \dots, n(\lambda, T)\}$, we put

$$\mu_{\lambda,\eta}(f) = \begin{cases} \inf\limits_{s\in\mathbb{N}} (|\lambda| - 1)^{-2s} \sum\limits_{i=0}^{l} \left\| \Delta^{s+i} f \right\|_{L^1(B_{r(T)})} & \text{if } |\lambda| > 1, \\ 0 & \text{if } |\lambda| \leqslant 1, \end{cases} \tag{9.124}$$

where l is defined by (9.101).

Theorem 9.11. *Let $f \in C^{\infty}_{k,j}(\mathring{B}_{r(T)})$ and assume that (9.120) holds for all $\nu \in \mathbb{Z}_+$. Suppose that $q \in \mathbb{N}$, $q \geqslant n + k + 3 + \operatorname{ord} T$, and*

$$\sum_{\lambda\in\mathcal{Z}_T} \sum_{\eta=0}^{n(\lambda,T)} \sigma^{\lambda,\eta}\big(\widetilde{T}\big) A_{\lambda,\eta}\big(r(T), \tau\big) \mu_{\lambda,\eta}(f) e^{r(T)|\operatorname{Im}\lambda|} < +\infty$$

for each $\tau \in \{0, \dots, q\}$. Then relation (9.122) holds, where $c_{\lambda,\eta} = \langle T_{\lambda,\eta,k,j}, f\rangle$, and the series converges in $C^q(\mathring{B}_{r(T)})$.

The proof nearly completely reproduces the proof of Theorem 9.10, and therefore we omit it.

Corollary 9.8. *Let $\alpha > 0$ and $T \in \mathfrak{G}_\alpha(\mathbb{R}^n)$. Then the following statements are equivalent.*

(i) *$f \in (C^{\infty}_{k,j} \cap G^{\alpha})(\mathring{B}_{r(T)})$, and (9.120) is fulfilled for each $\nu \in \mathbb{Z}_+$.*
(ii) *Condition (9.118) is satisfied with $c_{\lambda,\eta} = \langle T_{\lambda,\eta,k,j}, f\rangle$, and the series in (9.122) converges to f in $\mathcal{E}(\mathring{B}_{r(T)})$.*

This result can be proved in the same way as the corresponding statement for the case $n = 1$ (see Corollary 8.10, Theorem 9.11, and Proposition 9.20).

To conclude we state results similar to Propositions 9.18–9.20 for series of the form

$$\sum_{\lambda\in\mathcal{Z}_T} \sum_{\eta=0}^{n(\lambda,T)} c_{\lambda,\eta} \Psi_{\lambda,\eta,k,j}. \tag{9.125}$$

Proposition 9.21.

(i) *Assume that conditions (9.110) and (9.111) are satisfied. Then series (9.125) converges in $\mathcal{D}'(\mathbb{R}^n\backslash\{0\})$.*
(ii) *If conditions (9.110) and (9.113) hold, then series (9.125) converges in the space $\mathcal{E}(\mathbb{R}^n\backslash\{0\})$.*

(iii) *Suppose that* (9.110)*,* (9.115)*, and* (9.116) *are valid. Then series* (9.125) *converges in* $\mathcal{E}(\mathbb{R}^n\backslash\{0\})$ *to* $f \in \mathrm{QA}(\mathbb{R}^n\backslash\{0\})$.

(iv) *Let* $\alpha > 0$ *and assume that* (9.117) *and* (9.118) *hold. Then series* (9.125) *converges in* $\mathcal{E}(\mathbb{R}^n\backslash\{0\})$ *to* $f \in G^{\alpha}(\mathbb{R}^n\backslash\{0\})$.

Proof. To prove (i), first observe that

$$\langle \Psi_{\lambda,0,k,j}, \varphi \rangle = \frac{1}{(-\lambda^2)^q} \big\langle \Psi_{\lambda,0,k,j}, \Delta^q \varphi \big\rangle \tag{9.126}$$

for all $\lambda \in \mathcal{Z}_T\backslash\{0\}$, $q \in \mathbb{Z}_+$, and $\varphi \in \mathcal{D}(\mathbb{R}^n\backslash\{0\})$. Having (9.110) in mind, one has

$$\langle \Psi_{\lambda,\eta,k,j}, \varphi \rangle = O\big(|\lambda|^{-\gamma}\big) \quad \text{as } \lambda \to \infty$$

for each $\gamma > 0$ (see Proposition 9.5, (6.24), and (9.126)). Thus, using estimates (9.90) and (9.111), we arrive at (i). The proof of (ii)–(iv) is analogous to the proof of Propositions 9.18(ii), 9.19, and 9.20, only instead of Proposition 9.7, one applies Proposition 9.5. □

Chapter 10
The Case of Symmetric Spaces $X = G/K$ of Noncompact Type

The study of transmutation operators on symmetric spaces requires a much more complicated technique. In this chapter we consider the case of symmetric spaces $X = G/K$ of noncompact type. Our constructions are based on three main ingredients. The first is the Fourier decomposition on X, the second is the Eisenstein–Harish-Chandra integrals and their generalizations, and the third is the Helgason–Fourier transform. These themes and related questions are discussed in Sects. 10.1–10.5. In Sect. 10.6 we define and investigate the transmutation mapping $\mathfrak{A}_\delta$ associated with the inversion formula for the δ-spherical transform. If rank $X = 1$, we can work out our theory from Sect. 10.6 in more explicit and concrete form. This is done in Sect. 10.8. To apply $\mathfrak{A}_\delta$ to the theory of mean periodic functions on X (see Part III below), we introduce in Sect. 10.7 the class $\mathcal{E}'_{\natural\natural}(X)$ of K-invariant distributions on X with radial spherical transforms. It follows from the Paley–Wiener theorem for the spherical transform that $\mathcal{E}'_{\natural\natural}(X)$ is broad enough, and $\mathcal{E}'_{\natural\natural}(X) = \mathcal{E}'_{\natural}(X)$ if rank $X = 1$. We characterize the class $\mathcal{E}'_{\natural\natural}(X)$ in terms of the mean value theorem for eigenfunctions of the Laplace–Beltrami operator on X using the mapping $\mathfrak{A}_\delta$. In the case where the group G is complex we give very simple and explicit description of $\mathcal{E}'_{\natural\natural}(X)$ in Theorem 10.17. In Sect. 10.9, for distributions in the class $\mathcal{E}'_{\natural\natural}(X)$, we extend to X the principal results from Sect. 9.5.

10.1 Generalities

Here we list briefly the customary notation and facts from the theory of symmetric spaces and refer, for example, to Helgason [121–123] for more explicit definitions and for all unproved statements.

Let $X = G/K$ be a symmetric space of noncompact type, G being a connected semisimple Lie group with finite center, and K a maximal compact subgroup. Let $o = \{K\}$ be the origin in X and denote the action of G on X by $(g, x) \to gx$ for $g \in G$, $x \in X$. The Lie algebras of G and K are respectively denoted by $\mathfrak{g}$ and $\mathfrak{k}$. The adjoint representations of $\mathfrak{g}$ and G are respectively denoted by ad and Ad. Let

V.V. Volchkov, V.V. Volchkov, *Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group,* Springer Monographs in Mathematics,

$\langle\, , \rangle$ be the Killing form of $\mathfrak{g}_{\mathbb{C}}$, the complexification of $\mathfrak{g}$. The form $\langle\, , \rangle$ induces a G-invariant Riemannian structure on X with the corresponding distance function $d(\cdot,\cdot)$ and the Riemannian measure $\mathrm{d}x$. For $0 \leqslant r \leqslant R$, $y \in X$, we set

$$B_R(y) = \{x \in X \colon d(x, y) < R\}, \qquad B_R = B_R(o), \qquad B_{+\infty} = X,$$
$$\mathring{B}_R(y) = \{x \in X \colon d(x, y) \leqslant R\}, \qquad \mathring{B}_R = \mathring{B}_R(o),$$
$$S_R = \{x \in X \colon d(o, x) = R\}, \qquad B_{r,R} = \{x \in X \colon r < d(o, x) < R\},$$
$$\mathring{B}_{r,R} = \{x \in X \colon r \leqslant d(o, x) \leqslant R\}.$$

Let $\mathbf{D}(G)$ denote the algebra of left-invariant differential operators on G and $\mathbf{D}(X)$ the algebra of G-invariant differential operators on X. The Laplace–Beltrami operator on X is denoted by L.

Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to $\langle, \rangle$, and let $\mathfrak{a} \subset \mathfrak{p}$ be any maximal abelian subspace (all such subspaces have the same dimension). The dimension of $\mathfrak{a}$ is called the *real rank* of G and the *rank* of the space X. We shall write $\operatorname{rank} X = \dim \mathfrak{a}$.

Let $\mathfrak{a}^*$ be the dual of $\mathfrak{a}$, and $\mathfrak{a}^*_{\mathbb{C}}$ and $\mathfrak{a}_{\mathbb{C}}$ their respective complexifications. If $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, $\lambda = \xi + i\eta$, where $\xi, \eta \in \mathfrak{a}^*$, we set $\operatorname{Re}\lambda = \xi$, $\operatorname{Im}\lambda = \eta$, $\overline{\lambda} = \xi - i\eta$. Next, let $A_\lambda \in \mathfrak{a}_{\mathbb{C}}$ be determined by $\langle H, A_\lambda\rangle = \lambda(H)$ $(H \in \mathfrak{a})$ and put

$$\langle\lambda, \mu\rangle = \langle A_\lambda, A_\mu\rangle, \quad \lambda, \mu \in \mathfrak{a}^*_{\mathbb{C}}.$$

For $\lambda \in \mathfrak{a}^*$ and $P \in \mathfrak{p}$, put $|\lambda| = \langle\lambda, \lambda\rangle^{1/2}$, $|P| = \langle P, P\rangle^{1/2}$,

$$\mathfrak{g}_\lambda = \{U \in \mathfrak{g} : [H, U] = \lambda(H)U \text{ for all } H \in \mathfrak{a}\},$$

where $[\cdot, \cdot]$ is the bracket operation in $\mathfrak{g}$. If $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq \{0\}$, then λ is called a (*restricted*) *root*, and $m_\lambda = \dim \mathfrak{g}_\lambda$ is called its *multiplicity*. The spaces $\mathfrak{g}_\lambda$ are called *root subspaces*. The set of restricted roots will be denoted by Σ. A root $\lambda \in \Sigma$ is called *indivisible* if $c\lambda \in \Sigma \Rightarrow c = \pm 1, \pm 2$. Let Σ_0 denote the set of indivisible roots. A point $H \in \mathfrak{a}$ is called *regular* if $\lambda(H) \neq 0$ for all $\lambda \in \Sigma$. The subset $\mathfrak{a}' \subset \mathfrak{a}$ of regular elements consists of the complement of finitely many hyperplanes, and its components are called Weyl chambers. Fix a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ and let

$$\mathfrak{a}^*_+ = \left\{\lambda \in \mathfrak{a}^*_{\mathbb{C}} : A_\lambda \in \mathfrak{a}^+\right\}.$$

We call a root *positive* if it is positive on $\mathfrak{a}^+$. Let Σ^+ denote the set of positive roots; for $\alpha \in \Sigma^+$, we will also use the notation $\alpha > 0$ and put $\Sigma_0^+ = \Sigma_0 \cap \Sigma^+$. Now define

$$\rho = \frac{1}{2}\sum_{\alpha>0} m_\alpha \alpha \quad \text{and} \quad \mathfrak{n} = \sum_{\alpha>0} \mathfrak{g}_\alpha.$$

Let exp denote the exponential mapping of $\mathfrak{g}$ into G. As usual, we set $\operatorname{Exp} P = (\exp P)K \in X$ for each $P \in \mathfrak{p}$. Denote by log the inverse of the map $\exp : \mathfrak{a} \to A$.

Putting $N = \exp \mathfrak{n}$, $A = \exp \mathfrak{a}$, and $A^+ = \exp \mathfrak{a}^+$, we have the Iwasawa and Cartan decompositions,

$$G = KAN, \qquad G = K\overline{A^+}K,$$

where $\overline{A^+}$ is the closure of A^+ in G. The mapping

$$(k, a, n) \to kan \quad (k \in K, a \in A, n \in N)$$

is an analytic diffeomorphism of the product manifold $K \times A \times N$ onto G. If $g \in G$, we write

$$g = k(g) \exp H(g) n(g) = k_1 a k_2,$$

where $k(g) \in K$, $n(g) \in N$, $H(g) \in \mathfrak{a}$, $a \in \overline{A^+}$, are uniquely determined, and $k_1, k_2 \in K$. It can be shown that

$$|H(g)| \leqslant d(o, go) \quad \text{for each } g \in G, \tag{10.1}$$

and the equality in (10.1) holds if and only if $g \in KA$. This yields

$$|H(ak)| \leqslant |H(ka)| = |\log a|, \quad a \in A, k \in K. \tag{10.2}$$

Let M and M', respectively, denote the centralizer and normalizer of A in K. The factor group M'/M is the Weyl group W. Denote by $|W|$ the order of W. We point out that $|W| = 2$ if rank $X = 1$.

Let $B = K/M$ and denote the action of G on B by $(g, b) \to gb$ for $g \in G$, $b \in B$. The mapping $(kM, a) \to kaK$ is an analytic diffeomorphism of $B \times A^+$ onto an open dense set in G/K, the polar-coordinate decomposition of X. We put

$$A(gK, kM) = -H\big(g^{-1}k\big), \quad A(g) = -H\big(g^{-1}\big), \quad g \in G,\ k \in K.$$

Then $\exp A(x, b)$ is the complex distance from o to the horocycle in X through x with normal b and

$$A(gx, gb) = A(x, b) + A(go, gb), \quad x \in X,\ g \in G,\ b \in B. \tag{10.3}$$

In particular,

$$A(kx, b) = A\big(x, k^{-1}b\big), \quad x \in X,\ k \in K,\ b \in B. \tag{10.4}$$

For all $\mu \in \mathfrak{a}^*_{\mathbb{C}}$ and $b \in B$, the function $x \to e^{\mu(A(x,b))}$ is an eigenfunction of each G-invariant differential operator on X. We point out that

$$L\big(e^{(i\lambda+\rho)(A(x,b))}\big) = -\big(\langle\lambda, \lambda\rangle + |\rho|^2\big)e^{(i\lambda+\rho)(A(x,b))} \tag{10.5}$$

if $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, $x \in X$, $b \in B$.

The Killing form $\langle\, , \rangle$ induces Euclidean measures on A, $\mathfrak{a}$ and $\mathfrak{a}^*$. If $l = \dim A$, we multiply these measures by the factor $(2\pi)^{-l/2}$ and thereby obtain invariant measures $\mathrm{d}a$, $\mathrm{d}H$, and $\mathrm{d}\lambda$ on A, $\mathfrak{a}$, and $\mathfrak{a}^*$, respectively. Then the Fourier transform

$$\widehat{f}(\lambda) = \int_A f(a) e^{-i\lambda(\log a)}\, \mathrm{d}a, \quad f \in \mathcal{D}(A),\ \lambda \in \mathfrak{a}^*$$

is inverted by the formula

$$f(a) = \int_{\mathfrak{a}^*} \widehat{f}(\lambda) e^{i\lambda(\log a)}\, \mathrm{d}\lambda, \quad a \in A.$$

We normalize the Haar measure $\mathrm{d}k$ on K so that the total measure is 1. The Haar measures $\mathrm{d}g$ on G and $\mathrm{d}n$ on N are normalized so that

$$\int_G f(gK)\, \mathrm{d}g = \int_X f(x)\, \mathrm{d}x, \quad f \in L^1(X), \tag{10.6}$$

$$\int_G f(g)\, \mathrm{d}g = \int_{KAN} f(kan) e^{2\rho(\log a)}\, \mathrm{d}k\, \mathrm{d}a\, \mathrm{d}n, \quad f \in L^1(G). \tag{10.7}$$

Let Δ be defined on A by

$$\Delta(\exp H) = c \prod_{\alpha \in \Sigma^+} \big(\sinh \alpha(H)\big)^{m_\alpha}, \quad H \in \mathfrak{a},$$

the constant $c > 0$ being determined so that

$$\int_G f(g)\, \mathrm{d}g = \int_K \int_{A^+} \int_K f(k_1 a k_2) \Delta(a)\, \mathrm{d}k_1\, \mathrm{d}a\, \mathrm{d}k_2, \quad f \in L^1(G). \tag{10.8}$$

Let $\mathrm{d}b = \mathrm{d}k_M$ be the invariant measure on $B = K/M$ defined by

$$\int_B f(b)\, \mathrm{d}b = \int_K f(kM)\, \mathrm{d}k, \quad f \in L^1(B).$$

Let $\mathcal{O}$ be a nonempty open subset of X. By analogy with the Euclidean case, for $T \in \mathcal{E}'(X)$, we set

$$\begin{aligned} r(T) &= \inf\{r > 0 : \operatorname{supp} T \subset B_r(x) \text{ for some } x \in X\}, \\ r_0(T) &= \inf\{r > 0 : \operatorname{supp} T \subset B_r\}, \\ \mathcal{O}_T &= \big\{x \in X : \mathring{B}_{r(T)}(x) \subset \mathcal{O}\big\}. \end{aligned} \tag{10.9}$$

Let $*$ and $\times$ denote the convolutions on $\mathfrak{a}$ and X, respectively. We recall from Sect. 1.4 that if $f \in \mathcal{D}'(X)$ and $T \in \mathcal{E}'(X)$, then

$$\langle f \times T, u \rangle = \left\langle T(g_2 K), \left\langle f(g_1 K), \int_K u(g_1 k g_2 K)\, \mathrm{d}k \right\rangle\right\rangle, \quad u \in \mathcal{D}(X). \tag{10.10}$$

Relations (10.9) and (10.10) show that if T is K-invariant and $\mathcal{O}_T \neq \varnothing$, then for each $f \in \mathcal{D}'(\mathcal{O})$, the convolution $f \times T$ is a well-defined distribution in $\mathcal{D}'(\mathcal{O}_T)$.

Let $\mathfrak{W}(\mathcal{O})$ be an arbitrary subset of $\mathcal{D}'(\mathcal{O})$. If $\mathcal{O}$ is K-invariant, we shall write $\mathfrak{W}_\natural(\mathcal{O})$ for the set of all K-invariant distributions in $\mathfrak{W}(\mathcal{O})$.

For $0 < R \leqslant +\infty$, we denote $\mathcal{B}_R = \{H \in \mathfrak{a} : |H| < R\}$. Let $\mathcal{D}'_W(\mathcal{B}_R)$, $\mathcal{E}'_W(\mathcal{B}_R)$, $\mathcal{D}_W(\mathcal{B}_R)$, and $C^m_W(\mathcal{B}_R)$ with $m \in \mathbb{Z}_+$ or $m = \infty$ denote the sets of all W-invariant distributions in the classes $\mathcal{D}'(\mathcal{B}_R)$, $\mathcal{E}'(\mathcal{B}_R)$, $\mathcal{D}(\mathcal{B}_R)$, and $C^m(\mathcal{B}_R)$, respectively.

For the rest of the section, we assume that rank $X = 1$. Denote by γ the unique root in Σ_0^+ and put

$$\alpha_X = \frac{1}{2}(m_\gamma + m_{2\gamma} - 1), \qquad \beta_X = \frac{1}{2}(m_{2\gamma} - 1),$$

where $m_{2\gamma} = 0$ if $2\gamma \notin \Sigma$.

Let H_0 be the vector in $\mathfrak{a}^+$ such that $\gamma(H_0) = 1$. We set

$$\kappa = 1/|H_0|, \qquad \rho_X = \kappa(\alpha_X + \beta_X + 1).$$

Notice that

$$\kappa = (2m_\gamma + 8m_{2\gamma})^{-1/2}.$$

Let $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. Writing $\lambda(H_0)A_\lambda = \langle\lambda, \lambda\rangle H_0$, we deduce

$$\lambda(H_0)^2 = \kappa^{-2}\langle\lambda, \lambda\rangle, \qquad A_\lambda = \kappa^2 H_0, \tag{10.11}$$

whence

$$\langle\gamma, \gamma\rangle = \kappa^2. \tag{10.12}$$

In addition, by (10.11) and (10.12),

$$\langle\lambda, \gamma\rangle = \lambda(A_\gamma) = \kappa^2\lambda(H_0).$$

For $t \in \mathbb{R}^1$, we set $a_t = \exp(tH_0) \in A$. Then $a_{t_1+t_2} = a_{t_1}a_{t_2}$ for all $t_1, t_2 \in \mathbb{R}^1$. Using (10.2), we obtain

$$d(o, a_t o) = \kappa^{-1}|t|, \quad t \in \mathbb{R}^1. \tag{10.13}$$

Consider now the function $h : X \to \mathbb{R}^1$ defined as follows: if $x \in X$ and $x = go$, $g \in G$, then $h(x)H_0 = \kappa^{-1}A(g)$. Clearly, the so-defined function $h(x)$ is independent of our choice of g such that $x = go$.

Proposition 10.1.

(i) $|h(x)| \leqslant d(o, x)$ *for each* $x \in X$.
(ii) $h(nx) = h(x)$ *for all* $x \in X$, $n \in N$.
(iii) $h(ao) = \kappa^{-1}\gamma(\log a)$ *for each* $a \in A$.
(iv) $h(ax) = h(x) + h(ao)$ *for all* $x \in X$, $a \in A$.

Proof. Assertions (i)–(iii) follow immediately from (10.1) and the definition of h. To prove (iv) write $x = n_1 a_1 o$ for some $n_1 \in N$ and $a_1 \in A$. Since A normalizes N, there exists $n_2 \in N$ such that $n_2 a = a n_1$. Using (ii) and (iii), one has $h(ax) = h(a n_1 a_1 o) = h(n_2 a a_1 o) = h(a_1 o) + h(ao)$, which brings us to (iv). □

Let $A_X(R)$ denote the area of S_R. A calculation shows that

$$A_X(R) = \frac{2^{1-q}\pi^{(p+q+1)/2}}{\Gamma((p+q+1)/2)\kappa^{p+q}} \sinh^p(\kappa R) \sinh^q(2\kappa R), \tag{10.14}$$

where $p = m_\gamma$, $q = m_{2\gamma}$. If $0 \leqslant r < R \leqslant +\infty$ and $f \in L^1(B_{r,R})$, then

$$\int_{B_{r,R}} f(x)\,\mathrm{d}x = \int_r^R A_X(t) \int_K f(k a_{\kappa t} o)\,\mathrm{d}k\,\mathrm{d}t \tag{10.15}$$

because of (10.13). To conclude we note that if $f \in C^2(X)$ and $f(x) = F(d(o,x))$ for each $x \in X$, then

$$(Lf)(x) = \left(\frac{\mathrm{d}^2 F}{\mathrm{d}r^2} + \frac{A_X^{'}(r)}{A_X(r)}\frac{\mathrm{d}F}{\mathrm{d}r}\right)\Bigg|_{r=d(o,x)}. \tag{10.16}$$

10.2 Fourier Decompositions on G/K

As in Sect. 1.5, we write $\widehat{K}$ for the set of equivalence classes of finite-dimensional unitary irreducible representations of K. For each $\delta \in \widehat{K}$, denote by V_δ a vector space (with inner product $\langle\,,\rangle$) on which a representation of class δ is realized; let such a representation also be denoted by δ. Let $\widehat{K}_M$ denote the set of elements $\delta \in \widehat{K}$ for which

$$V_\delta^M = \{v \in V_\delta : \delta(m)v = v \text{ for all } m \in M\} \neq \{0\}.$$

We put

$$d(\delta) = \dim V_\delta, \qquad l(\delta) = \dim V_\delta^M,$$

and fix an orthonormal basis

$$v_1, \dots, v_{d(\delta)} \quad \text{of } V_\delta$$

such that

$$v_1, \dots, v_{l(\delta)} \quad \text{span } V_\delta^M.$$

As usual, let $\check{\delta}$ denote the contragredient representation of K on the dual space $V_\delta' = V_{\check{\delta}}$. We note that if $\delta \in \widehat{K}_M$, then $\check{\delta}$ also belongs to $\widehat{K}_M$.

Denote by $\mathrm{Hom}(V_\delta, V_\delta)$ (respectively $\mathrm{Hom}(V_\delta, V_\delta^M)$) the vector space of linear maps $V_\delta \to V_\delta$ (respectively $V_\delta \to V_\delta^M$).

Throughout the section, $\mathcal{O}$ is a nonempty open K-invariant subset of $X = G/K$. For $f \in C(\mathcal{O})$ and $\delta \in \widehat{K}$, we define

$$f^{\delta}(x) = d(\delta)\int_K f(kx)\delta\big(k^{-1}x\big)\,\mathrm{d}k, \quad x \in \mathcal{O}. \tag{10.17}$$

Then f is a continuous map from $\mathcal{O}$ to $\mathrm{Hom}(V_\delta, V_\delta)$ satisfying

$$f^{\delta}(kx) = \delta(k)f^{\delta}(x), \quad k \in K,\ x \in \mathcal{O}.$$

Next, we set

$$f_{\delta}(x) = \mathrm{Trace}\big(f^{\delta}(x)\big) = d(\delta)\int_K \chi\big(k^{-1}\big)f(kx)\,\mathrm{d}k, \quad x \in \mathcal{O}, \tag{10.18}$$

where χ is the character of δ. If $w \in \mathcal{D}(\mathcal{O})$, this yields

$$\int_{\mathcal{O}} f_\delta(x)w(x)\,\mathrm{d}x = \int_{\mathcal{O}} f(x)\overline{(\overline{w})_\delta(x)}\,\mathrm{d}x.$$

We now extend the definition of f_δ to distributions by the formula

$$\langle f_\delta, w\rangle = \big\langle f, \overline{(\overline{w})_\delta}\big\rangle, \quad f \in \mathcal{D}'(\mathcal{O}),\ w \in \mathcal{D}(\mathcal{O}). \tag{10.19}$$

Proposition 10.2.

(i) *If $f \in \mathcal{D}'(\mathcal{O})$ and $u \in C^{\infty}_{\natural}(\mathcal{O})$, then $(uf)_\delta = uf_\delta$.*
(ii) *If $f \in \mathcal{D}'(X)$ and $u \in \mathcal{E}'(X)$, then $(f \times u)_\delta = f_\delta \times u$.*
(iii) *If $f \in \mathcal{D}'(\mathcal{O})$ (respectively $f \in \mathcal{E}(\mathcal{O})$), then*

$$f = \sum_{\delta \in \widehat{K}_M} f_\delta,$$

where the series converges unconditionally in $\mathcal{D}'(\mathcal{O})$ (respectively $\mathcal{E}(\mathcal{O})$).

Proof. Part (i) is obvious from (10.19). To prove (ii), let $w \in \mathcal{D}(X)$. We define

$$\psi_1(x,y) = \int_K \overline{(\overline{w})_\delta}(gky)\,\mathrm{d}k$$

and

$$\psi_2(x,y) = \int_K w(gky)\,\mathrm{d}k,$$

where $x = go \in X$, $g \in G$, $y \in X$. Then

$$\psi_1(x,y) = d(\delta)\int_K \chi\big(\tau^{-1}\big)\int_K w(\tau gky)\,\mathrm{d}k\,\mathrm{d}\tau,$$

and hence

$$\big\langle f_\delta(x), \psi_2(x,y)\big\rangle = \big\langle f(x), \psi_1(x,y)\big\rangle \tag{10.20}$$

because of (10.18) and (10.19). Equalities (10.19), (10.18), and (10.10) yield

$$\big\langle (f \times u)_\delta, w\big\rangle = \big\langle f \times u, \overline{(\overline{w})_\delta}\big\rangle = \big\langle u(y), \big\langle f(x), \psi_1(x,y)\big\rangle\big\rangle. \tag{10.21}$$

On the other hand,

$$\langle f_\delta \times u, w\rangle = \big\langle u(y), \big\langle f_\delta(x), \psi_2(x,y)\big\rangle\big\rangle.$$

Combining this with (10.20) and (10.21), we obtain (ii).

For the proof of (iii), we refer the reader to [122, Chap. 5, Theorem 3.1]. □

Let $\mathfrak{W}(\mathcal{O})$ be an arbitrary subset of $\mathcal{D}'(\mathcal{O})$. For each $\delta \in \widehat{K}_M$, we define the set $\mathfrak{W}_{\check{\delta}}(\mathcal{O})$ by letting

$$\mathfrak{W}_{\check{\delta}}(\mathcal{O}) = \{f \in \mathfrak{W}(\mathcal{O}) : f = f_\delta\}.$$

If δ is the trivial representation, then $\mathfrak{W}_{\check{\delta}}(\mathcal{O})$ is just the set of all K-invariant distributions in $\mathfrak{W}(\mathcal{O})$.

Let $\mathcal{D}'_W(\mathcal{B}_R, \mathrm{Hom}(V_\delta, V_\delta^M))$ denote the set of all matrices $u = (u_{\mu,\nu})$ with $l(\delta)$ rows and $d(\delta)$ columns whose entries $u_{\mu,\nu}$ are distributions in $\mathcal{D}'_W(\mathcal{B}_R)$. The classes $\mathcal{E}'_W(\mathcal{B}_R, \mathrm{Hom}(V_\delta, V_\delta^M))$, $\mathcal{D}_W(\mathcal{B}_R, \mathrm{Hom}(V_\delta, V_\delta^M))$, and $C^m_W(\mathcal{B}_R, \mathrm{Hom}(V_\delta, V_\delta^M))$ with $m \in \mathbb{Z}_+$ or $m = \infty$ are defined likewise.

Let $u \in \mathcal{D}'_W(\mathcal{B}_R, \mathrm{Hom}(V_\delta, V_\delta^M))$. We set

$$\mathrm{supp}\, u = \bigcup_{\mu,\nu} \mathrm{supp}\, u_{\mu,\nu}.$$

If $v \in \mathcal{E}'(\mathcal{B}_R)$, we write $u * v$ for the matrix $(u_{\mu,\nu} * v)$. Also, for each nonempty open subset $\mathcal{G}$ of $\mathcal{B}_R$, we denote by $u|_{\mathcal{G}}$ the matrix with entries $u_{\mu,\nu}|_{\mathcal{G}}$. Similarly, for an arbitrary $\psi \in \mathcal{D}(\mathcal{B}_R)$, denote by $\langle u, \psi\rangle$ the matrix with entries $\langle u_{\mu,\nu}, \psi\rangle$. Next, for $u \in \mathcal{E}'_W(\mathcal{B}_R, \mathrm{Hom}(V_\delta, V_\delta^M))$, the Euclidean Fourier transform $\widehat{u}$ is the matrix such that the $(\mu\nu)$th matrix entry of $\widehat{u}$ is $\widehat{u_{\mu,\nu}}$.

For the rest of this section, we suppose that rank $X = 1$. Here we have Kostant's result, $l(\delta) = 1$ (see [123, Chap. 2, Corollary 6.8] and [122, Chap. 5, Theorem 3.5]).

Every $x \in X\backslash\{o\}$ has the expression

$$x = ka_t o, \quad \text{where } k \in K,\ t > 0. \tag{10.22}$$

Hence, there exists an open subset $A^+_{\mathcal{O}}$ in A^+ such that

$$\mathcal{O}\backslash\{o\} = \big\{x = kao : k \in K, a \in A^+_{\mathcal{O}}\big\}.$$

According to results of Sect. 1.5, we associate with each function $f \in L^{1,\mathrm{loc}}(\mathcal{O})$ its Fourier decomposition

$$f(x) \sim \sum_{\delta \in \widehat{K}_M} \sum_{j=1}^{d(\delta)} f_{\delta,j}(a) Y_j^\delta(kM), \quad x = kao \in \mathcal{O}\backslash\{o\}, \tag{10.23}$$

where

$$f_{\delta,j}(a) = d(\delta)\int_K f(kao)\overline{Y_j^{\delta}(kM)}\,\mathrm{d}k.$$

Thanks to the Fubini theorem, $f_{\delta,j} \in L^{1,\mathrm{loc}}(A^+_{\mathcal{O}})$. For $\delta \in \widehat{K}_M, i, j \in \{1, \dots, d(\delta)\}$, we set

$$f^{\delta,j,i}(x) = f_{\delta,j}(a)Y_i^{\delta}(kM), \quad x \in \mathcal{O}\backslash\{o\}. \tag{10.24}$$

Using (10.24) and (1.80), we obtain

$$f^{\delta,j,i}(x) = d(\delta)\int_K f\big(\tau^{-1}x\big)\overline{t_{i,j}^{\delta}(\tau)}\,\mathrm{d}\tau, \quad x \in \mathcal{O}\backslash\{o\}. \tag{10.25}$$

This, together with (1.77), shows that

$$\int_E \big|f^{\delta,j,i}(x)\big|\,\mathrm{d}x \leqslant \sqrt{d(\delta)}\int_E |f(x)|\,\mathrm{d}x \tag{10.26}$$

for each nonempty K-invariant compact subset E of $\mathcal{O}$.

Let us now consider the case of $o \in \mathcal{O}$ and $f \in C^m(\mathcal{O})$ for some $m \in \mathbb{Z}_+$. Formula (10.25) ensures us that $f^{\delta,j,i}$ can be defined for $x = o$ so that it becomes a function in the class $C^m(\mathcal{O})$. It what follows when considering expressions on the left-hand side of (10.25) for $f \in C^m(\mathcal{O})$, $o \in \mathcal{O}$, we shall assume that they are extended in $\mathcal{O}$ to functions in $C^m(\mathcal{O})$.

Following Sect. 9.1, we now extend the map $f \to f^{\delta,j,i}$ and decomposition (10.23) to distributions by the following way. For $f \in \mathcal{D}'(\mathcal{O})$ and $\delta \in \widehat{K}_M$, $i, j \in \{1, \dots, d(\delta)\}$, we set

$$\begin{aligned}\big\langle f^{\delta,j,i}, w\big\rangle &= \Big\langle f, d(\delta)\int_K w\big(\tau^{-1}x\big)t_{j,i}^{\delta}(\tau)\,\mathrm{d}\tau\Big\rangle\\ &= \Big\langle f, \overline{\big(\overline{w}\big)_{\delta,j}(a)Y_i^{\delta}(kM)}\Big\rangle, \quad w \in \mathcal{D}(\mathcal{O}),\end{aligned} \tag{10.27}$$

and associate with f the series

$$f \sim \sum_{\delta\in\widehat{K}_M}\sum_{j=1}^{d(\delta)} f^{\delta,j}, \tag{10.28}$$

where $f^{\delta,j} = f^{\delta,j,j}$.

Relation (10.27) shows that the mapping $f \to f^{\delta,j,i}$ is a continuous operator from $\mathcal{D}'(\mathcal{O})$ into $\mathcal{D}'(\mathcal{O})$ and that

$$\operatorname{ord} f^{\delta,j,i} \leqslant \operatorname{ord} f. \tag{10.29}$$

Next, let $T \in \mathcal{E}'(X)$ be such that the set $\mathcal{O}_T$ is nonempty. Then (10.10) and (10.27) give

$$(f \times T)^{\delta,j,i} = f^{\delta,j,i} \times T \quad \text{in } \mathcal{O}_T \tag{10.30}$$

for each $f \in \mathcal{D}'(\mathcal{O})$. In particular,

$$\big(p(L)f\big)^{\delta,j,i} = p(L)f^{\delta,j,i}$$

for any polynomial p.

To go further, we point out that $Y_j^{\delta} \in \mathrm{RA}(K/M)$. This fact will be proved later (see, for instance, Corollary 10.1). Now exactly as in the proof of Proposition 9.1 it follows that for each $f \in \mathcal{D}'(\mathcal{O})$, the set supp $f^{\delta,j,i}$ is K-invariant.

We note also that series (10.28) converges to f in $\mathcal{D}'(\mathcal{O})$ (respectively $\mathcal{E}(\mathcal{O})$) for each $f \in \mathcal{D}'(\mathcal{O})$ (respectively $\mathcal{E}(\mathcal{O})$).

By analogy with the case of Euclidean space, for an arbitrary set $\mathfrak{W}(\mathcal{O}) \subset \mathcal{D}'(\mathcal{O})$, we put

$$\mathfrak{W}_{\delta,j}(\mathcal{O}) = \big\{f \in \mathfrak{W}(\mathcal{O}) : f = f^{\delta,j}\big\}.$$

Notice that if $\delta \in \widehat{K}_M$ is trivial, then $d(\delta) = 1$ and $\mathfrak{W}_{\delta,1}(\mathcal{O}) = \mathfrak{W}_{\natural}(\mathcal{O})$.

10.3 Eisenstein–Harish-Chandra Integrals and Their Rank One Generalizations

We recall that a function $\varphi \in C^{\infty}(X)$ is called a *spherical function* if φ is K-invariant, $\varphi(o) = 1$, and for each $D \in \mathbf{D}(X)$, there exists $\lambda_D \in \mathbb{C}$ such that $D\varphi = \lambda_D \varphi$.

The following Harish-Chandra's result gives a complete description of the set of all spherical functions on X.

Theorem 10.1. *As λ runs through $\mathfrak{a}_{\mathbb{C}}^*$, the functions*

$$\varphi_\lambda(gK) = \int_K \mathrm{e}^{(\mathrm{i}\lambda+\rho)(A(kg))}\,\mathrm{d}k, \quad g \in G, \tag{10.31}$$

exhaust the class of spherical functions on X. Moreover, two such functions φ_μ and φ_λ are identical if and only if $\mu = s\lambda$ for some $s \in W$.

The proof of this theorem can be found in Helgason [122], Chap. 4, Theorem 4.3.

The function φ_λ will play an important role later. It satisfies the identity

$$\varphi_\lambda\big(g^{-1}hK\big) = \int_K \mathrm{e}^{(-\mathrm{i}\lambda+\rho)(A(kg))}\mathrm{e}^{(\mathrm{i}\lambda+\rho)(A(kg))}\,\mathrm{d}k, \quad g, h \in G, \tag{10.32}$$

see [122, Chap. 4, Lemma 4.4]. In particular, putting $h = e$, one has

$$\varphi_\lambda\big(g^{-1}K\big) = \varphi_{-\lambda}(gK), \quad g \in G. \tag{10.33}$$

Furthermore,

$$L\varphi_\lambda = -\big(\langle\lambda,\lambda\rangle + |\rho|^2\big)\varphi_\lambda$$

because of (10.31) and (10.5).

For $\mathrm{Re}(\mathrm{i}\lambda) \in \mathfrak{a}_+^*$, the behavior of φ_λ at infinity is given by

$$\lim_{t\to+\infty} \varphi_\lambda(\exp tH_0)\mathrm{e}^{(-\mathrm{i}\lambda+\rho)(tH_0)} = \mathbf{c}(\lambda), \tag{10.34}$$

where $\mathbf{c}(\lambda)$ is Harish-Chandra's $\mathbf{c}$-function. This function extends to the meromorphic function

$$\mathbf{c}(\lambda) = c_0 \prod_{\alpha\in\sum_0^+} \frac{2^{-\langle \mathrm{i}\lambda,\alpha_0\rangle}\Gamma(\langle \mathrm{i}\lambda,\alpha_0\rangle)}{\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + 1 + \langle \mathrm{i}\lambda,\alpha_0\rangle))\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + m_{2\alpha} + \langle \mathrm{i}\lambda,\alpha_0\rangle))}$$

on $\mathfrak{a}_\mathbb{C}^*$, where α_0 is the normalized root $\alpha/\langle\alpha,\alpha\rangle$, and the constant c_0 is given by the condition $\mathbf{c}(-\mathrm{i}\rho) = 1$. Notice that

$$|\mathbf{c}(\lambda)|^2 = \mathbf{c}(\lambda)\mathbf{c}(-\lambda) = |\mathbf{c}(s\lambda)|^2, \quad \lambda \in \mathfrak{a}^*, s \in W. \tag{10.35}$$

In addition, if $\mathrm{Re}(\mathrm{i}\lambda) \in \mathrm{Cl}(\mathfrak{a}_+^*)$, then

$$|\mathbf{c}(\lambda)|^{-1} \leqslant \gamma_1 + \gamma_2|\lambda|^p, \quad p = \frac{1}{2}\dim M, \tag{10.36}$$

where $\gamma_1, \gamma_2 > 0$ are independent of λ. If $\operatorname{rank} X = 1$, we have

$$|\mathbf{c}(\lambda)|^{-1} \leqslant \gamma_3(1 + |\lambda|)^{\alpha_X+1/2}, \quad \mathrm{Re}(\mathrm{i}\lambda) \in \mathrm{Cl}(\mathfrak{a}_+^*), \tag{10.37}$$

where $\gamma_3 > 0$ is independent of λ. For a proof of the above results, see Helgason [122], Chap. 4, Proposition 7.2.

Next, if the group G is complex, then

$$\varphi_\lambda(\operatorname{Exp} P) = J^{-1/2}(P) \int_K \mathrm{e}^{\mathrm{i}\langle A_\lambda, \mathrm{Ad}(k)P\rangle}\,\mathrm{d}k, \quad P \in \mathfrak{p}, \tag{10.38}$$

where J is defined by

$$J(P) = \det\left(\frac{\sinh \operatorname{ad} P}{\operatorname{ad} P}\right) \tag{10.39}$$

(see [122, Chap. 4, Propositions 4.8 and 4.10]). We note also that

$$\int_X f(x)\,\mathrm{d}x = \int_\mathfrak{p} f(\operatorname{Exp} P)J(P)\,\mathrm{d}P, \quad f \in L^1(X) \tag{10.40}$$

(see [122, Chap. 2, §3, (69)]).

Our concern from now on will be with some generalizations of spherical functions.

Let $\delta \in \hat{K}_M$ and $\lambda \in \mathfrak{a}_\mathbb{C}^*$. The mapping $\Phi_{\lambda,\delta} : X \to \operatorname{Hom}(V_\delta, V_\delta)$ given by

$$\Phi_{\lambda,\delta}(x) = \int_K \mathrm{e}^{(\mathrm{i}\lambda+\rho)(A(x,kM))}\delta(k)\,\mathrm{d}k \tag{10.41}$$

is called the *generalized spherical function of class* δ. This function is a special case of Eisenstein integrals considered by Harish-Chandra [107, 109].

If δ is the identity representation, (10.41) shows that $\Phi_{\lambda,\delta}$ and φ_λ coincide. In the general case we see from (10.41) that

$$\Phi_{\lambda,\delta}(kx) = \delta(k)\Phi_{\lambda,\delta}(x), \quad k \in K,\ x \in X, \tag{10.42}$$

and

$$\Phi_{\lambda,\delta}(x)\delta(m) = \Phi_{\lambda,\delta}(x), \quad m \in M,\ x \in X.$$

Next, relations (10.41) and (10.5) yield

$$L\Phi_{\lambda,\delta} = -\big(\langle\lambda,\lambda\rangle + |\rho|^2\big)\Phi_{\lambda,\delta}$$

and

$$\Phi_{\overline{\lambda},\delta}(x)^* = \int_K \mathrm{e}^{(-\mathrm{i}\lambda+\rho)(A(x,kM))}\delta\big(k^{-1}\big)\,\mathrm{d}k, \tag{10.43}$$

where $*$ denotes the adjoint on $\mathrm{Hom}(V_\delta, V_\delta)$.

Following [123, p. 236], denote by $S(\mathfrak{p}^*)$ the algebra of polynomial functions on $\mathfrak{p}$. Then we have

$$S(\mathfrak{p}^*) = I(\mathfrak{p}^*)H(\mathfrak{p}^*),$$

where $I(\mathfrak{p}^*)$ is the algebra of K-invariant polynomial functions on $\mathfrak{p}$, and $H(\mathfrak{p}^*)$ is the space of the corresponding harmonic polynomials (see [123, Chap. 3, §2, (8)]). Let H_δ be the space of $h \in H(\mathfrak{p}^*)$ of type δ, and let $S(\mathfrak{g})$ and $S(\mathfrak{p})$ denote the (complex) symmetric algebras over $\mathfrak{g}$ and $\mathfrak{p}$, respectively. Identifying $S(\mathfrak{p}^*)$ and $S(\mathfrak{p})$ via the Killing form of $\mathfrak{g}$, $I(\mathfrak{p}^*)$, $H(\mathfrak{p}^*)$ and H_δ become subspaces of $S(\mathfrak{g})$. Let H^* and H^*_δ denote the images of $H(\mathfrak{p}^*)$ and H_δ, respectively, under the symmetrization mapping of $S(\mathfrak{g})$ onto $\mathbf{D}(G)$ (see [123, p. 237]). It can be shown that the vector space

$$E_\delta = \mathrm{Hom}_K\big(V_\delta, H^*_\delta\big)$$

of linear maps $V_\delta \to H^*_\delta$ commuting with the action of K (i.e., δ and Ad) has dimension $l(\delta)$ [123, p. 238].

In view of the Iwasawa decomposition $\mathfrak{g} = \mathfrak{a}+\mathfrak{n}+\mathfrak{k}$, one has the following direct decomposition of the universal enveloping algebra $\mathbf{D}(G)$:

$$\mathbf{D}(G) = \mathbf{D}(A) \oplus \big(\mathfrak{n}\mathbf{D}(G) + \mathbf{D}(G)\mathfrak{k}\big)$$

[123, Chap. 3, §2, (18)]. Let $D \to q^D$ denote the corresponding projection of $\mathbf{D}(G)$ onto $\mathbf{D}(A)$. Since the function

$$\zeta_\lambda(g) = \mathrm{e}^{(-\mathrm{i}\lambda+\rho)(A(g))}, \quad g \in G,$$

satisfies

$$\zeta_\lambda(ngk) = \zeta_\lambda(g) = \zeta_\lambda(mg), \quad n \in N,\ k \in K,\ m \in M, \tag{10.44}$$

we obtain

$$(D\zeta_\lambda)(e) = \big(q^D\zeta_\lambda\big)(e) = q^D(\rho - \mathrm{i}\lambda). \tag{10.45}$$

For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, now define the linear map

$$Q^\delta(\lambda) : E_\delta \to V_{\check{\delta}}^M$$

by the formula

$$\big(Q^\delta(\lambda)(\varepsilon)\big)(v) = q^{\varepsilon(v)}(\rho - \mathrm{i}\lambda), \quad \varepsilon \in E_\delta,\ v \in V_\delta. \tag{10.46}$$

The right-hand side in (10.46) is indeed invariant under $v \to \delta(m)v$ because of (10.44) and (10.45).

Let $\varepsilon_1, \ldots, \varepsilon_{l(\delta)}$ be any basis of E_δ and, as before, $v_1, \ldots, v_{l(\delta)}$ an orthonormal basis of V_δ^M. For convenience, we shall often represent $Q^\delta(\lambda)$ by the $l(\delta) \times l(\delta)$ matrix

$$Q^\delta(\lambda)_{i,j} = q^{\varepsilon_j(v_i)}(\rho - \mathrm{i}\lambda), \tag{10.47}$$

whose entries are polynomial functions on $\mathfrak{a}_{\mathbb{C}}^*$. Denote by ν_δ the largest degree of these polynomials. It is known that $\det(Q^\delta(\lambda)) \not\equiv 0$ for each $\delta \in \hat{K}_M$ (see [123, p. 240]). If we change the basis (ε_j) to another one (η_ξ), where

$$\eta_\xi = \sum_{j=1}^{l(\delta)} c_{j,\xi}\varepsilon_j,$$

then the matrix $Q^\delta(\lambda)$ changes to $Q^\delta(\lambda)C$ with $C = (c_{j,\xi})$.

There exists a simple relationship between $Q^{\check{\delta}}(\lambda)$ and $Q^\delta(\lambda)$, which is best expressed in the matrix form (10.47). Corresponding to the direct decomposition

$$\mathbf{D}(G) = \mathbf{D}(G)_{\mathbb{R}} + \mathrm{i}\mathbf{D}(G)_{\mathbb{R}},$$

where $\mathbf{D}(G)_{\mathbb{R}}$ is the set of operators in $\mathbf{D}(G)$ with real coefficients, consider the conjugation

$$D \to \overline{D}, \quad D \in \mathbf{D}(G).$$

It can be shown that $(H^*)^- = H^*$ and $(H_\delta^*)^- = H_{\check{\delta}}^*$ [123, p. 255]. For $\varepsilon \in E_\delta$, define the map $\varepsilon' : V_{\check{\delta}} \to H^*$ by

$$\varepsilon'(v') = \varepsilon(v)^-, \quad v \in V_\delta.$$

Then $\varepsilon \to \varepsilon'$ is a complex-linear bijection of E_δ onto $E_{\check{\delta}}$, and relative to the bases (v_i), (v_i'), (ε_j), (ε_j') it follows that

$$Q^{\check{\delta}}(\lambda)_{i,j} = \big(Q^\delta(-\overline{\lambda})_{i,j}\big)^-,$$

see [123, Chap. 3, Lemma 3.6].

We now record one theorem which will be used later.

Theorem 10.2. *Let $a \in A$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, and $s \in W$. Then*

$$\Phi_{\lambda,\delta}(ao)Q^{\check{\delta}}(\lambda) = \Phi_{s\lambda}(ao)Q^{\check{\delta}}(s\lambda)$$

and

$$Q^{\check{\delta}}(\lambda)^{-1}\Phi_{\overline{\lambda},\delta}(ao)^* = Q^{\check{\delta}}(s\lambda)^{-1}\Phi_{\overline{s\lambda},\delta}(ao)^*. \tag{10.48}$$

Moreover, both sides in (10.48) are holomorphic on all of $\mathfrak{a}^*_{\mathbb{C}}$; in other words, $\Phi_{\overline{\lambda},\delta}(ao)^*$ is divisible by $Q^{\check{\delta}}(\lambda)$.

The proof of Theorem 10.2 is contained in [123, p. 289].

For the rest of the section, we suppose rank $X = 1$. Let us introduce the function

$$\varphi_{\lambda,\delta}(x) = \int_K \mathrm{e}^{(-\mathrm{i}\lambda+\rho)(A(x,kM))} Y_1^{\delta}(kM)\,\mathrm{d}k, \quad x \in X,\ \lambda \in \mathfrak{a}^*_{\mathbb{C}},\ \delta \in \hat{K}_M.$$

The basic properties of $\varphi_{\lambda,\delta}$ are worth studying. To begin with, we quote a result that determines $\varphi_{\lambda,\delta}$ in terms of the Jacobi function of the first kind (see Sect. 7.2).

As usual, let γ be the single root in Σ_0^+ and $\gamma_0 = \gamma/\langle\gamma,\gamma\rangle$. We determine $H_0 \in \mathfrak{a}^*_{\mathbb{C}}$ by $\gamma(H_0) = 1$ and put $a_t = \exp tH_0 \in A^+$ for $t \geqslant 0$.

Theorem 10.3. *For each $\delta \in \widehat{K}_M$, there exist unique numbers $s(\delta), s_1(\delta) \in \mathbb{Z}_+$ such that*

$$\varphi_{\lambda,\delta}(a_t o) = R_\delta(\lambda)(\sinh t)^{s(\delta)}(\cosh t)^{s_1(\delta)}\varphi^{(\xi,\eta)}_{\lambda(H_0)}(t), \quad t \geqslant 0,$$

where

$$\xi = s(\delta) + \alpha_X, \qquad \eta = s_1(\delta) + \beta_X, \tag{10.49}$$

$$R_\delta(\lambda) = \frac{\Gamma(\alpha_X + 1)}{\Gamma(\xi + 1)} p_\delta(\lambda) q_\delta(\lambda), \tag{10.50}$$

and p_δ and q_δ are given by

$$p_\delta(\lambda) = \frac{\Gamma(\frac{1}{2}(s(\delta) + s_1(\delta) + \langle \mathrm{i}\lambda + \rho, \gamma_0\rangle))}{\Gamma(\frac{1}{2}\langle \mathrm{i}\lambda + \rho, \gamma_0\rangle)}, \tag{10.51}$$

$$q_\delta(\lambda) = \frac{\Gamma(\frac{1}{2}(s(\delta) - s_1(\delta) + \langle \mathrm{i}\lambda + \rho, \gamma_0\rangle) - \beta_X)}{\Gamma(\frac{1}{2}\langle \mathrm{i}\lambda + \rho, \gamma_0\rangle - \beta_X)}. \tag{10.52}$$

In addition, if δ is identity representation, then $s(\delta) = s_1(\delta) = 0$ and

$$\varphi_{\lambda,\delta}(x) = \varphi_\lambda(x), \quad x \in X.$$

The proof of Theorem 10.3 can be found in [123, p. 344] and [120].

Denote by $\mathcal{P}$ the set of all pairs $(s(\delta), s_1(\delta))$ when δ runs through $\widehat{K}_M$. This set can be described in the following way [120]:

$$\mathcal{P} = \begin{cases} \{(p,q) : p \in \mathbb{Z}_+, q = 0\} & \text{if } m_{2\gamma} = 0, \\ \{(p,q) : p, q \in \mathbb{Z}_+ \text{ and } p \pm q \in 2\mathbb{Z}_+\} & \text{if } m_{2\gamma} > 0. \end{cases}$$

This, together with (10.50)–(10.52), shows that if $m_{2\gamma} > 0$, then p_δ and q_δ are polynomials. Furthermore, if $m_{2\gamma} = 0$, then R_δ is the polynomial

$$R_\delta(\lambda) = \frac{\Gamma(\alpha_X + 1)\Gamma(s(\delta) + \langle i\lambda + \rho, \gamma_0\rangle)}{2^{s(\delta)}\Gamma(\xi + 1)\Gamma(\langle i\lambda + \rho, \gamma_0\rangle)}, \tag{10.53}$$

see [123, p. 245].

It follows by Theorem 10.3 that

$$\overline{\varphi_{-\overline{\lambda},\delta}(ao)} = \varphi_{\lambda,\delta}(ao), \quad a \in A. \tag{10.54}$$

For future use, we note also that

$$\varphi_{\lambda,\delta}(ao)R_\delta(-\lambda) = \varphi_{-\lambda,\delta}(ao)R_\delta(\lambda), \quad a \in A. \tag{10.55}$$

Proposition 10.3. *If $a \in A$, $\tau \in K$, and $j \in \{1, \dots, d(\delta)\}$, then*

$$\varphi_{\lambda,\delta}(ao)Y_j^\delta(\tau M) = \int_K e^{(i\lambda+\rho)(A(\tau ao, kM))} Y_j^\delta(kM)\, dk$$
$$= \int_K e^{(i\lambda+\rho)(A(kao, \tau M))} Y_j^\delta(kM)\, dk.$$

Proof. The first equality was obtained in [123, Chap. 3, Lemma 11.1]. The second relation reduces to this one by means of (10.54) and Proposition 10.2(iii). □

Corollary 10.1. *Let $R > 0$ and $f \in \mathrm{RA}_\natural(-R\kappa, R\kappa)$. Then the function*

$$x \to t^{s(\delta)} f(t) Y_j^\delta(kM), \quad x = ka_t o \in B_R \backslash \{o\},$$

is in the class $\mathrm{RA}(B_R\backslash\{o\})$. Moreover, this function admits real analytic extension to o.

Proof. It follows by Proposition 10.3 that for each $\lambda \in \mathfrak{a}_\mathbb{C}^*$, the function

$$x \to \varphi_{\lambda,\delta}(a_t o) Y_j^\delta(kM), \quad x \in X\backslash\{o\},$$

is in $\mathrm{RA}(X\backslash\{o\})$ and admits real analytic continuation to o (see [122, Chap. 4, the proof of Lemma 2.1]). In addition, $\varphi_\lambda \in \mathrm{RA}(X)$ for all $\lambda \in \mathfrak{a}_\mathbb{C}^*$. Using now Theorem 10.3, we arrive at the desired statement. □

Next, let $\mu \in \mathbb{C}$, $\nu \in \mathbb{Z}_+$, and

$$\varkappa = \varkappa(\mu, \nu) = \begin{cases} \nu & \text{if } \mu \neq 0, \\ 2\nu & \text{if } \mu = 0. \end{cases} \tag{10.56}$$

Assume that $\delta \in \widehat{K}_M$, $j \in \{1, \dots, d(\delta)\}$, and let ξ, η denote the numbers determined in Theorem 10.3. Having (10.22) and (10.49) in mind, we define

$$\Phi_{\mu,\nu,\delta,j}(x) = \left(\frac{\partial}{\partial z}\right)^{\varkappa} \Phi_{z,0,\delta,j}(x)\bigg|_{z=\mu}, \quad x \in X\backslash\{o\}, \tag{10.57}$$

where

$$\Phi_{z,0,\delta,j}(x) = (\sinh t)^{s(\delta)}(\cosh t)^{s_1(\delta)}\varphi_{z/\kappa}^{(\xi,\eta)}(t)Y_j^{\delta}(kM). \tag{10.58}$$

Furthermore, we put

$$\Phi_{\mu,\nu,\delta,j}(o) = \begin{cases} 1 & \text{if } \nu = 0 \text{ and } \delta \text{ is trivial,} \\ 0 & \text{otherwise.} \end{cases} \tag{10.59}$$

Relations (10.57)–(10.59) and Corollary 10.1 ensure us that $\Phi_{\mu,\nu,\delta,j} \in \mathrm{RA}(X)$. If δ is trivial and $j = 1$, we shall write $\Phi_{\mu,\nu,\mathrm{triv}}$ instead of $\Phi_{\mu,\nu,\delta,j}$. We point out that

$$\Phi_{\mu,0,\mathrm{triv}} = \varphi_\lambda,$$

where $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $\lambda(H_0) = \mu/\kappa$.

Suppose now that $\mathrm{i}\mu/\kappa \notin \mathbb{N}$ and define

$$\Psi_{\mu,\nu,\delta,j}(x) = \left(\frac{\partial}{\partial z}\right)^{\varkappa} \Psi_{z,0,\delta,j}(x)\bigg|_{z=\mu}, \quad x \in X\backslash\{o\}, \tag{10.60}$$

where

$$\Psi_{z,0,\delta,j}(x) = (\sinh t)^{s(\delta)}(\cosh t)^{s_1(\delta)}\Phi_{z/\kappa}^{(\xi,\eta)}(t)Y_j^{\delta}(kM), \tag{10.61}$$

and $\Phi_{z/\kappa}^{(\xi,\eta)}$ is determined by (7.17).

The following result is the analog of Theorem 9.1.

Theorem 10.4.

(i) *If $q \in \mathbb{Z}_+$ then*

$$L^q\Phi_{0,\nu,\delta,j} = (-1)^q \sum_{l=\max\{0,\nu-q\}}^{\nu} \binom{2\nu}{2l}\binom{q}{\nu-l}(2\nu-2l)!\rho_X^{2(q-\nu+l)}\Phi_{0,l,\delta,j}$$

and

$$\left(L+\rho_X^2\right)^q\Phi_{0,\nu,\delta,j} = \begin{cases} (-1)^q(-2\nu)_{2q}\Phi_{0,\nu-q,\delta,j} & \text{if } q \leqslant \nu, \\ 0 & \text{if } q > \nu. \end{cases} \tag{10.62}$$

Moreover, for the functions $\Psi_{0,\nu,\delta,j}$, the same relations are true in $X\backslash\{o\}$.

(ii) *If* $q \in \mathbb{Z}_+$ *and* $\mu \in \mathbb{C}\backslash\{0\}$, *then*

$$L^q \Phi_{\mu,\nu,\delta,j} = (-1)^q \sum_{m=\max\{0,\nu-2q\}}^{\nu} \sum_{\frac{\nu-m}{2}\leqslant l\leqslant q} \binom{\nu}{m}\binom{q}{l} \frac{(2l)!\rho_X^{2(q-l)}\mu^{2l-\nu+m}}{(2l-\nu+m)!} \Phi_{\mu,m,\delta,j}, \tag{10.63}$$

$$\left(L+\rho_X^2\right)^q \Phi_{\mu,\nu,\delta,j} = (-1)^q \sum_{l=0}^{\min\{\nu,2q\}} \binom{l}{\nu} \frac{(2q)!}{(2q-l)!} \mu^{2q-l} \Phi_{\mu,\nu-l,\delta,j}, \tag{10.64}$$

and

$$\left(L+\rho_X^2+\mu^2\right)^q \Phi_{\mu,\nu,\delta,j} = \sum_{l=\max\{0,\nu-2q\}}^{\nu} \frac{\nu!2^{2q-\nu+l}(-q)_{2q-\nu+l}}{l!(2q-\nu+l)!} (-1)^{q+l-\nu} \mu^{2q-\nu+l} \Phi_{\mu,l,\delta,j}.$$

In addition, for the functions $\Psi_{\mu,\nu,\delta,j}$, *the same formulae are valid in* $X\backslash\{o\}$, *provided that* $\mathrm{i}\mu/\kappa \notin \mathbb{Z}_+$.

(iii) *If* $\mathrm{i}\mu/\kappa \notin \mathbb{N}$ *and* δ *is identity representation, then* $\Psi_{\mu,0,\delta,1} \in L^{1,\mathrm{loc}}(X)$ *and*

$$\left(L+\rho_X^2+\mu^2\right)\Psi_{\mu,0,\delta,1} = \frac{\mathrm{i}\mu 2^{\alpha_X+\beta_X-1-\mathrm{i}\mu/\kappa}\pi^{\alpha_X+1}\kappa^{-1-2\alpha_X}\Gamma(-\mathrm{i}\mu/\kappa)\delta_0}{\Gamma(\frac{1}{2}(\rho_X-\mathrm{i}\mu/\kappa))\Gamma(\frac{1}{2}(\alpha_X-\beta_X+1-\mathrm{i}\mu/\kappa))},$$

where δ_0 *is the Dirac measure supported at origin.*

Proof. It follows by (10.58) and (7.12) that

$$\left(L+\rho_X^2+\mu^2\right)\Phi_{\mu,0,\delta,j} = 0 \quad \text{in } X$$

for each $\mu \in \mathbb{C}$ (see [123, p. 346]). Similarly, the corresponding relation in which $\Phi_{\mu,0,\delta,j}$ is replaced by $\Psi_{\mu,0,\delta,j}$ is valid in $X\backslash\{o\}$, provided that $\mathrm{i}\mu/\kappa \notin \mathbb{N}$ (see (10.61) and (7.12)). Hence,

$$L^q \Phi_{\mu,\nu,\delta,j} = \left(\frac{\partial}{\partial z}\right)^{\varkappa} \left(\left(-\rho_X^2-z^2\right)^q \Phi_{z,0,\delta,j}\right)\Big|_{z=\mu}$$

and

$$\left(L+\rho_X^2\right)^q \Phi_{\mu,\nu,\delta,j} = \left(\frac{\partial}{\partial z}\right)^{\varkappa} \left(\left(-z^2\right)^q \Phi_{z,0,\delta,j}\right)\Big|_{z=\mu},$$

where $\varkappa$ is given by (10.56). Similar relations for the functions $\Psi_{\mu,\nu,\delta,j}$ hold in $X\backslash\{o\}$ if $\mathrm{i}\mu/\kappa \notin \mathbb{N}$. Now assertions (i) and (ii) follow just like the analogous statements in Theorem 9.1.

To prove (iii), first observe that $\Psi_{\mu,0,\delta,1} \in L^{1,\mathrm{loc}}(X)$ by virtue of (10.61), (10.14), (10.15), and Proposition 7.5(iv). Let $f \in \mathcal{D}(X)$ and

$$f^{\natural}(x) = \int_K f(kx)\,\mathrm{d}k, \quad x \in X.$$

Then $f^{\natural} \in \mathcal{D}_{\natural}(X)$ and $f^{\natural}(o) = f(o)$. For each $r \geqslant 0$, now define

$$u(r) = f^{\natural}(x) \quad \text{where } d(o,x) = r.$$

Using (10.15) and (10.16), we see that

$$\begin{aligned}\langle L\Psi_{\mu,0,\delta,1}, f\rangle &= \left\langle \Psi_{\mu,0,\delta,1}, Lf^{\natural}\right\rangle \\ &= \lim_{\varepsilon\to 0}\int_{X\setminus B_{\varepsilon}} \Psi_{\mu,0,\delta,1}(x)\big(Lf^{\natural}\big)(x)\,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty} \Phi_{\mu/\kappa}^{(\alpha_X,\beta_X)}(\kappa r)\frac{\mathrm{d}}{\mathrm{d}r}\left(A_X(r)\frac{\mathrm{d}}{\mathrm{d}r}u(r)\right)\mathrm{d}r.\end{aligned}$$

Integrating by parts twice, we get

$$\begin{aligned}\langle L\Psi_{\mu,0,\delta,1}, f\rangle &= -\big(\rho_X^2+\mu^2\big)\langle \Psi_{\mu,0,\delta,1}, f\rangle \\ &\quad + \lim_{\varepsilon\to 0} A_X(\varepsilon)\left(u(\varepsilon)\frac{\mathrm{d}}{\mathrm{d}r}\Phi_{\mu/\kappa}^{(\alpha_X,\beta_X)}(\kappa r)\bigg|_{r=\varepsilon} - \Phi_{\mu/\kappa}^{(\alpha_X,\beta_X)}(\kappa\varepsilon)u'(\varepsilon)\right),\end{aligned}$$

which, together with (10.14) and Proposition 7.5(iv), yields (iii). □

Corollary 10.2. *If $\mu \in \mathbb{C}$, then*

$$\big(L+\rho_X^2+\mu^2\big)^{\nu+1}\Phi_{\mu,\nu,\delta,j} = 0 \quad \text{in } X.$$

In addition, if $\mathrm{i}\mu/\kappa \notin \mathbb{N}$, then

$$\big(L+\rho_X^2+\mu^2\big)^{\nu+1}\Psi_{\mu,\nu,\delta,j} = 0 \quad \text{in } X\setminus\{o\}.$$

Proof. This is immediate from Theorem 10.4(i), (ii). □

Proposition 10.4. *Let $\{\mu_1,\dots,\mu_m\}$ be a set of complex numbers such that $\mathrm{i}\mu_l/\kappa \notin \mathbb{N}$ for each $l \in \{1,\dots,m\}$ and the numbers $\mu_1^2,\dots,\mu_m^2$ are distinct. Suppose that $\mathcal{O}$ is a nonempty open subset of X and let $p \in \mathbb{Z}_+$. Then the following statements are valid.*

(i) *If $o \notin \mathcal{O}$ and*

$$\sum_{l=1}^{m}\sum_{\nu=0}^{p} \alpha_{l,\nu}\Phi_{\mu_l,\nu,\delta,j} + \beta_{l,\nu}\Psi_{\mu_l,\nu,\delta,j} = 0 \quad \text{in } \mathcal{O}$$

for some $\alpha_{l,\nu}, \beta_{l,\nu} \in \mathbb{C}$, then $\alpha_{l,\nu} = \beta_{l,\nu} = 0$ for all l, ν.

(ii) *If* $o \in \mathcal{O}$ *and there is* $f \in C^{\infty}(\mathcal{O})$ *such that*

$$f = \sum_{l=1}^{m} \sum_{\nu=0}^{p} \gamma_{l,\nu} \Psi_{\mu_l,\nu,\delta,j} \quad \text{in } \mathcal{O}\backslash\{o\}$$

for some $\gamma_{l,\nu} \in \mathbb{C}$, *then* $\gamma_{l,\nu} = 0$ *for all* l, ν.

Proof. To show (i), first assume that $m = 1$ and $\mu_1 = 0$. If $p = 0$, the required statement is obvious from (7.15) and Proposition 7.5(iv). The case $p > 0$ reduces to this one in view of (10.62) and the corresponding relation for the functions $\Psi_{0,\nu,\delta,j}$. The rest of the proof of (i) follows analogously to that of Proposition 9.3(i).

Assertion (ii) can be proved in the same way as Proposition 9.3(ii) with attention to Proposition 7.5(iv) and Theorem 10.4(i), (ii). □

Important information concerning the asymptotic behavior of $\Phi_{\mu,\nu,\delta,j}$ is contained in the following propositions.

Proposition 10.5. *Let* $0 < \varepsilon < R$, $k \in K$, $a_t \in A^+$, $x = ka_t o \in B_{\varepsilon,R}$, *and let* $\mu \in \mathbb{C}$, $\operatorname{Re}\mu \geqslant 0$, $\theta > 2$. *Suppose that* $\nu \in \mathbb{Z}_+$ *and* $t|\mu| > \theta\nu$. *Then*

$$\Phi_{\mu,\nu,\delta,j}(x) = \gamma Y_j^{\delta}(kM)\bigg(\frac{\cos(t\mu/\kappa - \pi(2\xi - 2\nu + 1)/4)}{(\mu/\kappa)^{\xi+1/2}} + O\big(\mathrm{e}^{|\operatorname{Im}\mu|t/\kappa}(1+\nu)|\mu|^{-\xi-3/2}\big)\bigg),$$

where ξ *is defined by* (10.49),

$$\gamma = \frac{2^{\xi+1/2}\Gamma(\xi+1)(t/\kappa)^{\nu}}{\sqrt{\pi}(\sinh t)^{\alpha_X+1/2}(\cosh t)^{\beta_X+1/2}},$$

and the constant in O *depends only on* ε, R, θ, δ.

The proof follows from (10.57), (10.58), (7.20), and Proposition 6.13.

To continue, for $t > 0$, $\mu \in \mathbb{C}\backslash\{0\}$, $\operatorname{Im}\mu \geqslant 0$, $\nu \in \mathbb{Z}_+$, we set

$$\varphi(\mu,\nu,t) = 2^{-s(\delta)-s_1(\delta)} c_{\xi,\eta}(\mu)\left(\frac{\mathrm{i}t}{\kappa}\right)^{\nu} \exp\left(\frac{(\mathrm{i}\mu - \rho_X)t}{\kappa}\right)$$

and

$$\psi(\nu,t) = \frac{2^{1-s(\delta)-s_1(\delta)}}{(2\nu+1)\kappa^{2\nu}}(\mathrm{i}t)^{\nu+1} \exp\left(\frac{-\rho_X t}{\kappa}\right)\big(zc_{\xi,\eta}(z)|_{z=0}\big),$$

where ξ, η, $c_{\xi,\eta}$ are determined by (7.24) and (10.49).

Proposition 10.6. *Let* $k \in K$, $a_t \in A^+$, *and* $x = ka_t o \in X$. *Assume that* $\mu \in \mathbb{C}\backslash\{0\}$, $\operatorname{Im}\mu \geqslant 0$, $\nu \in \mathbb{Z}_+$. *Then*

$$\Phi_{\mu,\nu,\delta,j}(x) = Y_j^{\delta}(kM)\big(\varphi(\mu,\nu,t)\big(1 + O\big(t^{-1}\big)\big) + (-1)^{\nu}\varphi(-\mu,\nu,t)\big(1 + O\big(t^{-1}\big)\big)\big) \quad \text{as } t \to +\infty.$$

In addition,

$$\Phi_{0,\nu,\delta,j}(x) = Y_j^{\delta}(kM)\psi(\nu,t)\big(1 + O\big(t^{-1}\big)\big) \quad \textit{as } t \to +\infty.$$

The proof is obvious from (10.57), (10.58), and Corollary 7.1.

We end the consideration by presenting some results similar to Propositions 9.5–9.7.

Proposition 10.7. *Let* $R > 0$, $\varepsilon \in (0,1)$, $\mu \in \mathbb{C}$, $\nu, q \in \mathbb{Z}_+$. *Then the following assertions hold.*

(i) *If* $\nu < R|\mu|\varepsilon$, *then for each differential operator* D *of order* q *on* X,

$$\|D\Phi_{\mu,\nu,\delta,j}\|_{C(\overset{\bullet}{B}_R)} \leqslant \gamma_1\sqrt{1+\nu}(1+|\mu|)^{q-s(\delta)}R^{\nu}\mathrm{e}^{R|\mathrm{Im}\,\mu|}, \tag{10.65}$$

where $\gamma_1 > 0$ *is independent of* μ, ν.

(ii) *If* $\nu < R|\mu|\varepsilon$, *then*

$$\big\|L^q\Phi_{\mu,\nu,\delta,j}\big\|_{C(\overset{\bullet}{B}_R)} \leqslant \gamma_2\sqrt{1+\nu}(1+\rho_X)^{2q}(1+|\mu|)^{2q-s(\delta)}R^{\nu} \times \exp\bigg(R|\mathrm{Im}\,\mu| + \frac{2q\nu}{|\mu|}\bigg),$$

where $\gamma_2 > 0$ *is independent of* μ, ν, q.

(iii) *If* $x \in \overset{\bullet}{B}_R$, $\mu \neq 0$, *and* $q \in \mathbb{N}$, *then*

$$\big(L + \rho_X^2\big)^q\Phi_{\mu,\nu,\delta,j}(x) = (-1)^q(2q)^{\nu}\mu^{2q-\nu}\big(\Phi_{\mu,0,\delta,j}(x) + O\big(q^{-1}\big)\big),$$

where the constant in O *is independent of* q, x.

Proof. To prove (i) there is no loss of generality in assuming $R_\delta(\lambda) \neq 0$, where $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is such that $\lambda(H_0) = \mu$. By virtue of Theorem 10.3 and Proposition 10.3,

$$\Phi_{\mu,0,\delta,j}(x) = \frac{1}{R_\delta(\lambda)}\int_K \mathrm{e}^{(\mathrm{i}\lambda+\rho)(A(x,\tau M))}Y_j^{\delta}(\tau M)\,\mathrm{d}\tau, \quad x \in X. \tag{10.66}$$

Then there exist functions $f_l \in C^\infty(X \times K)$, $l \in \{0,\dots,q\}$, such that

$$D\Phi_{\mu,0,\delta,j}(x) = \int_K Y_j^{\delta}(\tau M)\sum_{l=0}^{q}\frac{((\mathrm{i}\lambda+\rho)(A(x,\tau M)))^l}{R_\delta(\lambda)} \times \mathrm{e}^{(\mathrm{i}\lambda+\rho)(A(x,\tau M))}f_l(x,\tau)\,\mathrm{d}\tau.$$

Applying (10.1) and taking (10.50)–(10.52) into account, one has

$$|D\Phi_{\mu,0,\delta,j}(x)| \leqslant \gamma_3(1+|\mu|)^{q-s(\delta)}\mathrm{e}^{R|\mathrm{Im}\,\mu|}, \quad x \in \overset{\bullet}{B}_R,$$

where $\gamma_3 > 0$ is independent of x, μ. Now Proposition 6.11 leads us to (i).

Turning to (ii), let

$$M_{\mu,\nu}(R) = \max_{0\leqslant m\leqslant \nu} \|\Phi_{\mu,m,\delta,j}\|_{C(\mathring{B}_R)}. \tag{10.67}$$

Relation (10.63) yields

$$\begin{aligned}\left\|L^q \Phi_{\mu,\nu,\delta,j}\right\|_{C(\mathring{B}_R)} &\leqslant M_{\mu,\nu}(R)\sum_{l=0}^{q}\binom{q}{l}\rho_X^{2(q-l)}\sum_{m=0}^{\nu}\binom{\nu}{m}\left(\frac{2q}{|\mu|}\right)^{\nu-m}|\mu|^{2q}\\ &\leqslant M_{\mu,\nu}(R)(1+\rho_X)^{2q}(1+|\mu|)^{2q}\left(1+\frac{2q}{|\mu|}\right)^{\nu}.\end{aligned}$$

This, together with (10.67) and (10.65), gives (ii). Part (iii) results from using (10.64), and we are done. □

Proposition 10.8. *Let* $0 < r < R$, $\mu \in \mathbb{C}$, $\mathrm{i}\mu/\kappa \notin \mathbb{N}$, $\nu, q \in \mathbb{Z}_+$, $\varepsilon \in (0,1)$.

(i) *If* $\operatorname{Im}\mu \geqslant 0$ *and* $\nu \leqslant R|\mu|\varepsilon$, *then for each differential operator* D *of order* q *on* X,

$$\|D\Psi_{\mu,\nu,\delta,j}\|_{C(\mathring{B}_{r,R})} \leqslant \gamma_1\sqrt{1+\nu}(1+|\mu|)^q R^{\nu}\mathrm{e}^{R|\operatorname{Im}\mu|}, \tag{10.68}$$

where $\gamma_1 > 0$ *is independent of* μ, ν.

(ii) *If* $\operatorname{Im}\mu \geqslant 0$ *and* $\nu \leqslant R|\mu|\varepsilon$, *then*

$$\begin{aligned}\left\|L^q \Psi_{\mu,\nu,\delta,j}\right\|_{C(\mathring{B}_{r,R})} &\leqslant \gamma_2\sqrt{1+\nu}(1+\rho_X)^{2q}(1+|\mu|)^{2q}R^{\nu}\\ &\quad\times\exp\left(R|\operatorname{Im}\mu| + \frac{2q\nu}{|\mu|}\right),\end{aligned}$$

where $\gamma_2 > 0$ *is independent of* μ, ν, q.

(iii) *If* $x \in \mathring{B}_{r,R}$, $\mathrm{i}\mu/\kappa \notin \mathbb{Z}_+$, *and* $q \in \mathbb{N}$, *then*

$$\left(L+\rho_X^2\right)^q \Psi_{\mu,\nu,\delta,j}(x) = (-1)^q(2q)^{\nu}\mu^{2q-\nu}\left(\Psi_{\mu,0,\delta,j}(x) + O\left(q^{-1}\right)\right),$$

where the constant in O *is independent of* q, x.

(iv) *If* $k \in K$, $a_t \in A^+$, *and* $x = ka_t o \in X$, *then*

$$\Psi_{\mu,\nu,\delta,j}(x) = 2^{-s(\delta)-s_1(\delta)}Y_j^{\delta}(kM)\left(\frac{\mathrm{i}t}{\kappa}\right)^{\varkappa}\exp\left(\frac{(\mathrm{i}\mu-\rho_X)t}{\kappa}\right)\left(1+O\left(t^{-1}\right)\right)$$

as $t \to +\infty$.

Proof. Regarding (i), first assume that

$$\mu \in \left\{z \in \mathbb{C} : \operatorname{Im} z \geqslant -\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}|\operatorname{Re} z|\right\}$$

(we do not yet suppose that $\mathrm{Im}\,\mu \geqslant 0$). Owing to (10.61) and (7.22),

$$\Psi_{\mu,0,\delta,j}(x) = Y_j^\delta(kM)\sum_{m=0}^{\infty}\Gamma_m(\mu)\mathrm{e}^{(\mathrm{i}\mu-\rho_X-m)t}(\sinh t)^{s(\delta)}(\cosh t)^{s_1(\delta)}$$

for all $x \in X\backslash\{o\}$. Then there are functions $f_{l,m} \in C^\infty(X\backslash\{o\})$, $l \in \{0,\dots,q\}$, $m \in \mathbb{Z}_+$, such that

$$D\Psi_{\mu,0,\delta,j}(x) = \sum_{l=0}^{q}\sum_{m=0}^{\infty}\Gamma_m(\mu)(\mathrm{i}\mu-\rho_X-m)^l\mathrm{e}^{(\mathrm{i}\mu-\rho_X-m)t}f_{l,m}(x).$$

Using (7.21) and (10.13) for $x \in \mathring{B}_{r,R}$, we obtain

$$|D\Psi_{\mu,0,\delta,j}(x)| \leqslant \gamma_3(1+|\mu|)^q\mathrm{e}^{R|\mathrm{Im}\,\mu|},$$

where $\gamma_3 > 0$ is independent of x, μ. Therefore, part (i) is a consequence of Proposition 6.11.

The proof of (ii) and (iii) now completely reproduces the proof of Proposition 10.7(ii), (iii). Finally, part (iv) immediately follows from (10.60), (10.61), and (7.23). ☐

10.4 The Helgason–Fourier Transform $\widetilde{f}(\lambda, b)$

For any distribution $f \in \mathcal{E}'(X)$, we define the *Fourier transform* $\widetilde{f}$ by letting

$$\widetilde{f}(\lambda,b) = \left\langle f, \mathrm{e}^{(-\mathrm{i}\lambda+\rho)(A(x,b))}\right\rangle, \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*,\ b \in B. \tag{10.69}$$

This transform was introduced by Helgason [116].

Using (10.69) and (10.4), we obtain

$$\widetilde{f\circ k}(\lambda,b) = \widetilde{f}(\lambda,kb) \quad \text{for each } k \in K. \tag{10.70}$$

In particular, if f is K-invariant, then the right-hand side of (10.70) is independent of k. Integrating it over k, we see from (10.31) that

$$\widetilde{f}(\lambda,b) = \left\langle f, \varphi_{-\lambda}(x)\right\rangle \tag{10.71}$$

for all $f \in \mathcal{E}'_\natural(X)$, $(\lambda,b) \in \mathfrak{a}_{\mathbb{C}}^* \times B$. We write $\widetilde{f}(\lambda)$ for the right-hand side of (10.71). This function is called the *spherical transform* of $f \in \mathcal{E}'_\natural(X)$. Thus, the transform (10.69) is an extension of the spherical transform from K-invariant compactly supported distributions to arbitrary distributions in $\mathcal{E}'(X)$.

We now establish some elementary properties of the Fourier transform.

Proposition 10.9.

(i) *If* $f \in \mathcal{E}'(X)$ *and* $T \in \mathcal{E}'_{\natural}(X)$, *then*

$$\widetilde{f \times T}(\lambda, b) = \widetilde{f}(\lambda, b)\widetilde{T}(\lambda). \tag{10.72}$$

In particular,

$$\widetilde{p(L)f}(\lambda, b) = p\big(-\langle\lambda, \lambda\rangle - |\rho|^2\big)\widetilde{f}(\lambda, b) \tag{10.73}$$

for each polynomial p.

(ii) *Let* $m \in \mathbb{Z}_+$, $f \in C^m(X)$, *and assume that* $\operatorname{supp} f \subset \mathring{B}_R$ *for some* $R > 0$. *Then*

$$\big|\widetilde{f}(\lambda, b)\big| < c(1 + |\lambda|)^{-m} \mathrm{e}^{R|\operatorname{Im}\lambda|}, \tag{10.74}$$

where the constant $c > 0$ *is independent of* λ, b. *In addition, if* m *is even, then*

$$\big|\widetilde{f}(\lambda, b)\big| \leqslant \frac{\mathrm{e}^{R(|\operatorname{Im}\lambda| + |\rho|)}}{|\langle\lambda, \lambda\rangle + |\rho|^2|^{m/2}} \int_{B_R} \big|L^{m/2} f(x)\big| \,\mathrm{d}x \tag{10.75}$$

for all $(\lambda, b) \in \mathfrak{a}^*_{\mathbb{C}} \times B$ *such that* $\langle\lambda, \lambda\rangle \neq -|\rho|^2$.

Proof. To prove (i), assume that $x \in X$ and $g \in G$. By (10.3) we have

$$A(gx, b) = A(go, b) + A\big(x, u^{-1}M\big),$$

where $u \in K$ is independent of x. This, together with (10.10), yields

$$\begin{aligned}
&\big\langle (f \times T)(x), \mathrm{e}^{(-\mathrm{i}\lambda+\rho)(A(x,b))}\big\rangle \\
&\quad = \big\langle T(x), \big\langle f(go), \mathrm{e}^{(-\mathrm{i}\lambda+\rho)(A(go,b))} \mathrm{e}^{(-\mathrm{i}\lambda+\rho)(A(x,u^{-1}M))}\big\rangle\big\rangle.
\end{aligned}$$

Since T is K-invariant, relation (10.72) is obvious from (10.4).

Assume now that $T = p(L)\delta_0$, where δ_0 is the Dirac measure supported at origin. Because the Laplace–Beltrami operator is self-adjoint, by (10.5) we find $\widetilde{T}(\lambda) = p(-\langle\lambda, \lambda\rangle - |\rho|^2)$. In view of (10.72), this gives (10.73).

As for (ii), first observe that for each $k \in K$,

$$\widetilde{f}(\lambda, kM) = \int_N \int_A f(knao) \mathrm{e}^{(-\mathrm{i}\lambda-\rho)(\log a)} \,\mathrm{d}a \,\mathrm{d}n \tag{10.76}$$

(see (10.69), (10.6), and (10.7)). By assumption on f the function

$$(k, n, a) \to f(knao)\mathrm{e}^{-\rho(\log a)}$$

is in the class $C_c^m(K \times N \times A)$. Bearing in mind (10.1) and (10.2) and integrating the integral over A in (10.76) by parts, we arrive at estimate (10.74). Finally, if $m \in \mathbb{Z}_+$ is even, then inequality (10.75) is an easy consequence of (10.73) and (10.1). □

We now turn to the problem of inverting the Fourier transform.

Theorem 10.5. *Let $f \in C_c^m(X)$ with $m = \dim M + \operatorname{rank} X + 1$. Then for all $x \in X$, the following equality holds:*

$$f(x) = \frac{1}{|W|} \int_{\mathfrak{a}^*} \int_B e^{(i\lambda+\rho)(A(x,b))} \widetilde{f}(\lambda, b) |\mathbf{c}(\lambda)|^{-2} \, d\lambda \, db. \tag{10.77}$$

Proof. For $f \in \mathcal{D}(X)$, the proof of (10.77) can be found in [123, Chap. 3, Theorem 1.3]. Next, for each $\varepsilon > 0$, let $\eta_\varepsilon \in \mathcal{D}_\natural(X)$ be nonnegative, with $\int_X \eta_\varepsilon(x) \, dx = 1$ and $\operatorname{supp} \eta_\varepsilon \subset \mathring{B}_\varepsilon$. Then

$$\widetilde{\eta}_\varepsilon(\lambda) = 1 + \int_{B_\varepsilon} \eta_\varepsilon(x) \big(\varphi_{-\lambda}(x) - 1\big) \, dx.$$

Given a compact set $U \subset \mathfrak{a}_{\mathbb{C}}^*$ and $\zeta > 0$, there exists $\varepsilon > 0$ such that $|\varphi_{-\lambda}(x) - 1| < \zeta$ for $d(o, x) < \varepsilon$, $\lambda \in U$. Thus, if $\varepsilon \to 0$, then $\widetilde{\eta}_\varepsilon \to 1$ uniformly on compact sets. In addition, using (10.75), we see that

$$\big|\widetilde{\eta}_\varepsilon(\lambda)\big| \leqslant e^{\varepsilon|\rho|} \quad \text{for all } \lambda \in \mathfrak{a}^*, \ \varepsilon > 0. \tag{10.78}$$

If $f \in C_c^m(X)$, then $f \times \eta_\varepsilon \in \mathcal{D}(X)$, and we have (10.77) for $f \times \eta_\varepsilon$ instead of f. Observe that $\widetilde{f \times \eta_\varepsilon}(\lambda, b) = \widetilde{f}(\lambda, b)\widetilde{\eta}_\varepsilon(\lambda)$ (see (10.72)) and suppose that $\varepsilon \to 0$. Applying (10.74), (10.78), and (10.36), by Lebesgue's dominated convergence theorem we obtain (10.77) in the general case. □

Remark 10.1. The proof of Theorem 10.5 shows that if $f \in \mathcal{E}'(X)$, $m = \dim M + \operatorname{rank} X + 1$, and

$$\sup_{(\lambda,b)\in\mathfrak{a}^*\times B} (1 + |\lambda|)^{l+m} \big|\widetilde{f}(\lambda, b)\big| < +\infty \quad \text{for some } l \in \mathbb{Z}_+,$$

then $f \in C_c^l(X)$ and (10.77) holds. For the rank one case, this is valid, provided that $m = 2\alpha_X + 3$ (see (10.37)).

Remark 10.2. Assume that $f \in (C^m \cap \mathcal{E}'_\natural)(X)$ for $m = \dim M + \operatorname{rank} X + 1$. Because of (10.31), equality (10.77) can be written

$$f(x) = \frac{1}{|W|} \int_{\mathfrak{a}^*} \widetilde{f}(\lambda) \varphi_\lambda(x) |\mathbf{c}(\lambda)|^{-2} \, d\lambda, \quad x \in X.$$

Let us now consider the Plancherel formula for the Fourier transform.

Theorem 10.6. *The Fourier transform defined on $C_c(X)$ by (10.69) extends to an isometry of $L^2(X)$ onto $L^2(\mathfrak{a}_+^* \times B)$ (with the measure $|\mathbf{c}(\lambda)|^{-2} \, d\lambda \, db$ on $\mathfrak{a}_+^* \times B$). Moreover,*

$$\int_X f_1(x) \overline{f_2(x)} \, dx = \frac{1}{|W|} \int_{\mathfrak{a}^* \times B} \widetilde{f}_1(\lambda, b) \overline{\widetilde{f}_2(\lambda, b)} |\mathbf{c}(\lambda)|^{-2} \, d\lambda \, db$$

for all $f_1, f_2 \in L^2(X)$.

For the proof, we refer the reader to [123, p. 227].

Corollary 10.3. *Let $\eta_n \in L^2(X)$, $n = 1, 2, \dots$, and assume that for almost all $(\lambda, b) \in \mathfrak{a}^* \times B$ (with respect to the measure $\mathrm{d}\lambda\,\mathrm{d}b$ on $\mathfrak{a}^* \times B$), the following assumptions are satisfied:*

(1) $\widetilde{\eta}_n(\lambda, b) \to 1$ *as* $n \to \infty$;
(2) $|\eta_n(\lambda, b)| \leqslant (2 + |\lambda|)^c$, *where the constant $c > 0$ is independent of n, λ, b.*

Then $\langle \eta_n, \psi \rangle \to \psi(o)$ as $n \to \infty$ for any $\psi \in \mathcal{D}(X)$.

Proof. Let $\psi \in \mathcal{D}(X)$. It follows by (10.36) and Proposition 10.9(ii) that for each $m \in \mathbb{Z}_+$, there exists $c_m > 0$ such that

$$\big|\widetilde{\psi}(\lambda, b)\big| |\mathbf{c}(\lambda)|^{-2} \leqslant c_m (1 + |\lambda|)^{-m}$$

for all $(\lambda, b) \in \mathfrak{a}^* \times B$. Applying Theorem 10.6, by Lebesgue's dominated convergence theorem and (10.77) we arrive at the desired result. □

We shall now give an explicit description of the ranges $\mathcal{D}(X)^{\sim}$ and $\mathcal{E}'(X)^{\sim}$ of $\mathcal{D}(X)$ and $\mathcal{E}'(X)$, respectively, under the Fourier transform.

Assume that $R \geqslant 0$. Let $H^R(\mathfrak{a}^*_{\mathbb{C}} \times B)$ denote the set of all C^∞ functions $\psi(\lambda, b)$ on $\mathfrak{a}^*_{\mathbb{C}} \times B$, holomorphic in λ and satisfying

$$\sup_{\lambda \in \mathfrak{a}^*_{\mathbb{C}}, b \in B} \mathrm{e}^{-R|\operatorname{Im}\lambda|} (1 + |\lambda|)^{\alpha} |\psi(\lambda, b)| < \infty \quad \text{for each } \alpha > 0. \tag{10.79}$$

We set $H(\mathfrak{a}^*_{\mathbb{C}} \times B) = \bigcup_{R \geqslant 0} H^R(\mathfrak{a}^*_{\mathbb{C}} \times B)$. Also $H(\mathfrak{a}^*_{\mathbb{C}} \times B)_W$ denotes the set of all functions $\psi \in H(\mathfrak{a}^*_{\mathbb{C}} \times B)$ satisfying

$$\int_B \mathrm{e}^{(\mathrm{i}s\lambda + \rho)(A(x,b))} \psi(s\lambda, b)\,\mathrm{d}b = \int_B \mathrm{e}^{(\mathrm{i}\lambda + \rho)(A(x,b))} \psi(\lambda, b)\,\mathrm{d}b \tag{10.80}$$

for all $s \in W$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, $x \in X$. Next, let $\mathcal{H}^R(\mathfrak{a}^*_{\mathbb{C}} \times B)$ denote the set of all C^∞ functions $\psi(\lambda, b)$ on $\mathfrak{a}^*_{\mathbb{C}} \times B$, holomorphic in λ and satisfying

$$|\psi(\lambda, b)| \leqslant \gamma_1 (1 + |\lambda|)^{\gamma_2} \mathrm{e}^{R|\operatorname{Im}\lambda|}, \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}},\ b \in B, \tag{10.81}$$

with constants $\gamma_1 > 0$, $\gamma_2 \in \mathbb{R}^1$ independent of λ and b. We define

$$\mathcal{H}\big(\mathfrak{a}^*_{\mathbb{C}} \times B\big) = \bigcup_{R \geqslant 0} \mathcal{H}^R\big(\mathfrak{a}^*_{\mathbb{C}} \times B\big)$$

and denote by $\mathcal{H}(\mathfrak{a}^*_{\mathbb{C}} \times B)_W$ the set of all functions $\psi \in \mathcal{H}(\mathfrak{a}^*_{\mathbb{C}} \times B)$ satisfying (10.80) for all $s \in W$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, $x \in X$. Also let $H_W(\mathfrak{a}^*_{\mathbb{C}})$ and $\mathcal{H}_W(\mathfrak{a}^*_{\mathbb{C}})$ be the spaces of functions in $H(\mathfrak{a}^*_{\mathbb{C}} \times B)$ and $\mathcal{H}(\mathfrak{a}^*_{\mathbb{C}} \times B)$, respectively, constant in $b \in B$ and W-invariant in $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$.

The following result is an analog of Theorem 6.3.

Theorem 10.7. *Let $\mathcal{F}$ denote the mapping $f \to \widetilde{f}$. Then the following assertions hold.*

(i) *$\mathcal{F}$ is a bijection of $\mathcal{D}(X)$ onto $H(\mathfrak{a}^*_{\mathbb{C}} \times B)_W$. Furthermore, if $f \in \mathcal{D}(X)$, then the function $\psi = \widetilde{f}$ satisfies* (10.79) *if and only if* $\operatorname{supp} f \subset \mathring{B}_R$. *In particular, $\mathcal{F}$ is a bijection of $\mathcal{D}_\natural(X)$ onto $H_W(\mathfrak{a}^*_{\mathbb{C}})$.*

(ii) *$\mathcal{F}$ is a bijection of $\mathcal{E}'(X)$ onto $\mathcal{H}(\mathfrak{a}^*_{\mathbb{C}} \times B)_W$. Moreover, for $f \in \mathcal{E}'(X)$, the function $\psi = \widetilde{f}$ satisfies* (10.81) *if and only if* $\operatorname{supp} f \subset \mathring{B}_R$. *In particular, $\mathcal{F}$ is a bijection of $\mathcal{E}'_\natural(X)$ onto $\mathcal{H}_W(\mathfrak{a}^*_{\mathbb{C}})$.*

The proof of this theorem can be found in [123, pp. 270, 281].

10.5 Action of $\widetilde{f}(\lambda, b)$ on the Space $\mathcal{E}'_{\check{\delta}}(X)$

As mentioned in the previous section, the Fourier transform $\widetilde{f}(\lambda, b)$ of $f \in \mathcal{E}'_\natural(X)$ is independent of b. In this section we shall investigate the Fourier transform on the spaces $\mathcal{E}'_{\check{\delta}}(X)$, $\delta \in \widehat{K}_M$.

Let $f \in \mathcal{E}'_{\check{\delta}}(X)$. By (10.18) we see that (10.69) can be written

$$\widetilde{f}(\lambda, kM) = d(\delta)\Big\langle f, \int_K e^{(-i\lambda+\rho)(A(x,\tau K))}\chi\big(k\tau^{-1}\big)\, d\tau\Big\rangle, \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}},\ k \in K. \tag{10.82}$$

This leads to the following definition of the δ-spherical transform.

For $f \in \mathcal{E}'_{\check{\delta}}(X)$, the δ-*spherical transform* $\widetilde{f}$ is defined by

$$\widetilde{f}(\lambda) = d(\delta)\big\langle f, \Phi_{\bar{\lambda},\delta}(x)^*\big\rangle, \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}}. \tag{10.83}$$

If δ is the trivial representation, (10.43) shows that $f \to \widetilde{f}$ is just the spherical transform of K-invariant distributions. In the general case, $\delta(m)\widetilde{f}(\lambda) = \widetilde{f}(\lambda)$ for all $m \in M$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. In addition, formulae (10.82), (10.83), and (10.43) yield

$$\widetilde{f}(\lambda) = d(\delta)\int_K \widetilde{f}(\lambda, kM)\delta\big(k^{-1}\big)\, dk, \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}}, \tag{10.84}$$

and

$$\widetilde{f}(\lambda, kM) = \operatorname{Trace}\big(\delta(k)\widetilde{f}(\lambda)\big), \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}},\ k \in K. \tag{10.85}$$

Thus, we conclude from (10.84) and Theorem 10.7(ii) that $\widetilde{f}$ belongs to the space of entire functions on $\mathfrak{a}^*_{\mathbb{C}}$ with values in $\operatorname{Hom}(V_\delta, V_\delta^M)$. By Theorem 10.7(ii) and (10.85) we infer that the δ-spherical transform is injective on $\mathcal{E}'_{\check{\delta}}(X)$. Furthermore, (10.84), together with Theorem 10.6, defines $\widetilde{f}$ on $\mathfrak{a}^*$ for each $f \in L^2_{\check{\delta}}(X)$.

Now we shall obtain the inversion formula and the Plancherel formula for the δ-spherical transform.

Theorem 10.8.

(i) *Let* $f \in (\mathcal{E}'_{\check{\delta}} \cap C^m)(X)$ *for* $m = \dim M + \operatorname{rank} X + 1$. *Then*

$$f(x) = \frac{1}{|W|} \operatorname{Trace}\left(\int_{\mathfrak{a}^*} \Phi_{\lambda,\delta}(x) \widetilde{f}(\lambda) |\mathbf{c}(\lambda)|^{-2} \, d\lambda\right) \quad \text{for all } x \in X. \tag{10.86}$$

(ii) *For each* $f \in L^2_{\check{\delta}}(X)$,

$$\int_X |f(x)|^2 \, dx = \frac{1}{|W| d(\delta)} \int_{\mathfrak{a}^*} \operatorname{Trace}\big(\widetilde{f}(\lambda) \widetilde{f}(\lambda)^*\big) |\mathbf{c}(\lambda)|^{-2} \, d\lambda, \tag{10.87}$$

where $*$ *denotes adjoint on* V_δ.

Proof. Equality (10.86) follows immediately from Theorem 10.5 and (10.85). On the other hand, (10.85), together with the orthogonality relations, implies

$$d(\delta) \int_K \big|\widetilde{f}(\lambda, kM)\big|^2 \, dk = \operatorname{Trace}\big(\widetilde{f}(\lambda) \widetilde{f}(\lambda)^*\big) \quad \text{on } \mathfrak{a}^*$$

for each $f \in L^2_{\check{\delta}}(X)$. Then by Theorem 10.5 we obtain (10.87). □

To state the analog of Theorem 10.7 for the δ-spherical transform, it will be most convenient to think of $\operatorname{Hom}(V_\delta, V_\delta^M)$ in terms of $d(\delta) \times l(\delta)$ matrices. Then $Q^{\check{\delta}}(\lambda)$ is an $l(\delta) \times l(\delta)$ matrix whose entries are polynomial functions on $\mathfrak{a}^*_{\mathbb{C}}$ (see Sect. 10.3).

Assume that $R \geqslant 0$. Denote by $H^{R,\delta}(\mathfrak{a}^*_{\mathbb{C}})$ the set of all entire functions ψ on $\mathfrak{a}^*_{\mathbb{C}}$ with values in $\operatorname{Hom}(V_\delta, V_\delta^M)$ such that $(Q^{\check{\delta}})^{-1}\psi$ is W-invariant and every matrix entry $\psi_{i,j}$ of ψ satisfies

$$\sup_{\lambda \in \mathfrak{a}^*_{\mathbb{C}}} e^{-R|\operatorname{Im} \lambda|} (1 + |\lambda|)^\alpha |\psi_{i,j}(\lambda)| < +\infty \quad \text{for each } \alpha > 0.$$

Now define $H^\delta(\mathfrak{a}^*_{\mathbb{C}}) = \bigcup_{R \geqslant 0} H^{R,\delta}(\mathfrak{a}^*_{\mathbb{C}})$. Next, let $\mathcal{H}^{R,\delta}(\mathfrak{a}^*_{\mathbb{C}})$ be the set of all entire functions $\psi : \mathfrak{a}^*_{\mathbb{C}} \to \operatorname{Hom}(V_\delta, V_\delta^M)$ such that $(Q^{\check{\delta}})^{-1}\psi$ is W-invariant and each matrix entry $\psi_{i,j}$ of ψ satisfies the estimate

$$|\psi_{i,j}(\lambda)| \leqslant \gamma_1 (1 + |\lambda|)^{\gamma_2} e^{R|\operatorname{Im} \lambda|}, \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}},$$

with constants $\gamma_1, \gamma_2 > 0$ independent of λ. Also let $\mathcal{H}^\delta(\mathfrak{a}^*_{\mathbb{C}}) = \bigcup_{R \geqslant 0} \mathcal{H}^{R,\delta}(\mathfrak{a}^*_{\mathbb{C}})$.

Theorem 10.9.

(i) *The* δ*-spherical transform is a bijection of* $\mathcal{D}_{\check{\delta}}(X)$ *onto* $H^\delta(\mathfrak{a}^*_{\mathbb{C}})$. *The same is true for the spaces* $\mathcal{E}'_{\check{\delta}}(X)$ *and* $\mathcal{H}^\delta(\mathfrak{a}^*_{\mathbb{C}})$.

(ii) *Let* $f \in \mathcal{E}'_{\check{\delta}}(X)$ *(respectively* $f \in \mathcal{D}_{\check{\delta}}(X)$*). Then* $\widetilde{f} \in \mathcal{H}^{R,\delta}(\mathfrak{a}^*_{\mathbb{C}})$ *(respectively* $\widetilde{f} \in H^{R,\delta}(\mathfrak{a}^*_{\mathbb{C}})$*) if and only* $\operatorname{supp} f \subset \mathring{B}_R$.

Proof. For the proof of (i) in the case of $\mathcal{D}_{\check{\delta}}(X)$ and $H^{\delta}(\mathfrak{a}^*_{\mathbb{C}})$, we refer the reader to [123, p. 285], where a refinement of this result is obtained. For the spaces $\mathcal{E}'_{\check{\delta}}(X)$ and $\mathcal{H}^{\delta}(\mathfrak{a}^*_{\mathbb{C}})$, the proof is similar.

Assertion (ii) is clear from (10.84), (10.85), and Theorem 10.7. □

We now extend the definitions of the Fourier transform and the δ-spherical transform on the class

$$L^1_{\check{\delta}}(X) + L^2_{\check{\delta}}(X) = \left\{f_1 + f_2 : f_1 \in L^1_{\check{\delta}}(X),\ f_2 \in L^2_{\check{\delta}}(X)\right\}.$$

First, assume that $f \in L^1_{\check{\delta}}(X)$ and put

$$\widetilde{f}(\lambda) = d(\delta)\int_X f(x)\Phi_{\bar{\lambda},\delta}(x)^*\,\mathrm{d}x, \quad \lambda \in \mathfrak{a}^*. \tag{10.88}$$

Using [123, Chap. 3, Theorem 2.7], we see that the right-hand side of (10.88) is well defined. Relation (10.88) shows that for $f \in (\mathcal{E}'_{\check{\delta}} \cap C)(X)$ and $\lambda \in \mathfrak{a}^*$, this definition of $\widetilde{f}(\lambda)$ coincides with (10.84). Moreover, Theorem 10.6 gives the same statement for $f \in (L^1_{\check{\delta}} \cap L^2_{\check{\delta}})(X)$.

Now define $\widetilde{f}(\lambda, kM)$ by (10.85) for all $f \in L^1_{\check{\delta}}(X)$, $\lambda \in \mathfrak{a}^*$, $k \in K$. We see from (10.85), (10.88), and [123, Chap. 3, Theorem 2.7] that for all $f \in L^1_{\check{\delta}}(X)$, $k \in K$, the function $\lambda \to \widetilde{f}(\lambda, kM)$ is in the class $(C \cap L^{\infty})(\mathfrak{a}^*)$.

Assume now that $f = f_1 + f_2$, where $f_1 \in L^1_{\check{\delta}}(X)$, $f_2 \in L^2_{\check{\delta}}(X)$. We define $\widetilde{f}$ on $\mathfrak{a}^*$ by

$$\widetilde{f} = (f_1 + f_2)^{\sim} = \widetilde{f_1} + \widetilde{f_2}. \tag{10.89}$$

Notice that for any other representation $f = g_1 + g_2$ with $g_1 \in L^1_{\check{\delta}}(X)$, $g_2 \in L^2_{\check{\delta}}(X)$, we have $f_1 - g_1 = g_2 - f_2 \in (L^1_{\check{\delta}} \cap L^2_{\check{\delta}})(X)$, and the above arguments show that $\widetilde{f_1} + \widetilde{f_2} = \widetilde{g_1} + \widetilde{g_2}$ on $\mathfrak{a}^*$. Thus, $\widetilde{f}$ is well defined on $\mathfrak{a}^*$ by (10.89). As before, we define $\widetilde{f}(\lambda, kM)$ on $\mathfrak{a}^*$ by (10.85) for all $k \in K$. By this definition we conclude that the function $\lambda \to \widetilde{f}(\lambda, kM)$ belongs to $L^{2,\mathrm{loc}}(\mathfrak{a}^*)$ for all $f \in L^1_{\check{\delta}}(X) + L^2_{\check{\delta}}(X)$, $k \in K$.

It is easy to see that $L^p_{\check{\delta}}(X) \subset L^1_{\check{\delta}}(X) + L^2_{\check{\delta}}(X)$ for each $p \in [1, 2]$. Therefore, $\widetilde{f}(\lambda)$ and $\widetilde{f}(\lambda, b)$ are well defined on $\mathfrak{a}^*$ and $\mathfrak{a}^* \times B$, respectively, for each $f \in L^p_{\check{\delta}}(X)$, $p \in [1, 2]$.

We shall now obtain the following useful result.

Proposition 10.10.

(i) *If $f \in \mathcal{E}'_{\check{\delta}}(X)$ and $T \in \mathcal{E}'_{\natural}(X)$, then $f \times T \in \mathcal{E}'_{\check{\delta}}(X)$ and*

$$\widetilde{f \times T}(\lambda) = \widetilde{f}(\lambda)\widetilde{T}(\lambda) \tag{10.90}$$

for all $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$.

(ii) *If* $f \in L^1_{\check{\delta}}(X) + L^2_{\check{\delta}}(X)$ *and* $T \in (\mathcal{E}'_\natural \cap L^1)(X)$*, then* $f \times T \in L^1_{\check{\delta}}(X) + L^2_{\check{\delta}}(X)$*. In addition,* (10.90) *holds on* $\mathfrak{a}^*$*, and* (10.72) *holds on* $\mathfrak{a}^* \times B$*.*

(iii) *The Fourier transform and the* δ*-spherical transform are injective on* $L^1_{\check{\delta}}(X) + L^2_{\check{\delta}}(X)$*.*

Proof. The first statement follows from Proposition 10.2(ii), (10.84), and (10.72). Next, let $p \in \{1, 2\}$, and let $T \in (\mathcal{E}'_\natural \cap L^1)(X)$. It is easy to see that $f \times T \in L^p_{\check{\delta}}(X)$ for each $f \in L^p_{\check{\delta}}(X)$. Since $(\mathcal{E}'_{\check{\delta}} \cap C)(X)$ is dense in $L^p_{\check{\delta}}(X)$, this, together with (i) and (10.72), proves (ii).

It is enough to prove the last statement for the δ-spherical transform (see (10.84)). Assume that $f = f_1 + f_2$, where $f_1 \in L^1_{\check{\delta}}(X)$, $f_2 \in L^2_{\check{\delta}}(X)$, and $\widetilde{f} = 0$ on $\mathfrak{a}^*$. Using (ii), for each $\eta \in \mathcal{D}_\natural(X)$, we find that $\widetilde{f_1 \times \eta}(\lambda, b) = -\widetilde{f_2 \times \eta}(\lambda, b)$ on $\mathfrak{a}^* \times B$. In addition, $f_1 \times \eta \in L^2(X)$ and $f_2 \times \eta \in L^2(X)$. Now Theorem 10.6 yields $f_1 \times \eta = -f_2 \times \eta$. Since $\eta \in \mathcal{D}_\natural(X)$ could be arbitrary, we have $f = 0$. This concludes the proof. □

For the rest of this section, we assume that rank $X = 1$ and that $\delta \in \widehat{K}_M$, $j \in \{1, \ldots, d(\delta)\}$, are fixed. We now consider the action of the Fourier transform on the space $\mathcal{E}'_{\delta,j}(X)$.

For each $f \in \mathcal{E}'_{\delta,j}(X)$, we define

$$\mathrm{F}^\delta_j(f)(\lambda) = d(\delta)\Big\langle f, \varphi_{-\lambda,\delta}(ao)\overline{Y^\delta_j(kM)}\Big\rangle, \quad \lambda \in \mathfrak{a}^*_\mathbb{C}, \tag{10.91}$$

where

$$x = kao, \quad k \in K, a \in \overline{A^+}.$$

By (10.91) and Theorem 10.3 we conclude that the function

$$\mathcal{F}^\delta_j(f)(\lambda) = \frac{\mathrm{F}^\delta_j(f)(\lambda)}{d(\delta)R_\delta(-\lambda)} = \big\langle f, \overline{\Phi_{\overline{z},0,\delta,j}}\big\rangle, \quad \lambda \in \mathfrak{a}^*_\mathbb{C}, z = \kappa\lambda(H_0), \tag{10.92}$$

is entire and even. The following result relates F^δ_j to the Fourier transform.

Theorem 10.10.

(i) *If* $f \in \mathcal{E}'_{\delta,j}(X)$*, then*

$$\widetilde{f}(\lambda, b) = \mathrm{F}^\delta_j(f)(\lambda)Y^\delta_j(b), \quad (\lambda, b) \in \mathfrak{a}^*_\mathbb{C} \times B. \tag{10.93}$$

(ii) *Assume that* $f \in \mathcal{E}'(X)$ *and*

$$\widetilde{f}(\lambda, b) = \psi(\lambda)Y^\delta_j(b), \quad (\lambda, b) \in \mathfrak{a}^*_\mathbb{C} \times B, \tag{10.94}$$

where the function ψ *is independent of* b*. Then* $f \in \mathcal{E}'_{\delta,j}(X)$*.*

Proof. Let $f \in \mathcal{E}'_{\delta,j}(X)$, and let $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, $b = kM$, $k \in K$. Then (10.69) yields

$$\widetilde{f}(\lambda, b) = d(\delta)\left\langle f, \int_K \mathrm{e}^{(-\mathrm{i}\lambda+\rho)(A(\tau^{-1}x,kM))} t^{\delta}_{j,j}(\tau)\,\mathrm{d}\tau\right\rangle.$$

Using (10.91), (1.80), and Proposition 10.3, we see that both sides in (10.93) actually coincide.

Assume now that $f \in \mathcal{E}'(X)$ and let f satisfy (10.94). For each $\varepsilon > 0$, let $\eta_\varepsilon \in \mathcal{D}_\natural(X)$ be nonnegative, with $\int_X \eta_\varepsilon(x)\,\mathrm{d}x = 1$ and $\operatorname{supp}\eta_\varepsilon \subset \dot{B}_\varepsilon$. By (10.72) and Theorem 10.5,

$$(f \times \eta_\varepsilon)(kao) = \frac{1}{2}\int_{\mathfrak{a}^*} \psi(\lambda)\widetilde{\eta}_\varepsilon(\lambda)|\mathbf{c}(\lambda)|^{-2}\int_B \mathrm{e}^{(\mathrm{i}\lambda+\rho)(A(kao,b))}Y^{\delta}_j(b)\,\mathrm{d}b\,\mathrm{d}\lambda$$

for all $k \in K$, $a \in A$. Because of Proposition 10.3, this equality can be written

$$(f \times \eta_\varepsilon)(kao) = \frac{1}{2}\int_{\mathfrak{a}^*} \psi(\lambda)\widetilde{\eta}_\varepsilon(\lambda)\varphi_{\lambda,\delta}(ao)|\mathbf{c}(\lambda)|^{-2}\,\mathrm{d}\lambda Y^{\delta}_j(kM).$$

This means that $f \times \eta_\varepsilon \in \mathcal{E}'_{\delta,j}(X)$. However, by assumptions on η_ε we deduce that $f \times \eta_\varepsilon \to f$ in $\mathcal{D}'(X)$ as $\varepsilon \to 0$. Assertion (ii) is thereby established. □

The Plancherel formula and the Paley–Wiener theorem for the transform F^{δ}_j are contained in the corresponding result for the Fourier transform (see Theorems 10.6 and 10.7 and (10.93)). We restrict ourselves to only the following analog of Theorem 10.7 for the transform $\mathcal{F}^{\delta}_j$.

Theorem 10.11.

(i) *If $f \in \mathcal{E}'_{\delta,j}(X)$, then*

$$\left|\mathcal{F}^{\delta}_j(f)(\lambda)\right| \leqslant \gamma_1(1+|\lambda|)^{\gamma_2}\mathrm{e}^{r(f)|\mathrm{Im}\,\lambda|}, \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}}, \tag{10.95}$$

where $\gamma_2 = \operatorname{ord} f - s(\delta)$, and $\gamma_1 > 0$ is independent of λ. Moreover, if $f \in C^m_{\delta,j}(X)$ for some $m \in \mathbb{Z}_+$, then (10.95) holds with $\gamma_2 = -m - s(\delta)$.

(ii) *Let $w : \mathfrak{a}^*_{\mathbb{C}} \to \mathbb{C}$ be an even entire function and suppose that*

$$|w(\lambda)| \leqslant \gamma_1(1+|\lambda|)^{\gamma_2}\mathrm{e}^{R|\mathrm{Im}\,\lambda|}, \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}}, \tag{10.96}$$

where $\gamma_1 > 0$, $\gamma_2 \in \mathbb{R}^1$, and $R \geqslant 0$ are independent of λ. Then there exists a unique $f \in \mathcal{E}'_{\delta,j}(X)$ such that $\mathcal{F}^{\delta}_j(f) = w$. In addition, $r(f) \leqslant R$ and $\operatorname{ord} f \leqslant \max\{0, \gamma_2 + 2\alpha_X + 5 + s(\delta)\}$. Next, if $\gamma_2 = -(2\alpha_X + 3 + s(\delta) + l)$ for some $l \in \mathbb{Z}_+$, then $f \in C^l_{\delta,j}(X)$ and

$$f(x) = \frac{1}{2}Y^{\delta}_j(kM)\int_{\mathfrak{a}^*} \mathrm{F}^{\delta}_j(f)(\lambda)\varphi_{\lambda,\delta}(ao)|\mathbf{c}(\lambda)|^{-2}\,\mathrm{d}\lambda \tag{10.97}$$

for each $x = kao \in X$.

Proof. For $f \in \mathcal{E}'_{\delta,j}(X)$, estimate (10.95) follows from (10.92) and (10.65). Next, if $f \in C^m_{\delta,j}(X)$, relations (10.92), (10.93), and (10.74) ensure us that (10.95) is satisfied with $\gamma_2 = -m - s(\delta)$. Thus, part (i) is established.

To prove (ii), first assume that $w : \mathfrak{a}^*_{\mathbb{C}} \to \mathbb{C}$ is an even entire function satisfying (10.96). Putting

$$\psi(\lambda, b) = w(\lambda) R_\delta(-\lambda) Y^\delta_j(b),$$

we see that $\psi \in C^\infty(\mathfrak{a}^*_{\mathbb{C}} \times B)$ and (10.81) holds. In addition, Proposition 10.3 and (10.55) yield (10.80). Then it follows by Theorem 10.7(ii) that $\psi(\lambda, b) = \widetilde{f}(\lambda, b)$ for some $f \in \mathcal{E}'(X)$ with supp $f \subset \mathring{B}_R$. Applying now Theorem 10.10, we deduce that

$$f \in \mathcal{E}'_{\delta,j}(X) \quad \text{and} \quad \mathcal{F}^\delta_j(f) = w. \tag{10.98}$$

Furthermore, the proof of Theorem 10.10, together with Remark 10.1, implies that if (10.96) holds with $\gamma_2 = -(2\alpha_X + 3 + s(\delta) + l)$ for some $l \in \mathbb{Z}_+$, then $f \in C^l_{\delta,j}(X)$ and (10.97) is fulfilled. Next, if (10.96) and (10.98) hold, then the above argument shows that there are functions $\varphi \in \mathcal{D}_{\delta,j}(X)$ and $\Phi \in C_{\delta,j}(X)$ such that

$$\mathcal{F}^\delta_j(f + \varphi)(\lambda) = \mathcal{F}^\delta_j(\Phi)(\lambda) p\big(-\langle\lambda, \lambda\rangle - |\rho|^2\big)$$

for some polynomial p of degree at most $\max\{0, \alpha_X + (s(\delta) + 5 + \gamma_2)/2\}$. This means that $f = p(L)\Phi - \varphi$, and therefore ord $f \leqslant \max\{0, \gamma_2 + 2\alpha_X + 5 + s(\delta)\}$. Hence the theorem. □

Corollary 10.4.

(i) *If $f \in \mathcal{E}'_{\delta,j}(X)$, $\varepsilon > 0$, and $f \in C(B_\varepsilon)$, then for each $d \in \mathbb{Z}_+$, there exists a polynomial q of degree d such that $q(L)F = f$ for some $F \in \mathcal{E}'_{\delta,j}(X)$.*

(ii) *If $R > 0$ and $f \in \mathcal{E}'_{\delta,j}(B_R)$, then for each polynomial q, there exists $F \in \mathcal{E}'_{\delta,j}(X)$ such that $q(L)F = f$ in B_R.*

Proof. To show (i) it is enough to consider the case $r(f) > 0$. Then the function $\mathcal{F}^\delta_j(f)$ has infinitely many zeros. Hence, for each $d \in \mathbb{Z}_+$, there is a a polynomial q of degree d such that the function

$$w(\lambda) = \mathcal{F}^\delta_j(f)(\lambda) \big/ q\big(-\langle\lambda, \lambda\rangle - |\rho|^2\big)$$

is entire. Define $F \in \mathcal{E}'_{\delta,j}(X)$ by the relation $\mathcal{F}^\delta_j(f) = w$ (see Theorem 10.11(ii)). Then $q(L)F = f$, and (i) is proved.

Turning to (ii), we select $\varphi \in \mathcal{D}_{\delta,j}(X)$ so that supp $\varphi \subset X \backslash \mathring{B}_R$ and the function $u(\lambda) = \mathcal{F}^\delta_j(f + \varphi)(\lambda)/q(-\langle\lambda, \lambda\rangle - |\rho|^2)$ is entire. Then $\mathcal{F}^\delta_j(F) = u$ for some $F \in \mathcal{E}'_{\delta,j}(X)$ because of Theorem 10.11(ii). By the definition of φ we conclude that $q(L)F = f$ in B_R, as contended. □

The following result is an analog of Proposition 9.9.

Proposition 10.11. *Let E be an infinite subset of $\mathbb{C}$, and let $A(E, \delta, j)$ be the set of all finite linear combinations of the functions $\Phi_{\mu,0,\delta,j}$ with $\mu \in E$. Then for each $R > 0$, the set $A(E, \delta, j)$ is dense in $C^\infty_{\delta,j}(B_R)$ with the topology induced by $C^\infty(B_R)$.*

Proof. The argument in the proof of Proposition 9.9 is applicable with minor modifications (see (10.92) and (10.27)). □

By analogy with Sect. 9.3 we put

$$\mathrm{conj}\big(\mathcal{E}'_{\delta,j}(X)\big) = \big\{f \in \mathcal{E}'(X) : \overline{f} \in \mathcal{E}'_{\delta,j}(X)\big\}.$$

Theorems 6.3 and 10.11(i) show that for each $T \in \mathrm{conj}(\mathcal{E}'_{\delta,j}(X))$, there is a unique distribution $\Lambda^{\delta,j}(T) \in \mathcal{E}'_\natural(\mathbb{R}^1)$ such that

$$\widehat{\Lambda^{\delta,j}(T)}(z) = \langle T, \Phi_{z,0,\delta,j}\rangle, \quad z \in \mathbb{C}.$$

Moreover,

$$\mathrm{ord}\, \Lambda^{\delta,j}(T) \leqslant \max\{0, \mathrm{ord}\, T - s(\delta) + 1\}, \qquad r\big(\Lambda^{\delta,j}(T)\big) = r(T), \tag{10.99}$$

and the transform $\Lambda^{\delta,j} : T \to \Lambda^{\delta,j}(T)$ sets up a bijection between $\mathrm{conj}(\mathcal{E}'_{\delta,j}(X))$ and $\mathcal{E}'_\natural(\mathbb{R}^1)$. If δ is the trivial representation and $j = 1$, we shall write Λ instead of $\Lambda^{\delta,j}$. For future use, we note also that

$$d_{\Lambda(T)} \leqslant 2 + \mathrm{ord}\, T, \quad \text{provided that } r(T) > 0, \tag{10.100}$$

because of (10.99) and (8.3).

10.6 The Transmutation Mapping $\mathfrak{A}_\delta$ Related to the Inversion Formula for the δ-Spherical Transform

As before, assume that $\delta \in \widehat{K}_M$. For an arbitrary $f \in \mathcal{E}'_{\check{\delta}}(X)$, we define $\mathfrak{A}_\delta(f) \in \mathcal{D}'_W(\mathfrak{a}, \mathrm{Hom}(V_\delta, V_\delta^M))$ by the formula

$$\big\langle \mathfrak{A}_\delta(f), \psi\big\rangle = \frac{1}{|W|}\int_{\mathfrak{a}^*} Q^{\check{\delta}}(\lambda)^* \widetilde{f}(\lambda)|\mathbf{c}(\lambda)|^{-2}\int_{\mathfrak{a}} \psi(H)\mathrm{v}_\lambda(H)\, \mathrm{d}H\, \mathrm{d}\lambda, \quad \psi \in \mathcal{D}(\mathfrak{a}), \tag{10.101}$$

where $\widetilde{f}$ is the δ-spherical transform of f, and

$$\mathrm{v}_\lambda(H) = \frac{1}{|W|}\sum_{s\in W} \mathrm{e}^{\mathrm{i}\lambda(sH)}, \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}}, \ H \in \mathfrak{a}. \tag{10.102}$$

We now establish some elementary but basic properties of the mapping $\mathfrak{A}_\delta$.

For $T \in \mathcal{E}'_\natural(X)$, we define the distributions $\Lambda_+(T), \Lambda_-(T) \in \mathcal{E}'_W(\mathfrak{a})$ by

$$\widehat{\Lambda_+(T)}(\lambda) = \big\langle \Lambda_+(T), \mathrm{e}^{-\mathrm{i}\lambda(H)}\big\rangle = \widetilde{T}(\lambda) = \widehat{\Lambda_-(T)}(-\lambda), \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}}. \tag{10.103}$$

Theorems 6.3 and 10.7 show that the mappings Λ_+ and Λ_- are bijections of $\mathcal{E}'_\natural(X)$ onto $\mathcal{E}'_W(\mathfrak{a})$ and

$$r(T) = r\big(\Lambda_+(T)\big) = r\big(\Lambda_-(T)\big). \tag{10.104}$$

Assume now that $a \in A^+$. Let $C(\log a)$ denote the convex hull of the set $\{H \in \mathfrak{a} : sH = \log a$ for some $s \in W\}$. Because of Theorems 10.2 and 6.3, there exists a matrix $m_{a,\delta} = (m_{a,\delta,\mu,\nu})$ with $d(\delta)$ rows and $l(\delta)$ columns whose entries $m_{a,\delta,\mu,\nu}$ are distributions in $\mathcal{E}'_W(\mathfrak{a})$ with the following properties:

(a) $\operatorname{supp} m_{a,\delta,\mu,\nu} \subset C(\log a)$ for all μ, ν;
(b) $\Phi_{\lambda,\delta}(ao)(Q^{\check{\delta}}(\overline{\lambda})^*)^{-1} = \langle m_{a,\delta}, \mathrm{e}^{\mathrm{i}\lambda(H)}\rangle$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$,

where the right-hand side is the matrix with entries $\langle m_{a,\delta,\mu,\nu}, \mathrm{e}^{\mathrm{i}\lambda(H)}\rangle$.
Also let ν_δ be defined as in Sect. 10.3.

Proposition 10.12.

(i) *Let $l \in \mathbb{Z}_+$ and suppose that $f \in (\mathcal{E}'_{\check{\delta}} \cap C^m)(X)$ with $m = \nu_\delta + \dim M +$ rank $X + l + 1$. Then $\mathfrak{A}_\delta(f) \in C^l_W(\mathfrak{a}, \mathrm{Hom}(V_\delta, V_\delta^M))$ and*

$$\mathfrak{A}_\delta(f)(H) = \frac{1}{|W|}\int_{\mathfrak{a}^*} Q^{\check{\delta}}(\lambda)^* \tilde{f}(\lambda)|\mathbf{c}(\lambda)|^{-2}\mathrm{v}_\lambda(H)\,\mathrm{d}\lambda, \quad H \in \mathfrak{a}. \tag{10.105}$$

(ii) *Let $f \in \mathcal{E}'_{\check{\delta}}(X)$, and let $u_1, u_2 \in \mathcal{D}_\natural(X)$. Then*

$$(f \times u_1 \times u_2)(ao) = \mathrm{Trace}\bigg(\int_{\mathfrak{a}^+}\big(m_{a,\delta} * \Lambda_-(u_1)\big)(H)\mathfrak{A}_\delta(f \times u_2)(H)\,\mathrm{d}H\bigg) \tag{10.106}$$

for all $a \in A^+$.
(iii) *If $f \in \mathcal{E}'_{\check{\delta}}(X)$ and $T \in \mathcal{E}'_\natural(X)$, then*

$$\mathfrak{A}_\delta(f \times T) = \mathfrak{A}_\delta(f) * \Lambda_+(T). \tag{10.107}$$

Proof. To prove (i), first observe that

$$\big|\widetilde{f}(\lambda, b)\big| \leqslant c(1 + |\lambda|)^{-m} \quad \text{for all } (\lambda, b) \in \mathfrak{a}^* \times B,$$

where the constant $c > 0$ is independent of λ, b (see Proposition 10.9(ii)). The first assertion is now clear from (10.101), (10.102), (10.36), and (10.84).

Turning to (ii), we have by (10.90) and Theorem 10.8(i)

$$(f \times u_1 \times u_2)(ao) = \frac{1}{|W|}\mathrm{Trace}\bigg(\int_{\mathfrak{a}^*}\Phi_{\lambda,\delta}(ao)\widetilde{u}_1(\lambda)\widetilde{f \times u_2}(\lambda)|\mathbf{c}(\lambda)|^{-2}\,\mathrm{d}\lambda\bigg), \quad a \in A^+. \tag{10.108}$$

It follows by the definition of $m_{a,\delta}$ and (10.102), (10.103) that

$$\Phi_{\lambda,\delta}(ao)\big(Q^{\check{\delta}}(\lambda)^*\big)^{-1}\widetilde{u}_1(\lambda) = \int_{\mathfrak{a}^+}\big(m_{a,\delta} * \Lambda_-(u_1)\big)(H)\mathrm{v}_\lambda(H)\,\mathrm{d}H, \quad \lambda \in \mathfrak{a}^*. \tag{10.109}$$

Using (10.109) together with (10.108) and (10.105), we obtain (10.106).

As for (iii), note that

$$\big\langle \mathfrak{A}_\delta(f) * \Lambda_+(T), \psi\big\rangle = \big\langle \mathfrak{A}_\delta(f)(H), \big\langle \Lambda_+(T)(\cdot), \psi(\cdot + H)\big\rangle\big\rangle \tag{10.110}$$

for each $\psi \in \mathcal{D}(\mathfrak{a})$. On the other hand, relations (10.101), (10.90), and (10.103) yield

$$\big\langle \mathfrak{A}_\delta(f \times T), \psi\big\rangle = \frac{1}{|W|}\int_{\mathfrak{a}^*} Q^{\check{\delta}}(\lambda)^*\widetilde{f}(\lambda)\widehat{\Lambda_+(T)}(\lambda)|\mathbf{c}(\lambda)|^{-2}\int_{\mathfrak{a}}\psi(H)\mathrm{v}_\lambda(H)\,\mathrm{d}H\,\mathrm{d}\lambda. \tag{10.111}$$

Comparing (10.110) with (10.111) and using (10.101), we arrive at (10.107). This completes the proof. □

We now prove that the mapping $\mathfrak{A}_\delta$ is injective on $\mathcal{E}'_{\check{\delta}}(X)$.

Proposition 10.13. *Let $f \in \mathcal{E}'_{\check{\delta}}(X)$ and let $r \in (0, +\infty]$. Then the following statements are equivalent.*

(i) $f = 0$ *in* B_r.
(ii) $\mathfrak{A}_\delta(f) = 0$ *in* $\mathcal{B}_r$.

Proof. (i)→(ii). Let $\varepsilon \in (0, r/2)$ and suppose that $u_1, u_2 \in \mathcal{D}_\natural(B_\varepsilon)$. By assumption on f we infer that $f \times u_1 \times u_2 = 0$ in $B_{r-2\varepsilon}$. Using Theorem 10.8(i), [123, Chap. 3, Proposition 5.10], (10.42), and (10.90), we find

$$\int_{\mathfrak{a}^*}\Phi_{\lambda,\delta}(x)\widetilde{u}_1(\lambda)\widetilde{f \times u_2}(\lambda)|\mathbf{c}(\lambda)|^{-2}\,\mathrm{d}\lambda = 0, \quad x \in B_{r-2\varepsilon}.$$

This gives, by (10.109) and (10.105), that

$$\int_{\mathfrak{a}^+}\big(m_{a,\delta} * \Lambda_-(u_1)\big)(H)\mathfrak{A}_\delta(f \times u_2)(H)\,\mathrm{d}H = 0, \quad |\log a| < r - 2\varepsilon. \tag{10.112}$$

Let $\psi \in \mathcal{D}'_W(\mathfrak{a}, \mathrm{Hom}(V_\delta, V_\delta^M))$ with supp $\psi \subset B_{r-2\varepsilon}$. Then there exists $\zeta \in \mathcal{D}_{\check{\delta}}(X)$ such that supp $\zeta \subset B_{r-2\varepsilon}$ and

$$\widetilde{\zeta}(\lambda) = Q^{\check{\delta}}(\lambda)\widehat{\psi}(\lambda), \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}} \tag{10.113}$$

(see Theorems 6.3 and 10.9). It follows by (10.83) and (10.8) that

$$\widetilde{\zeta}(\lambda) = \int_{A^+}\Phi_{\overline{\lambda},\delta}(ao)^*\zeta^\delta(ao)\Delta(a)\,\mathrm{d}a, \tag{10.114}$$

where, as usual, ζ^δ is defined by (10.17). Comparing (10.113) with (10.114), we find

$$\widehat{\psi}^*(-\lambda) = \int_{A^+} \zeta^\delta(ao)^* \Phi_{\lambda,\delta}(ao)\big(Q^{\check{\delta}}(\overline{\lambda})^*\big)^{-1} \Delta(a)\, \mathrm{d}a. \tag{10.115}$$

Now (10.115) and the properties of $m_{a,\delta}$ yield

$$\big(\psi * \overline{\Lambda\_(u_1)}\big)(H)^* = \int_{A^+} \zeta^\delta(ao)^* \big(m_{a,\delta} * \Lambda\_(u_1)\big)(H)\Delta(a)\, \mathrm{d}a.$$

In view of (10.112), this gives

$$\int_{\mathfrak{a}^+} \big(\psi * \overline{\Lambda\_(u_1)}\,\big)(H)\mathfrak{A}_\delta(f \times u_2)(H)\, \mathrm{d}H = 0. \tag{10.116}$$

Since $\varepsilon, u_1, u_2, \psi$ above were arbitrary, relations (10.116) and (10.107) imply that $\mathfrak{A}_\delta(f) = 0$ in $\mathcal{B}_r$.

(ii)→(i). As above, assume that $u_1, u_2 \in \mathcal{D}_\natural(B_\varepsilon)$ for some $\varepsilon \in (0, r/2)$. By (10.107), $\mathfrak{A}_\delta(f \times u_2) = \mathfrak{A}_\delta(f) * \Lambda_+(u_2)$. Then (10.104) and the assumption on $\mathfrak{A}_\delta(f)$ show that $\mathfrak{A}_\delta(f \times u_2) = 0$ in $\mathcal{B}_{r-\varepsilon}$. Using (10.106), (10.104), and the properties of $m_{a,\delta}$, we have $(f \times u_1 \times u_2)(ao) = 0$ when $|\log a| < r - 2\varepsilon$. Since $f \times u_1 \times u_2 \in \mathcal{D}_{\check{\delta}}(X)$ (see Proposition 10.2(ii)), we obtain $f \times u_1 \times u_2 = 0$ in $B_{r-2\varepsilon}$. Again, ε, u_1, u_2 being arbitrary, this shows that $f = 0$ in B_r. □

We shall now extend the mapping $\mathfrak{A}_\delta$ to the space $\mathcal{D}'_{\check{\delta}}(B_R)$, $R \in (0, +\infty]$. For $\psi \in \mathcal{E}'(\mathfrak{a})$, we set $r_0(\psi) = \sup\{r > 0 : \operatorname{supp} \psi \subset \mathcal{B}_r\}$.

Let $f \in \mathcal{D}'_{\check{\delta}}(B_R)$. We define $\mathfrak{A}_\delta(f) \in \mathcal{D}'_W(\mathcal{B}_R, \operatorname{Hom}(V_\delta, V_\delta^M))$ by the relation

$$\big\langle \mathfrak{A}_\delta(f), \psi \big\rangle = \big\langle \mathfrak{A}_\delta(f\eta), \psi \big\rangle, \quad \psi \in \mathcal{D}(\mathcal{B}_R), \tag{10.117}$$

where $\eta \in \mathcal{D}_\natural(B_R)$ and $\eta = 1$ in $B_{r_0(\psi)+\varepsilon}$ for some $\varepsilon \in (0, R - r_0(\psi))$. Propositions 10.2(i) and 10.13 show that $f\eta \in \mathcal{E}'_{\check{\delta}}(X)$ and that the right-hand side in (10.117) is independent of our choice of η. Moreover, for each $r \in (0, R]$, we have $f|_{B_r} \in \mathcal{D}'_{\check{\delta}}(B_r)$ and

$$\mathfrak{A}_\delta(f|_{B_r}) = \mathfrak{A}_\delta(f)|_{\mathcal{B}_r}.$$

We shall now obtain the following basic result concerning the properties of the mapping $\mathfrak{A}_\delta$.

Theorem 10.12. *Let $R \in (0, +\infty]$. Then the following assertions hold.*

(i) *The mapping $\mathfrak{A}_\delta : \mathcal{D}'_{\check{\delta}}(B_R) \to \mathcal{D}'_W(\mathcal{B}_R, \operatorname{Hom}(V_\delta, V_\delta^M))$ is continuous.*

(ii) *If $f \in \mathcal{D}'_{\check{\delta}}(B_R)$, $T \in \mathcal{E}'_\natural(X)$ and $r(T) < R$, then (10.107) holds in $\mathcal{B}_{R-r(T)}$.*

(iii) *Assume that $f_1, f_2 \in \mathcal{D}'_{\check{\delta}}(B_R)$ and let $r \in (0, R]$. Then $f_1 = f_2$ in B_r if and only if $\mathfrak{A}_\delta(f_1) = \mathfrak{A}_\delta(f_2)$ in $\mathcal{B}_r$.*

(iv) *Let $l \in \mathbb{Z}_+$, and let $f \in C^m_{\check{\delta}}(B_R)$ with $m = \nu_\delta + \dim M + \operatorname{rank} X + l + 1$. Then $\mathfrak{A}_\delta(f) \in C^l_W(\mathcal{B}_R, \operatorname{Hom}(V_\delta, V_\delta^M))$.*

(v) *Let* $f \in \mathcal{D}'_{\check{\delta}}(B_R)$ *be a distribution of finite order. Then every matrix entry of* $\mathfrak{A}_\delta(f)$ *is a distribution of finite order.*

Proof. For (i), assume that $\psi \in \mathcal{D}(\mathcal{B}_R)$, $\eta \in \mathcal{D}_\natural(B_R)$, and $\eta = 1$ in $B_{r_0(\psi)+\varepsilon}$ for some $\varepsilon \in (0, R - r_0(\psi))$. By (10.36) and (6.34), for each $\alpha > 0$, there exists $c_1 > 0$ such that

$$|\mathbf{c}(\lambda)|^{-2}\left|\int_{\mathfrak{a}} \psi(H)\mathrm{v}_\lambda(H)\,\mathrm{d}H\right| \leqslant c_1(1+|\lambda|)^{-\alpha} \quad \text{for all } \lambda \in \mathfrak{a}^*. \tag{10.118}$$

Suppose that $f_n \in \mathcal{D}'_{\check{\delta}}(B_R)$, $n = 1, 2, \ldots$, and let $f_n \to 0$ in $\mathcal{D}'(B_R)$ as $n \to \infty$. Formulae (10.117) and (10.101) yield

$$\langle \mathfrak{A}_\delta(f_n), \psi\rangle = \frac{1}{|W|}\int_{\mathfrak{a}^*} Q^{\check{\delta}}(\lambda)^* \widetilde{f_n\eta}(\lambda)|\mathbf{c}(\lambda)|^{-2}\int_{\mathfrak{a}} \psi(H)\mathrm{v}_\lambda(H)\,\mathrm{d}H\,\mathrm{d}\lambda. \tag{10.119}$$

By assumption on f_n one has $\widetilde{f_n\eta}(\lambda) \to 0$ for each $\lambda \in \mathfrak{a}^*$. In addition, there exist differential operators $D_1, \ldots, D_q$ on X such that for each $(\lambda, b) \in \mathfrak{a}^* \times B$,

$$\left|\widetilde{f_n\eta}(\lambda, b)\right| \leqslant c_2 \sum_{\nu=1}^{q} \sup_{x \in B_{r_0(\psi)}} \left|D_\nu\left(\eta(x)\mathrm{e}^{(-\mathrm{i}\lambda+\rho)(A(x,b))}\right)\right|,$$

where $c_2 > 0$ and $q \in \mathbb{N}$ are independent of n, λ, b (see [126, Theorem 2.1.8]). Then

$$\left|\widetilde{f_n\eta}(\lambda, b)\right| \leqslant (2+|\lambda|)^{c_3}, \tag{10.120}$$

where $c_3 > 0$ is independent of n, λ, b. Bearing (10.119) in mind and using (10.118), (10.120), and (10.84), by Lebesgue's dominated convergence theorem we obtain $\langle \mathfrak{A}_\delta(f_n), \psi\rangle \to 0$ as $n \to \infty$. This brings us to assertion (i).

The second assertion is an easy consequence of the definition of $\mathfrak{A}_\delta(f)$ for $f \in \mathcal{D}'_{\check{\delta}}(B_R)$ and Proposition 10.12(iii).

To prove (iii) we define $f = f_1 - f_2$. Applying (10.117) together with Proposition 10.13, we arrive at the desired result.

Turning to (iv), let $\varepsilon \in (0, R)$. We set $f_1 = f\eta_\varepsilon$, where $\eta_\varepsilon \in \mathcal{D}_\natural(B_R)$ and $\eta_\varepsilon = 1$ in $B_{R-\varepsilon}$. Owing to (iii), $\mathfrak{A}_\delta(f_1) = \mathfrak{A}_\delta(f)$ in $\mathcal{B}_{R-\varepsilon}$. Therefore,

$$\mathfrak{A}_\delta(f) \in C^l_W\left(\mathcal{B}_{R-\varepsilon}, \mathrm{Hom}\left(V_\delta, V_\delta^M\right)\right)$$

(see Proposition 10.12(i)). Since $\varepsilon \in (0, R)$ could be arbitrary, this proves (iv). Finally, it is not difficult to adapt the argument in the proof of (i) to show (v). □

For the rest of this section, we assume that δ is the trivial representation. In this case, for brevity, we shall write $\mathfrak{A}$ instead of $\mathfrak{A}_\delta$. We now turn the problem of inverting the mapping $\mathfrak{A}$.

Theorem 10.13. *For each* $R \in (0, +\infty]$, *the following statements are valid.*

(i) *Let* $f \in \mathcal{D}'_\natural(B_R)$, $m \in \mathbb{Z}_+$, *and assume that* $\mathfrak{A}(f) \in C^m_W(\mathcal{B}_R)$. *Then* $f \in C^m_\natural(\mathcal{B}_R)$ *and*

$$f(gK) = \int_K \mathfrak{A}(f)\big(H(gK)\big)e^{-\rho(H(gK))}\,dk \tag{10.121}$$

for each $g \in G$ such that $gK \in B_R$.

(ii) *Let $l \in \mathbb{Z}_+$, and let $f \in C^m_\natural(B_R)$ with $m = \dim M + \operatorname{rank} X + l + 1$. Then $\mathfrak{A}(f) \in C^l_W(\mathcal{B}_R)$, and (10.121) holds.*

(iii) $\mathfrak{A}(\varphi_\lambda) = \mathrm{v}_\lambda$ *for each $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$.*

To prove the theorem we begin with the following auxiliary statement.

Lemma 10.1. *Let $R > 0$, $u \in C_W(\mathcal{B}_R)$, and let*

$$\int_K u\big(H\big(a^{-1}k\big)\big)e^{-\rho(H(a^{-1}k))}\,dk = 0, \quad a \in A^+,\ |\log a| < R. \tag{10.122}$$

Then $u = 0$.

Proof. Let $\varepsilon \in (0, R)$ and assume that $\psi_1, \psi_2 \in \mathcal{D}_W(\mathfrak{a})$ have the following properties:

(a) $\operatorname{supp}\psi_1 \subset \mathcal{B}_R$ and $\psi_1 = 1$ in $\mathcal{B}_{R-\varepsilon/2}$;
(b) $\operatorname{supp}\psi_2 \subset \mathcal{B}_{R-\varepsilon}$.

We define $u_1 \in C_W(\mathfrak{a})$ by letting $u_1(x) = u(x)\psi_1(x)$ if $x \in \mathcal{B}_R$ and $u_1(x) = 0$ if $x \notin \mathcal{B}_R$. Owing to Theorems 6.3 and 10.7, there exists $v \in \mathcal{D}_\natural(X)$ such that $\operatorname{supp} v \subset B_{R-\varepsilon}$ and

$$\widetilde{v}(\overline{\lambda}) = \overline{\widehat{\psi_2}(\lambda)} \quad \text{for all } \lambda \in \mathfrak{a}^*_{\mathbb{C}}.$$

Next, if $\lambda \in \mathfrak{a}^*$, then

$$\overline{\widetilde{v}(\lambda)} = \int_{A^+} \overline{v(ao)}\Delta(a)\int_K e^{(-i\lambda-\rho)(H(a^{-1}k))}\,dk\,da$$

because of (10.8) and (10.31). Hence,

$$\begin{aligned}\int_{\mathfrak{a}^*} \widehat{u_1}(\lambda)\overline{\widehat{\psi_2}(\lambda)}\,d\lambda &= \int_{A^+} v(ao)\Delta(a) \\ &\quad\times \int_K\int_{\mathfrak{a}^*} \widehat{u_1}(\lambda)e^{i\lambda(H(a^{-1}k))}e^{-\rho(H(a^{-1}k))}\,d\lambda\,dk\,da \\ &= \int_{A^+} v(ao)\Delta(a)\int_K u_1\big(H\big(a^{-1}k\big)\big)e^{-\rho(H(a^{-1}k))}\,dk\,da.\end{aligned}$$

Since $v(ao) = 0$ for $|\log a| > R-\varepsilon$, this, together with (10.122) and the Plancherel formula for the Fourier transform on $\mathfrak{a}^*$, yields

$$\int_{\mathfrak{a}^*} u_1(H)\overline{\psi_2(H)}\,dH = 0.$$

Bearing in mind that u_1 and ψ_2 are W-invariant, one obtains $u_1 = 0$ in $\mathcal{B}_{R-\varepsilon}$ from the arbitrariness of ψ_2. As $u_1 = u$ in $\mathcal{B}_{R-\varepsilon/2}$ and $\varepsilon \in (0, R)$ is arbitrary, the desired conclusion follows. □

Proof of Theorem 10.13. To prove (i), suppose that $\eta \in \mathcal{D}_\natural(B_R)$ and $\eta = 1$ in $B_{R-\varepsilon/2}$ for some $\varepsilon \in (0, R)$. Assume that $\psi_n \in \mathcal{D}_W(\mathfrak{a})$ and

$$\psi_n \geqslant 0, \qquad \int_{\mathfrak{a}} \psi_n(H)\,\mathrm{d}H = 1, \qquad n \in \mathbb{N}. \tag{10.123}$$

Now define $f_n = f\eta \times w_n$, where $w_n = \Lambda_+^{-1}(\psi_n)$. Then $f_n \in \mathcal{D}_\natural(X)$ and, by Proposition 10.12(i),

$$\mathfrak{A}(f_n)(H) = \frac{1}{|W|}\int_{\mathfrak{a}^*} \widetilde{f_n}(\lambda)|\mathbf{c}(\lambda)|^{-2}\mathrm{v}_\lambda(H)\,\mathrm{d}\lambda, \qquad H \in \mathfrak{a}.$$

Thus, in view of Theorem 10.1,

$$\int_K \mathfrak{A}(f_n)\big(H(gk)\big)\mathrm{e}^{-\rho(H(gk))}\,\mathrm{d}k = \frac{1}{|W|}\int_{\mathfrak{a}^*} \widetilde{f_n}(\lambda)|\mathbf{c}(\lambda)|^{-2}\varphi_\lambda(gK)\,\mathrm{d}\lambda, \qquad g \in G.$$

Now Remark 10.2 and this last equality give us

$$\int_K \big(\mathfrak{A}(f\eta) * \psi_n\big)\big(H(gk)\big)\mathrm{e}^{-\rho(H(gk))}\,\mathrm{d}k = \big((f\eta) \times w_n\big)(gK), \qquad g \in G. \tag{10.124}$$

By assumption on $\mathfrak{A}(f)$, Theorem 10.12(iii), and (10.123), we see that $\mathfrak{A}(f\eta) * \psi_n$ converges to $\mathfrak{A}(f)$ uniformly on $\mathcal{B}_{R-\varepsilon}$. In addition, $|\widetilde{w_n}(\lambda)| = |\widehat{\psi_n}(\lambda)| \leqslant \widehat{\psi_n}(0) = 1$ for each $\lambda \in \mathfrak{a}^*$, whence $f\eta \times w_n \to f\eta$ in $\mathcal{D}'(X)$ as $n \to \infty$ (see Corollary 10.3). Thus, (10.124) and (10.1) imply (10.121) for each $g \in G$ such that $gK \in B_{R-\varepsilon}$. In particular, $f \in C_\natural^m(B_{R-\varepsilon})$. Since $\varepsilon \in (0, R)$ above was arbitrary, this proves (i).

Assertion (ii) is a direct consequence of (i) and Theorem 10.12(iv).

As for (iii), observe that for all $g \in G$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$,

$$\varphi_\lambda(gK) = \int_K \mathfrak{A}(\varphi_\lambda)\big(H(gk)\big)\mathrm{e}^{-\rho(H(gk))}\,\mathrm{d}k = \int_K \mathrm{v}_\lambda\big(H(gk)\big)\mathrm{e}^{-\rho(H(gk))}\,\mathrm{d}k$$

(see (10.121) and Theorem 10.1). Assertion (iii) is now obvious from Lemma 10.1. □

To continue, for each $F \in \mathcal{E}'_W(\mathfrak{a})$, we define the distribution $\mathfrak{B}(F) \in \mathcal{D}'_\natural(X)$ acting in $\mathcal{D}(X)$ by the formula

$$\big\langle \mathfrak{B}(F), w\big\rangle = \int_{\mathfrak{a}^*} \widehat{F}(\lambda)\widetilde{w}(-\lambda)\,\mathrm{d}\lambda, \qquad w \in \mathcal{D}(X), \tag{10.125}$$

where $\widetilde{w}(\lambda) = \int_X w(x)\varphi_{-\lambda}(x)\,\mathrm{d}x$. Theorems 10.7 and 6.3 show that the right-hand side in (10.125) is well defined. In addition, if $F \in \mathcal{D}_W(\mathfrak{a})$, then $\mathfrak{B}(F) \in C_\natural^\infty(X)$

and

$$\mathfrak{B}(F)(x) = \int_{\mathfrak{a}^*} \widehat{F}(\lambda)\varphi_\lambda(x)\,\mathrm{d}\lambda, \quad x \in X \tag{10.126}$$

(see Theorem 10.1).

Proposition 10.14. (i) *Let $F \in \mathcal{E}'_W(\mathfrak{a})$ and $T \in \mathcal{E}'_\natural(X)$. Then*

$$\mathfrak{B}(F) \times T = \mathfrak{B}\big(F * \Lambda_+(T)\big). \tag{10.127}$$

(ii) *Let $F \in \mathcal{D}_W(\mathfrak{a})$. Then*

$$\mathfrak{B}(F)(gK) = \int_K F\big(H(gk)\big)\mathrm{e}^{-\rho(H(gk))}\,\mathrm{d}k \tag{10.128}$$

for all $g \in G$.

(iii) *Assume that $F \in \mathcal{E}'_W(\mathfrak{a})$ and let $R \in (0, +\infty]$. Then $F = 0$ in $\mathcal{B}_R$ if and only if $\mathfrak{B}(F) = 0$ in B_R.*

Proof. Using (10.10), (10.125), and (10.6), we have in (i)

$$\big\langle \mathfrak{B}(F) \times T, w\big\rangle = \int_{\mathfrak{a}^*} \widehat{F}(\lambda)\Big\langle T, \int_G \varphi_\lambda(gK)w(gx)\,\mathrm{d}g\Big\rangle \mathrm{d}\lambda \tag{10.129}$$

for each $w \in \mathcal{D}(X)$. Next, taking (10.31) and (10.32) into account, we find

$$\begin{aligned}\Big\langle T, \int_G \varphi_\lambda(gK)w(gx)\mathrm{d}g\Big\rangle &= \int_K \big\langle T(hK), \mathrm{e}^{(\mathrm{i}\lambda+\rho)(A(kh^{-1}))}\big\rangle \\ &\quad \times \int_G w(gK)\mathrm{e}^{(-\mathrm{i}\lambda+\rho)(A(kg^{-1}))}\,\mathrm{d}g\,\mathrm{d}k \\ &= \widetilde{w}(-\lambda)\int_K \big\langle T(hK), \mathrm{e}^{(\mathrm{i}\lambda+\rho)(A(kh^{-1}))}\big\rangle\,\mathrm{d}k \\ &= \widetilde{w}(-\lambda)\widetilde{T}(\lambda).\end{aligned}$$

Now (10.129) and (10.125) yield

$$\big\langle \mathfrak{B}(F) \times T, w\big\rangle = \int_{\mathfrak{a}^*} \widehat{F}(\lambda)\widetilde{T}(\lambda)\widetilde{w}(-\lambda)\,\mathrm{d}\lambda = \big\langle \mathfrak{B}\big(F * \Lambda_+(T)\big), w\big\rangle,$$

as contended.

In (ii), first note that

$$F(H) = \int_{\mathfrak{a}^*} \widehat{F}(\lambda)\mathrm{e}^{\mathrm{i}\lambda(H)}\,\mathrm{d}\lambda, \quad H \in \mathfrak{a}.$$

Therefore, by (10.31) we obtain

$$\int_K F\big(H(gk)\big)\mathrm{e}^{-\rho(H(gk))}\,\mathrm{d}k = \int_{\mathfrak{a}^*} \widehat{F}(\lambda)\varphi_\lambda(gK)\,\mathrm{d}\lambda, \quad g \in G.$$

This, together with (10.126), implies (10.128).

To prove (iii) it is enough to consider the case where $F \in \mathcal{D}_W(\mathfrak{a})$. The general case reduces to this one by means of the standard smoothing trick (see (10.127)). Now if $F = 0$ in $\mathcal{B}_R$, then (10.128) and (10.1) give us $\mathfrak{B}(F) = 0$ in B_R. The converse statement follows by (10.128) and Lemma 10.1. □

Now we can extend the mapping $\mathfrak{B}$ to the space $\mathcal{D}'_W(\mathcal{B}_R)$, $R \in (0, +\infty]$.

Let $F \in \mathcal{D}'_W(\mathcal{B}_R)$. We define $\mathfrak{B}(F) \in \mathcal{D}'_\natural(B_R)$ by the relation

$$\big\langle \mathfrak{B}(F), w \big\rangle = \big\langle \mathfrak{B}(F\eta), w \big\rangle, \quad w \in \mathcal{D}(B_R), \tag{10.130}$$

where $\eta \in \mathcal{D}_W(\mathcal{B}_R)$ and $\eta = 1$ in $\mathcal{B}_{r_0(w)+\varepsilon}$ for some $\varepsilon \in (0, R - r_0(w))$. Owing to Proposition 10.14(iii), the right-hand side in (10.130) is independent of our choice of η. Furthermore, we see that

$$\mathfrak{B}(F|_{\mathcal{B}_r}) = \mathfrak{B}(F)|_{B_r} \quad \text{for any } r \in (0, R].$$

Theorem 10.14. *For each $R \in (0, +\infty]$, the following statements are true.*

(i) *The mapping $\mathfrak{B} : \mathcal{D}'_W(\mathcal{B}_R) \to \mathcal{D}'_\natural(B_R)$ is continuous.*
(ii) *If $F \in \mathcal{D}'_W(\mathcal{B}_R)$, $T \in \mathcal{E}'_\natural(X)$, and $r(T) < R$, then (10.127) holds in $B_{R-r(T)}$.*
(iii) *Assume that $F_1, F_2 \in \mathcal{D}'_W(\mathcal{B}_R)$ and let $r \in (0, R]$. Then $F_1 = F_2$ in $\mathcal{B}_r$ if and only if $\mathfrak{B}(F_1) = \mathfrak{B}(F_2)$ in B_r.*
(iv) *Let $F \in C^m_W(\mathcal{B}_R)$ for some $m \in \mathbb{Z}_+$. Then $\mathfrak{B}(F) \in C^m_\natural(B_R)$, and (10.128) holds for each $g \in G$ such that $gK \in B_R$.*

Proof. The proof of (i) is analogous to that of a similar result about the mapping $\mathfrak{A}_\delta$ (see Theorem 10.12(i)). Assertions (ii) and (iii) follow by the definition of $\mathfrak{B}$ on $\mathcal{D}'_W(\mathcal{B}_R)$ and Proposition 10.14(i), (iii). Let us prove (iv). Assume that $\varepsilon \in (0, R)$, $\eta \in \mathcal{D}_W(\mathcal{B}_R)$, and $\eta = 1$ in $\mathcal{B}_{R-\varepsilon/2}$. We set

$$F_n = F\eta * \psi_n,$$

where the functions $\psi_n \in \mathcal{D}_W(\mathcal{B}_{\varepsilon/n})$ satisfy (10.123). Then $F_n \in \mathcal{D}_W(\mathfrak{a})$, and by Proposition 10.14(ii),

$$\mathfrak{B}(F_n)(gK) = \int_K F_n\big(H(gk)\big)\mathrm{e}^{-\rho(H(gk))}\,\mathrm{d}k \tag{10.131}$$

for all $g \in G$. It follows by (10.123) that F_n converges to F as $n \to \infty$ uniformly on $\mathcal{B}_{R-\varepsilon}$. Using now (i), (iii), (10.1), and (10.131), we obtain (10.128) for every $g \in G$ such that $gK \in B_{R-\varepsilon}$. Since $\varepsilon \in (0, R)$ is arbitrary, this, together with (10.1), gives (iv). Hence the theorem. □

The following result relates the mapping $\mathfrak{A}$ to the mapping $\mathfrak{B}$.

Theorem 10.15. *Let $R \in (0, +\infty]$. Then the transform $\mathfrak{A}$ sets up a homeomorphism between:*

(i) $\mathcal{D}'_{\natural}(B_R)$ *and* $\mathcal{D}'_W(\mathcal{B}_R)$*;*
(ii) $C^\infty_{\natural}(B_R)$ *and* $C^\infty_W(\mathcal{B}_R)$*.*

In addition, $\mathfrak{A}^{-1} = \mathfrak{B}$*.*

Proof. Let $F \in \mathcal{D}'_W(\mathcal{B}_R)$, and let $\varepsilon \in (0, R)$. We define

$$F_n = F * \psi_n,$$

where the functions $\psi_n \in \mathcal{D}_W(\mathcal{B}_{\varepsilon/n})$ satisfy (10.123). It is clear that $F_n \in C^\infty_W(\mathcal{B}_{R-\varepsilon/n})$ and $F_n \to F$ in $\mathcal{D}'(\mathcal{B}_R)$ as $n \to \infty$. Because of Theorem 10.14(iv), equality (10.131) holds for all $g \in G$ such that $gK \in B_{R-\varepsilon/n}$. Combining this with Theorem 10.13(ii), we deduce

$$\int_K \big(F_n\big(H(gk)\big) - \mathfrak{A}\big(\mathfrak{B}(F_n)\big)\big(H(gk)\big)\big)\mathrm{e}^{-\rho(H(gk))}\,\mathrm{d}k = 0$$

for any $g \in G$ such that $gK \in B_{R-\varepsilon/n}$. Then Lemma 10.1 ensures us that $\mathfrak{A}(\mathfrak{B}(F_n)) = F_n$ in $\mathcal{B}_{R-\varepsilon/n}$. Letting $n \to \infty$ and applying Theorem 10.14(i) and Theorem 10.12(i), we get $\mathfrak{A}(\mathfrak{B}(F)) = F$ in $\mathcal{B}_R$. Since $\mathfrak{A}$ is injective on $\mathcal{D}'_{\natural}(B_R)$, this gives the desired result for the spaces $\mathcal{D}'_{\natural}(B_R)$ and $\mathcal{D}'_W(\mathcal{B}_R)$. The case of $C^\infty_{\natural}(B_R)$ and $C^\infty_W(\mathcal{B}_R)$ can be proved by an obvious modification of the above argument. □

10.7 The Class $\mathcal{E}'_{\natural\natural}(X)$ of Distributions with Radial Spherical Transform. Mean Value Characterization. Explicit Form for $X = G/K$ (G Complex)

Let $\mathcal{E}'_{\natural\natural}(X)$ be the set of all distributions $T \in \mathcal{E}'_{\natural}(X)$ with the following property:

$$\widetilde{T}(\lambda) = \widetilde{T}(\mu) \quad \text{for all } \lambda, \mu \in \mathfrak{a}^* \text{ such that } |\lambda| = |\mu|. \tag{10.132}$$

From the Paley–Wiener theorem for the spherical transform (see Theorem 10.7) it follows that the class $\mathcal{E}'_{\natural\natural}(X)$ is broad enough (see also Theorem 10.17 below). We point out that

$$\mathcal{E}'_{\natural\natural}(X) = \mathcal{E}'_{\natural}(X), \quad \text{provided that } \operatorname{rank} X = 1.$$

The class $\mathcal{E}'_{\natural\natural}(X)$ will play an important role later on in this book. In particular, in Part III we shall see that for the equation $f \times T = 0$ with $T \in \mathcal{E}'_{\natural\natural}(X)$, there exist natural analogs of some Euclidean results from Sect. 14.2 and [225, Part III].

Let us give the following characterization of the class $\mathcal{E}'_{\natural\natural}(X)$.

Theorem 10.16. *If* $T \in \mathcal{E}'_{\natural}(X)$*, then the following assertions are equivalent.*

(i) $T \in \mathcal{E}'_{\natural\natural}(X)$.

(ii) *For each* $\lambda \in \mathfrak{a}^*$, *every solution* $f \in C^\infty(X)$ *of the equation*

$$Lf = -\big(|\lambda|^2 + |\rho|^2\big) f \tag{10.133}$$

satisfies the equality

$$f \times T = \widetilde{T}(\lambda) f. \tag{10.134}$$

We note that if rank $X = 1$, then this result is the well-known mean value theorem for eigenfunctions of the operator L. In the general case equality (10.134) holds for each $T \in \mathcal{E}'_\natural(X)$ if f belongs to the corresponding joint eigenspace of all G-invariant differential operators on X (see Proposition 1.3).

Proof of Theorem 10.16. For brevity, we set $n = \text{rank}\, X$. In view of what has been said above, it is enough to consider the case $n \geqslant 2$. We choose a basis of $\mathfrak{a}$ such that for each $H = (H_1, \dots, H_n) \in \mathfrak{a}$, it follows that $|H|^2 = \sum_{i=1}^n H_i^2$. Then the Laplace–Beltrami operator $L_\mathfrak{a}$ on $\mathfrak{a}$ has the form $L_\mathfrak{a} = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, the Riemannian structure on $\mathfrak{a}$ being defined by the Killing form $\langle , \rangle$ restricted to $\mathfrak{a}$. If $\delta \in \widehat{K}_M$, then

$$\mathfrak{A}_\delta(Lu) = \big(L_\mathfrak{a} - |\rho|^2\big)\mathfrak{A}_\delta(u) \tag{10.135}$$

for each $u \in \mathcal{D}'_{\check{\delta}}(X)$ (see Theorem 10.12(ii)).

Assume now that (i) is true and let $f \in C^\infty(X)$ satisfy (10.133) with some $\lambda \in \mathfrak{a}^*$. For $\delta \in \widehat{K}_M$, relation (10.133) yields

$$Lu = -\big(|\lambda|^2 + |\rho|^2\big)u,$$

where $u = f_\delta$. Bearing (10.135) in mind, we obtain $L_\mathfrak{a}\mathfrak{A}_\delta(u) = -|\lambda|^2\mathfrak{A}_\delta(u)$. Using (10.132) and (10.103), one sees that $\Lambda_+(T)$ is radial, whence

$$\mathfrak{A}_\delta(u) * \Lambda_+(T) = \widetilde{T}(\lambda)\mathfrak{A}_\delta(u)$$

(see [225, Part I, (7.10)]). Thus, $u \times T = \widetilde{T}(\lambda)u$ because of Theorem 10.12(iii). Since $\delta \in \widehat{K}_M$ above was arbitrary, this, together with Proposition 10.2(iii), gives (10.134).

The proof of the implication (ii)→(i) requires the following auxiliary fact.

Lemma 10.2. *Let* $U \in \mathcal{E}'(\mathbb{R}^n)$, $n \geqslant 2$, *and assume that for each* $\mu > 0$, *there exists* $c_\mu \in \mathbb{C}$ *such that* $\Phi_{\mu,0,0,1} * U = c_\mu \Phi_{\mu,0,0,1}$ (*see* (9.13)). *Then* $U \in \mathcal{E}'_\natural(\mathbb{R}^n)$.

Proof. For $\tau \in \text{O}(n)$ and $\mu > 0$, we define the distribution $u_{\tau,\mu} \in \mathcal{E}'(\mathbb{R}^n)$ by the formula

$$\langle u_{\tau,\mu}, \psi\rangle = \big\langle U, \psi\big(\tau^{-1}x\big)\big\rangle - c_\mu \psi(0), \quad \psi \in \mathcal{D}\big(\mathbb{R}^n\big).$$

Then $\Phi_{\mu,0,0,1} * u_{\tau,\mu} = 0$ for all $\tau \in \text{O}(n)$. Applying [225, Part I, Proposition 5.7(1)] together with recursion relations for the Bessel functions (see (7.3)), we infer that $\Phi_{\mu,0,k,j} * u_{\tau,\mu} = 0$ for all $k \in \mathbb{Z}_+$, $j \in \{1, \dots, d(n,k)\}$. In particular,

$$(\Phi_{\mu,0,k,j} * U)(0) = 0, \quad \text{provided that } k \geqslant 1. \tag{10.136}$$

Let $\eta \in \mathbb{S}^{n-1}$, and let $f(x) = e^{i\langle\mu\eta,x\rangle_{\mathbb{R}}}$, where $x \in \mathbb{R}^n$, and $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ is the inner product in $\mathbb{R}^n$. According to (5.2), every term of decomposition (9.9) has the form $f^{k,j} = \gamma_{k,j,\mu,\eta}\Phi_{\mu,0,k,j}$ for some $\gamma_{k,j,\mu,\eta} \in \mathbb{C}$. Therefore, $\widehat{U}(\mu\eta) = (f * U)(0) = (f^{0,1} * U)(0)$ in view of (10.136). Hence, $\widehat{U}(\mu\eta)$ is independent of η, and the lemma is thereby established. □

Coming back to the proof of the implication (ii)→(i), we define $f \in C^\infty_\natural(X)$ by the relation $\mathfrak{A}(f) = \Phi_{\mu,0,0,1}$, where $\mu = |\lambda|$ for some $\lambda \in \mathfrak{a}^*$ (see Theorem 10.14(iv)). Owing to (9.30), $L_{\mathfrak{a}}\mathfrak{A}(f) = -|\lambda|^2\mathfrak{A}(f)$. Hence, (10.133) is satisfied because of (10.135) and Theorem 10.12(iii). Equality (10.134) and Theorem 10.12(ii) imply that $\mathfrak{A}(f) * \Lambda_+(T) = \widetilde{T}(\lambda)\mathfrak{A}(f)$. Since $\lambda \in \mathfrak{a}^*$ could be arbitrary, Lemma 10.2 shows that $\Lambda_+(T)$ is radial. Taking (10.103) into account, we obtain (10.132). Thus, $T \in \mathcal{E}'_{\natural\natural}(X)$, and the proof of Theorem 10.16 is complete. □

To continue, let $T \in \mathcal{E}'_{\natural\natural}(X)$. Condition (10.132) and Theorem 10.7 ensure us that there exists an even entire function $\overset{\circ}{T} : \mathbb{C} \to \mathbb{C}$ such that

$$\widetilde{T}(\lambda) = \overset{\circ}{T}\big(\sqrt{\langle\lambda,\lambda\rangle}\big) \quad \text{for all } \lambda \in \mathfrak{a}^*_{\mathbb{C}}. \tag{10.137}$$

As an application of Theorem 10.7, we now obtain the following analog of Corollary 6.2.

Proposition 10.15. *Let $T \in \mathcal{E}'_{\natural\natural}(X)$. Then the following statements are equivalent.*

(i) $r(T) = 0$.
(ii) $\overset{\circ}{T}$ *is a polynomial.*
(iii) $T = p(L)\delta_0$ *for some polynomial p, where δ_0 is the Dirac measure supported at origin.*

Proof. (i)→(ii). By Theorem 10.7(ii), $|\overset{\circ}{T}(z)| \leqslant \gamma_1(1+|z|)^{\gamma_2}$, $z \in \mathbb{C}$, where the constants $\gamma_1, \gamma_2 > 0$ are independent of z. Now it follows by Liouville's theorem that assertion (ii) is valid.

(ii)→(iii). Define the polynomial p by $p(z) = \overset{\circ}{T}(\sqrt{-z-|\rho|^2})$. Then (10.73) and Theorem 10.7(ii) imply (iii).

Since the implication (iii)→(i) is trivial, this concludes the proof. □

In the case where the group G is complex, the following result gives an explicit form for distributions in the class $(\mathcal{E}'_{\natural\natural} \cap L^1)(X)$.

Theorem 10.17. *Let $X = G/K$ be a symmetric space of noncompact type with complex group G, and let $T \in (\mathcal{E}'_\natural \cap L^1)(X)$. Then the following assertions are equivalent.*

(i) *T has the form*

$$T(x) = \big(J\big(\mathrm{Exp}^{-1}x\big)\big)^{-1/2}u\big(d(o,x)\big), \quad x \in X,$$

for some function $u : [0, +\infty) \to \mathbb{C}$ (*see* (10.39)).
(ii) $T \in \mathcal{E}'_{\natural\natural}(X)$.

Proof. First, observe that for each $T \in (\mathcal{E}'_{\natural} \cap L^1)(X)$, formulae (10.40) and (10.38) yield

$$\begin{aligned}\widetilde{T}(\lambda) &= \int_{\mathfrak{p}} T(\operatorname{Exp} P)\varphi_{-\lambda}(\operatorname{Exp} P)J(P)\,\mathrm{d}P \\ &= \int_K \int_{\mathfrak{p}} T(\operatorname{Exp} P)\big(J(P)\big)^{1/2}\mathrm{e}^{-\mathrm{i}\langle A_\lambda, \operatorname{Ad}(k)P\rangle}\,\mathrm{d}P\,\mathrm{d}k \\ &= \int_K \int_{\mathfrak{p}} T\big(\operatorname{Exp}\operatorname{Ad}(k)P\big)\big(J\big(\operatorname{Ad}(k)P\big)\big)^{1/2}\mathrm{e}^{-\mathrm{i}\langle A_\lambda, \operatorname{Ad}(k)P\rangle}\,\mathrm{d}P\,\mathrm{d}k \\ &= \int_{\mathfrak{p}} T(\operatorname{Exp} P)\big(J(P)\big)^{1/2}\mathrm{e}^{-\mathrm{i}\langle A_\lambda, P\rangle}\,\mathrm{d}P, \quad \lambda \in \mathfrak{a}^*. \end{aligned} \tag{10.138}$$

If (i) is true, relation (10.138) can be written

$$\widetilde{T}(\lambda) = \int_{\mathfrak{a}} \mathrm{e}^{-\mathrm{i}\lambda(H)} \int_{\mathfrak{q}} u(|H + Q|)\,\mathrm{d}Q\,\mathrm{d}H,$$

where $\mathfrak{q}$ is the orthogonal complement to $\mathfrak{a}$ in $\mathfrak{p}$. Since $|H + Q| = \sqrt{|H|^2 + |Q|^2}$, this, together with (10.132), implies (ii).

Let us prove the implication (ii)$\to$(i). By regularization we may assume that $T \in (\mathcal{E}'_{\natural\natural} \cap \mathcal{D})(X)$. In view of (10.132), there exists a function $u : [0, +\infty) \to \mathbb{C}$ such that

$$\widetilde{T}(\lambda) = \int_{\mathfrak{p}} u(|P|)\mathrm{e}^{-\mathrm{i}\langle A_\lambda, P\rangle}\,\mathrm{d}P, \quad \lambda \in \mathfrak{a}^*.$$

Combining this with (10.138) and using the fact that $T(\operatorname{Exp} P)(J(P))^{1/2} - u(|P|)$ is a K-invariant function, we arrive at (i). Hence the theorem. □

Assume that $T \in \mathcal{E}'_{\natural\natural}(X)$, $T \neq 0$ and $\mathcal{Z}(\overset{\circ}{T}) \neq \varnothing$. We set $n(\lambda, T) = n_\lambda(\overset{\circ}{T}) - 1$ if $\lambda \neq 0$ and $n(\lambda, T) = n_\lambda(\overset{\circ}{T})/2 - 1$ if $\lambda = 0 \in \mathcal{Z}(\overset{\circ}{T})$, where $n_\lambda(\overset{\circ}{T})$ is the multiplicity of $\lambda \in \mathcal{Z}(\overset{\circ}{T})$. Thanks to Theorem 10.7, for each $\lambda \in \mathcal{Z}(\overset{\circ}{T})$, there exists $T_{(\lambda)} \in \mathcal{E}'_{\natural\natural}(X)$ such that $r(T_{(\lambda)}) = r(T)$ and

$$\overset{\circ}{T}_{(\lambda)}(z)\big(z^2 - \lambda^2\big)^{n(\lambda,T)+1} = \overset{\circ}{T}(z), \quad z \in \mathbb{C}. \tag{10.139}$$

Using (10.139), one sees that

$$\big(-L - \lambda^2 - |\rho|^2\big)^{n(\lambda,T)+1} T_{(\lambda)} = T.$$

Our next task is to prove the following analog of Corollary 8.6.

Theorem 10.18. *Let* $T \in \mathcal{E}'_{\natural\natural}(X)$, $T \neq 0$, $\mathcal{Z}(\overset{\circ}{T}) \neq \varnothing$, $R > r(T)$, $f \in \mathcal{D}'(B_R)$, *and let*

$$f \times T_{(\lambda)} = 0 \quad \text{in } B_{R-r(T)} \quad \text{for all } \lambda \in \mathcal{Z}\big(\overset{\circ}{T}\big). \tag{10.140}$$

Then $f = 0$.

Proof. By regularization it is enough to consider the case $f \in C^\infty(B_R)$. It follows by (10.140), (10.18), and Theorem 10.12(ii), (iii) that

$$\mathfrak{A}_\delta(f_\delta) * \Lambda_+(T_{(\lambda)}) = 0 \quad \text{in } \mathcal{B}_{R-r(T)}$$

for all $\delta \in \widehat{K}_M$ and $\lambda \in \mathcal{Z}(\overset{\circ}{T})$. Since $\Lambda_+(T_{(\lambda)}) = (\Lambda_+(T))_{(\lambda)}$, this, together with Corollary 8.6, Theorem 10.12(iii), and [123, Chap. 3, Proposition 5.10], brings us to the desired result. □

We now define the mapping $\Lambda : \mathcal{E}'_{\natural\natural}(X) \to \mathcal{E}'_\natural(\mathbb{R}^1)$ by the relation

$$\overset{\circ}{T}(z) = \widehat{\Lambda(T)}(z), \quad T \in \mathcal{E}'_{\natural\natural}(X),\ z \in \mathbb{C}. \tag{10.141}$$

It follows by Theorems 10.7 and 6.3 that the transform $\Lambda : T \to \Lambda(T)$ sets up a bijection between $\mathcal{E}'_{\natural\natural}(X)$ and $\mathcal{E}'_\natural(\mathbb{R}^1)$.

Suppose that $\alpha > 0$ and let $\mathfrak{W}(\mathbb{R}^1)$ denote one of the classes $\mathfrak{M}(\mathbb{R}^1)$, $\mathfrak{U}(\mathbb{R}^1)$, $\mathfrak{N}(\mathbb{R}^1)$, $\mathfrak{E}(\mathbb{R}^1)$, $\mathfrak{G}_\alpha(\mathbb{R}^1)$, $\mathrm{Inv}_+(\mathbb{R}^1)$ (see Sect. 8.1). Set

$$\mathfrak{W}(X) = \big\{T \in \mathcal{E}'_{\natural\natural}(X) : \Lambda(T) \in \mathfrak{W}\big(\mathbb{R}^1\big)\big\}.$$

In addition, we put $\mathfrak{D}_\alpha(X) = (\mathfrak{G}_\alpha \cap \mathfrak{U})(X)$.

The introduced classes of distributions play an important role in the theory of mean periodic functions (see Chap. 15 in Part III).

10.8 Some Rank One Results on the Mapping $\mathfrak{A}_\delta$

Throughout the section we assume that rank $X = 1$ and that $\delta \in \widehat{K}_M$ and $j \in \{1, \dots, d(\delta)\}$ are fixed. We shall now investigate the mapping $\mathfrak{A}_\delta$ in greater detail.

According to results in Sect. 10.3, we define the differential operator $\mathbf{D}^\delta$ of order $s(\delta)$ by the relation

$$\mathbf{D}^\delta e^{i\lambda' t} = R_\delta(\lambda) e^{i\lambda' t}, \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}},\ t \in \mathbb{R}^1, \tag{10.142}$$

where $\lambda' = \lambda(H_0)/|H_0|$. For $r > 0$, $\zeta \in (-r, r)$, let

$$c_\delta(r) = \frac{\kappa 2^{1/2-\xi}\Gamma(\xi+1)}{\sqrt{\pi}\,\Gamma(\xi+1/2)}(\sinh r)^{s(\delta)-2\xi}(\cosh r)^{-s(\delta)-\alpha_X-\beta_X}$$

and

$$w_\delta(r,\zeta) = c_\delta(r)(\cosh 2r - \cosh 2\zeta)^{\xi-1/2} \times F\left(\xi+\eta, \xi-\eta; \xi+1/2; \frac{\cosh r - \cosh\zeta}{2\cosh r}\right), \quad (10.143)$$

where ξ and η are given by (10.49).

As a preparatory of development in this section, we now prove existence and uniqueness results for solutions of integral equations of a special form.

Theorem 10.19. *Let $R > 0$, $u \in C^{s(\delta)}(-R,R)$, $k_0 \in K$, and $Y_j^\delta(k_0M) \neq 0$. Then the following items are equivalent.*

(i) *The function $\mathbf{D}^\delta u$ is odd.*
(ii) *For each $t \in (0,\kappa R)$,*

$$\int_K u\big(h\big(\tau^{-1}k_0a_to\big)\big)\mathrm{e}^{\rho_X h(\tau^{-1}k_0a_to)}Y_j^\delta(\tau M)\,\mathrm{d}\tau = 0,$$

where $a_t = \exp(tH_0)$, and h is defined as in Sect. 10.1.

The proof starts with the following auxiliary result.

Lemma 10.3. *Let $x = ka_ro$ where $k \in K$ and $r > 0$. Then*

$$\int_K u\big(h\big(\tau^{-1}x\big)\big)\mathrm{e}^{\rho_X h(\tau^{-1}x)}Y_j^\delta(\tau M)\,\mathrm{d}\tau = Y_j^\delta(kM)\int_{-r/\kappa}^{r/\kappa}\big(\mathbf{D}^\delta u\big)(\zeta)w_\delta(r,\kappa\zeta)\,\mathrm{d}\zeta \quad (10.144)$$

for each $u \in C^{s(\delta)}[-r/\kappa, r/\kappa]$.

Proof. As λ runs trough $\mathfrak{a}_{\mathbb{C}}^*$, the set of all linear combinations of the functions

$$u_\lambda(t) = \mathrm{e}^{\mathrm{i}\lambda' t}, \quad t \in [-r/\kappa, r/\kappa]$$

is dense in $C^{s(\delta)}[-r/\kappa, r/\kappa]$. Hence, there is no loss of generality in assuming that $u = u_\lambda$ for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Then the left-hand side of (10.144) changes into the function $\varphi_{\lambda,\delta}(a_ro)Y_j^\delta(kM)$ (see Proposition 10.3). The desired statement now follows from Theorem 10.3, (10.142), (10.143), and Proposition 7.3(ii). □

Proof of Theorem 10.19. The implication (i)→(ii) is obvious from Lemma 10.3. Suppose now that (ii) is true. Let v_+ and v_- be functions in the class $C^{s(\delta)}(-R,R)$ such that

$$\big(\mathbf{D}^\delta v_\pm\big)(\zeta) = \frac{1}{2}\Big(\big(\mathbf{D}^\delta u\big)(\zeta) \pm \big(\mathbf{D}^\delta u\big)(-\zeta)\Big), \quad \zeta \in (-R,R). \quad (10.145)$$

Since $\mathbf{D}^\delta v_-$ is odd, we infer from (ii), Lemma 10.3, and (10.145) that

$$\int_K u\big(h\big(\tau^{-1}x_t\big)\big)\mathrm{e}^{\rho_X h(\tau^{-1}x_t)}Y_j^\delta(\tau M)\,\mathrm{d}\tau = \int_K v_+\big(h\big(\tau^{-1}x_t\big)\big)\mathrm{e}^{\rho_X h(\tau^{-1}x_t)}Y_j^\delta(\tau M)\,\mathrm{d}\tau, \quad (10.146)$$

where $x_t = k_0 a_t o$, $t \in (0, \kappa R)$. Let $\varepsilon \in (0, R)$ and suppose that $\psi_1 \in \mathcal{D}_\natural(\mathbb{R}^1)$, $\operatorname{supp}\psi_1 \subset (-R, R)$, and $\psi_1 = 1$ in $(-R + \varepsilon/2, R - \varepsilon/2)$. We define $v \in (\mathcal{E}' \cap C^{s(\delta)})(\mathbb{R}^1)$ by letting

$$v(\zeta) = v_+(\zeta)\psi_1(\zeta)$$

if $\zeta \in (-R, R)$ and $v(\zeta) = 0$ if $\zeta \in \mathbb{R}^1\backslash[-R, R]$. Next, let $\psi_2 \in \mathcal{D}_\natural(\mathbb{R}^1)$ with $\operatorname{supp}\psi_2 \subset (-R + \varepsilon, R - \varepsilon)$. Owing to Theorem 10.11(i) and (6.34), there exists $f \in \mathcal{D}_{\delta,j}(X)$ such that $\operatorname{supp} f \subset B_{R-\varepsilon}$ and

$$\overline{\mathcal{F}_j^\delta(f)(-\lambda)} = \widehat{\psi}_2(\lambda')$$

for each $\lambda \in \mathfrak{a}^*$. Hence,

$$R_\delta(\lambda)\overline{\widehat{\psi}_2(\lambda')} = \int_{A^+} \psi(a)\varphi_{\lambda,\delta}(ao)\,\mathrm{d}a \tag{10.147}$$

for some $\psi \in L^1(A^+)$ with $\operatorname{supp}\psi \subset \{a \in A^+ : d(ao, o) \leqslant R - \varepsilon\}$ (see (10.91), (10.92) and (10.8)). Assume now that $\psi_3 \in \mathcal{D}_\natural(\mathbb{R}^1)$ and $\operatorname{supp}\psi_3 \subset (-\varepsilon/2, \varepsilon/2)$. Using (10.142), (10.147), and Proposition 10.3, one has

$$\begin{aligned}\int_{\mathfrak{a}^*} \widehat{\mathbf{D}^\delta v}(\lambda')\overline{\widehat{\psi}_2(\lambda')}\widehat{\psi}_3(\lambda')\,\mathrm{d}\lambda &= \int_{\mathfrak{a}^*} R_\delta(\lambda)\widehat{v}(\lambda')\overline{\widehat{\psi}_2(\lambda')}\widehat{\psi}_3(\lambda')\,\mathrm{d}\lambda\\ &= \int_{A^+} \psi(a)\int_{\mathfrak{a}^*} \widehat{v * \psi_3}(\lambda')\varphi_{\lambda,\delta}(ao)\,\mathrm{d}\lambda\,\mathrm{d}a\\ &= \int_{A^+} \frac{\psi(a)}{Y_j^\delta(k_0M)}\int_K (v * \psi_3)\big(h\big(\tau^{-1}k_0ao\big)\big)\\ &\qquad\times \mathrm{e}^{\rho_X h(\tau^{-1}k_0ao)}Y_j^\delta(\tau M)\,\mathrm{d}\tau\,\mathrm{d}a.\end{aligned}$$

As ψ_3 is arbitrary, we conclude from (ii), (10.146), and the properties of ψ_1 that

$$\int_{\mathbb{R}^1} \widehat{\mathbf{D}^\delta v}(\zeta)\overline{\widehat{\psi}_2(\zeta)}\,\mathrm{d}\zeta = 0.$$

This yields

$$\int_0^{R-\varepsilon} \big(\mathbf{D}^\delta v\big)(\zeta)\overline{\psi_2(\zeta)}\,\mathrm{d}\zeta = 0$$

by virtue of the Plancherel formula and the evenness of $\mathbf{D}^\delta v$ and ψ_2. Now, from the arbitrariness of ψ_2 it follows that $\mathbf{D}^\delta v = \mathbf{D}^\delta v_+ = 0$ in $(-R + \varepsilon, R - \varepsilon)$. Next, $\varepsilon \in (0, R)$ being arbitrary, this shows that $\mathbf{D}^\delta v_+ = 0$ in $(-R, R)$. Therefore, $\mathbf{D}^\delta u$ is odd because of (10.145). Hence the theorem. □

Remark 10.3. It is essential in Theorem 10.19 that $Y_j^\delta(k_0M) \neq 0$. Otherwise, it follows by Proposition 10.3 that (ii) holds for each $u \in C(-R, R)$ (see the proof of Lemma 10.3).

One corollary of Theorem 10.19 is worth recording.

Corollary 10.5. *Let $R > 0$, $u \in C^{s(\delta)}(-R, R)$, assume that $\mathbf{D}^{\delta} u$ is even, and let*

$$\int_K u\big(h(\tau^{-1}x)\big)\mathrm{e}^{\rho_X h(\tau^{-1}x)} Y_j^{\delta}(\tau M)\,\mathrm{d}\tau = 0$$

for all $x \in B_R$. Then $\mathbf{D}^{\delta} u = 0$.

The proof follows at once from Theorem 10.19.

To continue, let $m \in \mathbb{N}$, $m \geqslant 2\alpha_X + 3$, and $f \in (\mathcal{E}'_{\delta,j} \cap C^m)(X)$. In view of (10.74) and (10.93),

$$\left|\mathrm{F}_j^{\delta}(f)(\lambda)\right| \leqslant \gamma(1 + |\lambda|)^{-m}, \quad \lambda \in \mathfrak{a}^*, \tag{10.148}$$

where the constant $\gamma > 0$ is independent of λ. Let us now define the function $U_f : \mathbb{R}^1 \to \mathbb{C}$ by the formula

$$U_f(t) = \frac{1}{2}\int_{\mathfrak{a}^*} \mathrm{F}_j^{\delta}(f)(\lambda)\mathrm{e}^{\mathrm{i}\lambda' t}|\mathbf{c}(\lambda)|^{-2}\,\mathrm{d}\lambda, \quad t \in \mathbb{R}^1. \tag{10.149}$$

Estimates (10.148) and (10.37) show that $U_f \in C^{m-2\alpha_X-3}(\mathbb{R}^1)$. If $m \geqslant s(\delta) + 2\alpha_X + 3$, then

$$\big(\mathbf{D}^{\delta} U_f\big)(t) = \frac{1}{2}\int_{\mathfrak{a}^*} \mathcal{F}_j^{\delta}(f)(\lambda)R_{\delta}(-\lambda)R_{\delta}(\lambda)\mathrm{e}^{\mathrm{i}\lambda' t}|\mathbf{c}(\lambda)|^{-2}\mathrm{d}\lambda, \tag{10.150}$$

which, together with (10.35), implies that $\mathbf{D}^{\delta} U_f$ is even. Next, we have

$$\int_K U_f\big(h(\tau^{-1}x)\big)\mathrm{e}^{\rho_X h(\tau^{-1}x)} Y_j^{\delta}(\tau M)\,\mathrm{d}\tau = f(x) \tag{10.151}$$

because of (10.149), Proposition 10.3, (10.92), and Theorem 10.11.

Theorem 10.20. *Let $R \in (0, +\infty]$ and $f \in C^m_{\delta,j}(B_R)$, where $m \geqslant s(\delta) + 2\alpha_X + 3$. Then there exists $u \in C^{m-2\alpha_X-3}(-R, R)$ such that $\mathbf{D}^{\delta} u$ is even and*

$$\int_K u\big(h(\tau^{-1}x)\big)\mathrm{e}^{\rho_X h(\tau^{-1}x)} Y_j^{\delta}(\tau M)\,\mathrm{d}\tau = f(x), \quad x \in B_R.$$

Proof. Let $r_1, r_2, \ldots$ be an increasing sequence of positive numbers such that $\lim_{n\to\infty} r_n = R$. For each $n \in \mathbb{N}$, we set $f_n = f\eta_n$, where $\eta_1, \eta_2, \ldots$ is a sequence of functions in the class $\mathcal{D}_{\natural}(B_R)$ such that $\eta_n = 1$ in B_{r_n}. Then $f_n \in (\mathcal{E}'_{\delta,j} \cap C^m)(B_R)$ and

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for each } x \in B_R. \tag{10.152}$$

We put $u_n = U_{f_n}$. Using (10.151), one derives from the definition of f_n and Corollary 10.5 that

$$\mathbf{D}^{\delta} u_n = \mathbf{D}^{\delta} u_{n+1} \quad \text{in } (-r_n, r_n),\ n \in \mathbb{N}.$$

Then there exist $v_n \in C^\infty(\mathbb{R}^1)$ such that $\mathbf{D}^\delta v_n = 0$ and $v_n = u_{n+1} - u_n$ in $(-r_n, r_n)$. Now relations (10.151), (10.152) and Corollary 10.5 ensure us that the function

$$u(\zeta) = \lim_{n\to\infty}\left(u_{n+1}(\zeta) - \sum_{q=1}^{n} v_q(\zeta)\right), \quad \zeta \in (-R, R),$$

has all the properties stated in the theorem. □

To go further, for $f \in \mathcal{E}'_{\delta,j}(X)$, we define the distribution $\mathfrak{A}_{\delta,j}(f) \in \mathcal{D}'(\mathbb{R}^1)$ by the formula

$$\langle \mathfrak{A}_{\delta,j}(f), \psi\rangle = \frac{1}{2}\int_{\mathfrak{a}^*} \mathrm{F}_j^\delta(f)(\lambda) R_\delta(\lambda)\widehat{\psi}(\lambda')|\mathbf{c}(\lambda)|^{-2}\,\mathrm{d}\lambda, \quad \psi \in \mathcal{D}(\mathbb{R}^1) \quad (10.153)$$

(see Theorems 10.11 and 6.3). It is evident from (10.153) and (10.35) that $\mathfrak{A}_{\delta,j}(f)$ is even. In addition,

$$\mathfrak{A}_{\delta,j}(f) = \mathbf{D}^\delta U_f \quad \text{for } f \in \left(\mathcal{E}'_{\delta,j} \cap C^{s(\delta)+2\alpha_X+3}\right)(X) \quad (10.154)$$

in view of (10.153), (10.149), and (10.142).

The distributions $\mathfrak{A}_{\delta,j}(f)$ can be interpreted as the matrix entries for $\mathfrak{A}_\delta(f)$ (see (10.153), (10.101), and Helgason [123], Chap. III, Corollary 11.3). On the other hand, the operator $\mathfrak{A}_{\delta,j}$ in an analogue of the operator $\mathfrak{A}_{k,j}$ from the Euclidean case (see Sect. 9.4). It has a number of important applications in the theory of mean periodic functions on domains in X (see Parts III and IV). Our main purpose in this section is to investigate basic properties of $\mathfrak{A}_{\delta,j}$.

Lemma 10.4. *For $f \in \mathcal{E}'_{\delta,j}(X)$, the following assertions are true.*

(i) *If $T \in \mathcal{E}'_\natural(X)$, then*

$$\mathfrak{A}_{\delta,j}(f \times T) = \mathfrak{A}_{\delta,j}(f) * \Lambda(T). \quad (10.155)$$

(ii) *If $l \in \mathbb{Z}_+$, $m = l + s(\delta) + 2\alpha_X + 3$, and $f \in (\mathcal{E}'_{\delta,j} \cap C^m)(X)$, then $\mathfrak{A}_{\delta,j}(f) \in C^l(\mathbb{R}^1)$ and*

$$f_{\delta,j}(a_t o) = 2\int_0^{t/\kappa} \mathfrak{A}_{\delta,j}(f)(\zeta) w_\delta(t, \kappa\zeta)\,\mathrm{d}\zeta \quad (10.156)$$

for each $t > 0$.

(iii) *Let $r \in (0, +\infty]$. Then $f = 0$ in B_r if and only if $\mathfrak{A}_{\delta,j}(f) = 0$ in $(-r, r)$.*

Proof. To verify (i) it is enough to combine (10.153), (10.72), (10.93), and (10.141). Part (ii) is a direct consequence of (10.154), (10.151), and Lemma 10.3.

In (iii), first assume that $f \in (\mathcal{E}'_{\delta,j} \cap C^\infty)(X)$. Then the required conclusion follows by (ii) and Theorem 10.19. The general case can be reduced to this one by means of the standard regularization (see (i)). Thus, the proof is complete. □

According to Lemma 10.4(iii), we can extend $\mathfrak{A}_{\delta,j}(f)$ to the space $\mathcal{D}'_{\delta,j}(B_R)$, $R \in (0, +\infty]$, by the relation

$$\langle \mathfrak{A}_{\delta,j}(f), \psi \rangle = \langle \mathfrak{A}_{\delta,j}(f\eta), \psi \rangle, \quad f \in \mathcal{D}'_{\delta,j}(B_R), \ \psi \in \mathcal{D}(-R, R),$$

where $\eta \in \mathcal{D}_\natural(B_R)$ is an arbitrary function such that $\eta = 1$ in $B_{r_0(\psi)+\varepsilon}$ for some $\varepsilon \in (0, R - r_0(\psi))$. Then $\mathfrak{A}_{\delta,j}(f) \in \mathcal{D}'_\natural(-R, R)$ and

$$\mathfrak{A}_{\delta,j}(f|_{B_r}) = \mathfrak{A}_{\delta,j}(f)|_{(-r,r)}$$

for each $r \in (0, R]$.

The following result generalizes Lemma 10.4 and corresponds to Theorem 9.3.

Theorem 10.21. *Let $R \in (0, +\infty]$, $l \in \mathbb{Z}_+$, and $m = l + s(\delta) + 2\alpha_X + 3$. Then the following statements hold.*

(i) *If $f \in \mathcal{D}'_{\delta,j}(B_R)$, $T \in \mathcal{E}'_\natural(X)$, and $r(T) < R$, then (10.155) is true on $(r(T) - R, R - r(T))$. In particular,*

$$\mathfrak{A}_{\delta,j}(p(L)f) = p\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \rho_X^2\right)\mathfrak{A}_{\delta,j}(f)$$

for any polynomial p.

(ii) *Let $f \in \mathcal{D}'_{\delta,j}(B_R)$, $r \in (0, R]$. Then $f = 0$ in B_r if and only if $\mathfrak{A}_{\delta,j}(f) = 0$ in $(-r, r)$.*

(iii) *If $f \in C^m_{\delta,j}(B_R)$, then $\mathfrak{A}_{\delta,j}(f) \in C^l(-R, R)$, and (10.156) is valid for $t \in (0, \kappa R)$. In addition,*

$$\mathfrak{A}_{\delta,j}(f)(0) = \lim_{t \to 0} f_{\delta,j}(a_t o) t^{-s(\delta)}. \tag{10.157}$$

(iv) *The mapping $\mathfrak{A}_{\delta,j}$ is continuous from $\mathcal{D}'_{\delta,j}(B_R)$ into $\mathcal{D}'_\natural(-R, R)$ and from $C^m_{\delta,j}(B_R)$ into $C^l_\natural(-R, R)$.*

(v) *If $f \in \mathcal{D}'_{\delta,j}(B_R)$ and* ord $f = l$, *then* ord $\mathfrak{A}_{\delta,j}(f) \leqslant m$.

(vi) *Let $f \in C^m_{\delta,j}(B_R)$ have all derivatives of order $\leqslant m$ vanishing at 0. Then $\mathfrak{A}_{\delta,j}(f)^{(q)}(0) = 0$ for each $q \in \{0, \dots, l\}$.*

(vii) *If $\mu \in \mathbb{C}$ and $\nu \in \mathbb{Z}_+$, then*

$$\mathfrak{A}_{\delta,j}(\Phi_{\mu,\nu,\delta,j}) = u_{\mu,\nu},$$

where

$$u_{\mu,\nu}(t) = \begin{cases} \dfrac{1}{2}\left(\mathrm{e}^{\mu,\nu}(t) + \mathrm{e}^{\mu,\nu}(-t)\right), & \mu \neq 0, \\ (-1)^\nu t^{2\nu}, & \mu = 0. \end{cases}$$

(viii) *Assume that $T \in$ conj$(\mathcal{E}'_{\delta,j}(X))$, $r(T) < R$, and $f \in C^q_{\delta,j}(B_R)$, where $q =$* $\max\{s(\delta) + 2\alpha_X + 3, \operatorname{ord} T + 2\alpha_X + 4\}$. *Then*

$$\langle T, f \rangle = \langle \Lambda^{\delta,j}(T), \mathfrak{A}_{\delta,j}(f) \rangle.$$

Proof. Parts (i), (ii) and the first assertion in (iii) are clear from the definition of $\mathfrak{A}_{\delta,j}$ on $\mathcal{D}'_{\delta,j}(B_R)$ and Lemma 10.4. Equality (10.157) is a consequence of (10.156) and (10.143). Next, it is not difficult to adapt the argument in the proof of Theorem 9.3(iv), (v) to verify (iv) and (v); we just have to use (10.93), (10.74), (10.37), and (10.153). Part (vi) follows from (ii) and (iv), taking [122, Chap. 2, Lemma 1.3] into account. Concerning (vii), first observe that

$$\Phi_{\mu,0,\delta,j}(x) = 2Y_j^\delta(kM)\int_0^{t/\kappa}\cos(\mu\zeta)w_\delta(t,\kappa\zeta)\,\mathrm{d}\zeta, \tag{10.158}$$

where $x = ka_t o$, $k \in K$, $t > 0$. To prove (10.158) it is enough to combine Proposition 10.3 with (10.144), where $u(t) = \mathrm{e}^{\mathrm{i}\mu t}$. Bearing (iii) in mind, we see from (10.158) that

$$\int_0^{t/\kappa}\big(\mathfrak{A}_{\delta,j}(\Phi_{\mu,0,\delta,j})(\zeta) - \cos(\mu\zeta)\big)w_\delta(t,\kappa\zeta)\,\mathrm{d}\zeta = 0$$

for all $t \in (0,\kappa R)$. Lemma 10.3 and Corollary 10.5 now give

$$\mathfrak{A}_{\delta,j}(\Phi_{\mu,0,\delta,j})(\zeta) = \cos\mu\zeta, \quad \zeta \in (-R,R). \tag{10.159}$$

Differentiating (10.159) with respect to μ, we obtain (vii). Finally, part (viii) follows like in the proof of Theorem 9.3(viii) (see (10.159), Proposition 10.11, part (iv), and (10.99)). □

Remark 10.4. Let $R \in (0,+\infty)$, $l \in \mathbb{Z}_+$, and $m = l + s(\delta) + 2\alpha_X + 3$. For each $f \in C^m_{\delta,j}(\mathring{B}_R)$, we set

$$\mathfrak{A}_{\delta,j}(f) = \mathfrak{A}_{\delta,j}(f_1)|_{[-R,R]},$$

where $f_1 \in C^m_{\delta,j}(X)$ is selected so that $f_1|_{\mathring{B}_R} = f$. Theorem 10.21(ii), (iii) shows that $\mathfrak{A}_{\delta,j}(f)$ is independent of the choice of f_1 and $\mathfrak{A}_{\delta,j}(f) \in C^l_\natural[-R,R]$. Moreover, the mapping $\mathfrak{A}_{\delta,j} : C^m_{\delta,j}(\mathring{B}_R) \to C^l_\natural[-R,R]$ is continuous.

Theorem 10.22. *Let $R \in (0,+\infty)$ and $q = 2 + [\alpha_X + s(\delta)/2]$. Then there exists a constant $c > 0$ such that*

$$\int_{-R}^{R}\big|\mathfrak{A}_{\delta,j}(f)^{(l)}(t)\big|\,\mathrm{d}t \leqslant c\sum_{i=0}^{q}\int_{B_R}\big|\big(L+\rho_X^2\big)^{[(l+1)/2]+i}f(x)\big|\,\mathrm{d}x$$

for all $l \in \mathbb{Z}_+$ and $f \in C^\nu_{\delta,j}(\mathring{B}_R)$, where $\nu = 2q + 2[(l+1)/2]$.

Proof. We can essentially use the same arguments as in the proof of Theorem 9.4 with the mappings $\Lambda^{k,j}$ and $\mathcal{F}^k_j$ replaced by $\Lambda^{\delta,j}$ and $\mathcal{F}^\delta_j$, respectively (see Theorems 10.21(viii) and 10.11). The change is that, instead of (7.10) and (9.68), we now use Lemma 7.2 and the function

$$w(x) = \begin{cases} \sum_{i=1}^{q} c_i \overline{\Phi_{\mu_i,0,\delta,j}(x)} & \text{if } x \in B_R, \\ 0 & \text{if } x \in X \backslash B_R \end{cases}$$

for suitable $c_i \in \mathbb{C}$, $\mu_i > 0$. We leave for the reader to examine the details of the proof. □

Let us now turn to the problem of inverting the mapping $\mathfrak{A}_{\delta,j}$.
Following Sect. 9.4, for $F \in \mathcal{E}'_{\natural}(\mathbb{R}^1)$, we set

$$\langle \mathfrak{B}_{\delta,j}(F), w \rangle = \frac{1}{\pi} \int_0^{\infty} \widehat{F}(\mu) \langle w, \Phi_{\mu,0,\delta,j} \rangle \, d\mu, \quad w \in \mathcal{D}(X).$$

Together, (10.92), (10.27), and Theorem 10.11(i) show that $\mathfrak{B}_{\delta,j}(F) \in \mathcal{D}'_{\delta,j}(X)$ and the mapping $\mathfrak{B}_{\delta,j} : \mathcal{E}'_{\natural}(\mathbb{R}^1) \to \mathcal{D}'_{\delta,j}(X)$ is continuous. Now we prove a result similar to Lemma 9.3.

Lemma 10.5.

(i) *If* $F \in \mathcal{E}'_{\natural}(\mathbb{R}^1)$ *and* $T \in \mathcal{E}'_{\natural}(X)$, *then*

$$\mathfrak{B}_{\delta,j}(F) \times T = \mathfrak{B}_{\delta,j}\big(F * \Lambda(T)\big). \tag{10.160}$$

(ii) *Let* $F \in (\mathcal{E}'_{\natural} \cap C^l)(\mathbb{R}^1)$ *for some* $l \geqslant 2$. *Then* $\mathfrak{B}_{\delta,j}(F) \in C^{l+s(\delta)-2}_{\delta,j}(X)$ *and*

$$\mathfrak{B}_{\delta,j}(F)(ka_t o) = 2Y_j^{\delta}(kM) \int_0^{t/\kappa} F(\zeta) w_{\delta}(t, \kappa\zeta) \, d\zeta \tag{10.161}$$

for all $k \in K$, $t > 0$.

(iii) *Let* $F \in \mathcal{E}'_{\natural}(\mathbb{R}^1)$ *and* $r \in (0, +\infty]$. *Then* $F = 0$ *in* $(-r, r)$ *if and only if* $\mathfrak{B}_{\delta,j}(F) = 0$ *in* B_r.

Proof. Relation (10.160) can be easily verified by using (10.92) and (10.30). Next, by the definition of $\mathfrak{B}_{\delta,j}$ and (10.65) one sees that

$$\mathfrak{B}_{\delta,j}(F)(x) = \frac{1}{\pi} \int_0^{\infty} \widehat{F}(\mu) \Phi_{\mu,0,\delta,j}(x) \, d\mu$$

for all $F \in (\mathcal{E}'_{\natural} \cap C^2)(\mathbb{R}^1)$, $x \in X$. Combining this with (10.158), we obtain (ii). To prove (iii) there is no loss of generality in assuming $F \in (\mathcal{E}'_{\natural} \cap C^{\infty})(\mathbb{R}^1)$ (see (10.160)). Now part (iii) follows from (ii), Lemma 10.3, and Corollary 10.5. □

Having Lemma 10.5(iii) in mind, we extend the mapping $\mathfrak{B}_{\delta,j}$ to the space $\mathcal{D}'_{\natural}(-R, R)$, $R \in (0, +\infty]$ by the formula

$$\langle \mathfrak{B}_{\delta,j}(F), w \rangle = \langle \mathfrak{B}_{\delta,j}(F\eta), w \rangle, \quad F \in \mathcal{D}'_{\natural}(-R, R), \ w \in \mathcal{D}(B_R),$$

where $\eta \in \mathcal{D}_\natural(-R, R)$ and $\eta = 1$ on $(-r_0(w) - \varepsilon, r_0(w) + \varepsilon)$ for some $\varepsilon \in (0, R - r_0(w))$. Then $\mathfrak{B}_{\delta,j}(F) \in \mathcal{D}'_{\delta,j}(B_R)$ and

$$\mathfrak{B}_{\delta,j}(F|_{(-r,r)}) = \mathfrak{B}_{\delta,j}(F)|_{B_r}$$

for each $r \in (0, R]$. For the case where δ is trivial and $j = 1$, we set

$$\mathfrak{B}_{\delta,j} = \mathfrak{B}_{\text{triv}}.$$

As in the Euclidean case, Lemma 10.5 and Theorem 10.21 lead to the following analog of Theorem 9.5.

Theorem 10.23. *For $R \in (0, +\infty]$ and $l \in \{2, 3, \ldots\}$, the following assertions are valid.*

(i) *Let $F \in \mathcal{D}'_\natural(-R, R)$, $r \in (0, R]$. Then $F = 0$ on $(-r, r)$ if and only if $\mathfrak{B}_{\delta,j}(F) = 0$ in B_r.*

(ii) *If $F \in C^l_\natural(-R, R)$, then $\mathfrak{B}_{\delta,j}(F) \in C^{l+s(\delta)-2}_{\delta,j}(B_R)$, and (10.161) holds for all $k \in K$, $t \in (0, \kappa R)$. In addition,*

$$\lim_{t\to 0} t^{-s(\delta)}\mathfrak{B}_{\delta,j}(F)(ka_t o) = Y^\delta_j(kM)F(0).$$

(iii) *The map $\mathfrak{B}_{\delta,j}$ is continuous from $\mathcal{D}'_\natural(-R, R)$ into $\mathcal{D}'_{\delta,j}(B_R)$ and from $C^l_\natural(-R, R)$ into $C^{l+s(\delta)-2}_{\delta,j}(B_R)$.*

(iv) *Let $F \in \mathcal{D}'_\natural(-R, R)$. Then* $\operatorname{ord}\mathfrak{B}_{\delta,j}(F) \leqslant \max\{0, \operatorname{ord} F - s(\delta) + 3\}$.

(v) *Suppose that $F \in C^l_\natural(-R, R)$ and $F^{(\nu)}(0) = 0$ for all $\nu \in \{0, \ldots, l\}$. Then $\mathfrak{B}_{\delta,j}(F)$ has all derivatives of order $\leqslant l + s(\delta) - 2$ vanishing at* 0.

(vi) *For $F \in \mathcal{D}'_\natural(-R, R)$, one has*

$$\mathfrak{A}_{\delta,j}\big(\mathfrak{B}_{\delta,j}(F)\big) = F.$$

(vii) *Assume that $T \in \operatorname{conj}(\mathcal{E}'_{\delta,j}(X))$, $r(T) < R$ and $F \in C^q_\natural(-R, R)$, where $q = \max\{2, \operatorname{ord} T - s(\delta) + 2\}$. Then*

$$\big\langle T, \mathfrak{B}_{\delta,j}(F)\big\rangle = \big\langle \Lambda^{\delta,j}(T), F\big\rangle.$$

(viii) *Let $F \in \mathcal{D}'_\natural(-R, R)$, $T \in \mathcal{E}'_\natural(X)$, and $r(T) < R$. Then (10.160) is true in $B_{R-r(T)}$. In particular,*

$$p(L)\mathfrak{B}_{\delta,j}(F) = \mathfrak{B}_{\delta,j}\left(p\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \rho_X^2\right) f\right)$$

for each polynomial p.

Theorems 10.21 and 10.23 imply the following consequence.

Corollary 10.6. *For each $R \in (0, +\infty]$, the transform $f \to \mathfrak{A}_{\delta,j}(f)$ defines a homeomorphism between:*

(i) $\mathcal{D}'_{\delta,j}(B_R)$ *and* $\mathcal{D}'_{\natural}(-R, R)$.

(ii) $C^{\infty}_{\delta,j}(B_R)$ *and* $C^{\infty}_{\natural}(-R, R)$.

In addition,

$$\mathfrak{A}^{-1}_{\delta,j} = \mathfrak{B}_{\delta,j}.$$

Remark 10.5. For $F \in C^l_{\natural}[-R, R]$, $l \geqslant 2$, $R \in (0, +\infty)$, we set $\mathfrak{B}_{\delta,j}(F) = \mathfrak{B}_{\delta,j}(F_1)|_{\mathring{B}_R}$, where $F_1 \in C^l_{\natural}(\mathbb{R}^1)$ and $F_1|_{[-R,R]} = F$. Theorem 10.23(i), (ii) shows that $\mathfrak{B}_{\delta,j}(F)$ is independent of the choice of F_1 and $\mathfrak{B}_{\delta,j}(F) \in C^{l+s(\delta)-2}_{\delta,j}(\mathring{B}_R)$. Moreover, the mapping $F \to \mathfrak{B}_{\delta,j}(F)$ is continuous from $C^l_{\natural}[-R, R]$ into $C^{l+s(\delta)-2}_{\delta,j}(\mathring{B}_R)$ because of Theorem 10.23(iii).

The following is an analogue of Theorem 9.6.

Theorem 10.24. *Let $R \in (0, +\infty)$. Then there exists a constant $c > 0$ such that for all $l \in \mathbb{Z}_+$ and $F \in C^{2l+2}_{\natural}[-R, R]$,*

$$\int_{B_R} \left|\left(L + \rho_X^2\right)^l \mathfrak{B}_{\delta,j}(F)(x)\right| \mathrm{d}x \leqslant c \int_{-R}^{R} \left(\left|F^{(2l)}(t)\right| + \left|F^{(2l+2)}(t)\right|\right) \mathrm{d}t.$$

Proof. The proof runs like the one for Theorem 9.6 (see Theorem 10.23(vii), (viii)). ◻

Corollary 10.7. *Let $R \in (0, +\infty]$, $f \in \mathcal{D}'_{\delta,j}(B_R)$, and $\alpha > 0$. Then $f \in (\mathcal{D}'_{\delta,j} \cap G^{\alpha})(B_R)$ if and only if $\mathfrak{A}_{\delta,j}(f) \in (\mathcal{D}'_{\natural} \cap G^{\alpha})(-R, R)$.*

The proof is obvious from Theorems 10.24 and 10.22.

To continue, let $f \in \mathcal{E}'_{\delta,j}(X)$. Taking (10.91), (10.55), Theorem 10.7, and Theorem 10.10 into account, we infer that there exists $\mathcal{A}^{\delta}_j(f) \in \mathcal{E}'_{\natural}(X)$ such that

$$\begin{aligned} \widetilde{f}(\lambda, b) R_\delta(\lambda) &= \widetilde{\mathcal{A}^{\delta}_j(f)}(\lambda) Y^{\delta}_j(b) \\ &= \mathrm{F}^{\delta}_j(f)(\lambda) R_\delta(\lambda) Y^{\delta}_j(b), \quad (\lambda, b) \in \mathfrak{a}^*_{\mathbb{C}} \times B. \end{aligned} \tag{10.162}$$

Next, bearing in mind that the polynomials $R_\delta(\lambda)$ and $R_\delta(-\lambda)$ are relatively prime (see [123, p. 348]), we define $\mathcal{B}^{\delta}_j(f) \in \mathcal{E}'_{\natural}(X)$ by the formula

$$\widetilde{\mathcal{A}^{\delta}_j(f)}(\lambda) = R_\delta(\lambda) R_\delta(-\lambda) \widetilde{\mathcal{B}^{\delta}_j(f)}(\lambda), \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}}. \tag{10.163}$$

Let us now investigate basic properties of the mappings $\mathcal{A}^{\delta}_j$ and $\mathcal{B}^{\delta}_j$.

Proposition 10.16. *For each $f \in \mathcal{E}'_{\delta,j}(X)$, the following statements are valid.*

(i) $r(\mathcal{A}^{\delta}_j(f)) = r(\mathcal{B}^{\delta}_j(f)) = r(f)$.

(ii) *For all* $u \in \mathcal{E}'_\natural(X)$,

$$\mathcal{A}_j^\delta(f \times u) = \mathcal{A}_j^\delta(f) \times u \tag{10.164}$$

and

$$\mathcal{B}_j^\delta(f \times u) = \mathcal{B}_j^\delta(f) \times u. \tag{10.165}$$

(iii) *If* $w \in \mathcal{D}(X)$, *then*

$$\langle \mathcal{A}_j^\delta(f), w \rangle = \frac{1}{2} \int_{\mathfrak{a}^*} \mathrm{F}_j^\delta(f)(\lambda) R_\delta(\lambda) |\mathbf{c}(\lambda)|^{-2} \int_X w(x)\varphi_\lambda(x)\, \mathrm{d}x\, \mathrm{d}\lambda.$$

In particular, if $l \in \mathbb{Z}_+$ *and* $f \in (\mathcal{E}'_{\delta,j} \cap C^m)(X)$ *with* $m = l + s(\delta) + 2\alpha_X + 3$, *then* $\mathcal{A}_j^\delta(f) \in (\mathcal{E}'_\natural \cap C^l)(X)$ *and*

$$\mathcal{A}_j^\delta(f)(x) = \frac{1}{2} \int_{\mathfrak{a}^*} \mathrm{F}_j^\delta(\lambda) R_\delta(\lambda) \varphi_\lambda(x) |\mathbf{c}(\lambda)|^{-2}\, \mathrm{d}\lambda \tag{10.166}$$

for all $x \in X$.

(iv) $\mathcal{A}_j^\delta = p(L)\mathcal{B}_j^\delta$ *for some polynomial* p *of degree* $s(\delta)$.

Proof. Part (i) follows from (10.162), (10.163), and Theorem 10.7(ii). As for (ii), note that by (10.162), (10.163), and Proposition 10.9(i) both sides in (10.164) and (10.165) have the same Fourier transform. This yields (ii). To prove (iii), first consider the case $f \in (\mathcal{E}'_{\delta,j} \cap C^m)(X)$. Using Remarks 10.1 and 10.2, we deduce from (10.148) and (10.162) that $\mathcal{A}_j^\delta(f) \in (\mathcal{E}'_\natural \cap C^l)(X)$ and (10.166) holds. In the general case there is obviously no loss of generality in assuming that $w \in \mathcal{D}_\natural(X)$. For each $\varepsilon > 0$, let $\eta_\varepsilon \in \mathcal{D}_\natural(X)$ be nonnegative, with $\int_X \eta_\varepsilon(x)\, \mathrm{d}x = 1$ and $\operatorname{supp} \eta_\varepsilon \subset \mathring{B}_\varepsilon$. Using now (10.10), (10.164), (10.72), and (10.166), we obtain

$$\begin{aligned} \langle \mathcal{A}_j^\delta(f), w \times \eta_\varepsilon \rangle &= \langle \mathcal{A}_j^\delta(f \times \eta_\varepsilon), w \rangle \\ &= \frac{1}{2} \int_{\mathfrak{a}^*} \mathrm{F}_j^\delta(f)(\lambda) R_\delta(\lambda) \widetilde{\eta}_\varepsilon(\lambda) \widetilde{w}(\lambda) |\mathbf{c}(\lambda)|^{-2}\, \mathrm{d}\lambda. \end{aligned} \tag{10.167}$$

The proof of Theorem 10.5 shows that (10.78) holds and $\widetilde{\eta}_\varepsilon(\lambda) \to 1$ as $\varepsilon \to 0$ for any $\lambda \in \mathfrak{a}^*$. In addition, if $\varepsilon \to 0$, then $w \times \eta_\varepsilon \to w$ in $\mathcal{D}(X)$. Applying Theorem 10.11, (10.37), (10.74), and (10.167), by Lebesgue's dominated convergence theorem we arrive at (iii). Part (iv) follows from (10.163), (10.50)–(10.52), and (10.53). □

The following corollary relates $\mathcal{A}_j^\delta$ to the transform $\mathfrak{A}_{\delta,j}$.

Corollary 10.8. *If* $f \in (\mathcal{E}'_{\delta,j} \cap C^{s(\delta)+2\alpha_X+3})(X)$, *then*

$$\mathcal{A}_j^\delta(f)(x) = \int_K \mathfrak{A}_{\delta,j}(f)\big(h(\tau x)\big) \mathrm{e}^{\rho_X h(\tau x)}\, \mathrm{d}\tau \tag{10.168}$$

for all $x \in X$.

The proof follows at once from (10.166), (10.31), (10.154), and (10.150).

Lemma 10.6. *Let $f \in \mathcal{E}'_{\delta,j}(X)$. Then the following assertions hold.*

(i) *If $\mathcal{A}_j^\delta(f) = 0$ in B_r for some $r \in (0, +\infty]$, then $f = 0$ in B_r.*
(ii) *Assume that $f = 0$ in $B_{r_1,r_2} = \{x \in X : r_1 < d(o,x) < r_2\}$ for some $r_1 \in \mathbb{R}^1$, $r_2 \in (0, +\infty]$. Then $\mathcal{A}_j^\delta(f) = 0$ in B_{r_1,r_2}.*
(iii) *If $\mathcal{B}_j^\delta(f) = 0$ in B_r for some $r \in (0, +\infty]$, then $f = 0$ in B_r.*

Proof. We can assume, without loss of generality, that $f \in \mathcal{D}_{\delta,j}(X)$. The general case reduces to this one by means of the standard smoothing procedure (see (10.164)). Corollaries 10.8 and 10.5 now lead to the conclusion that $\mathcal{A}_j^\delta(f) = 0$ in B_r if and only if $f = 0$ in B_r proving (i). Next, for $r_1 < 0$ or $r_2 = +\infty$, part (ii) is a direct consequence of (i) and Proposition 10.16(i). Let $0 \leqslant r_1 < r_2 < +\infty$. Writing $f = f_1 + f_2$ where $f_1, f_2 \in \mathcal{E}'_{\delta,j}(X)$ such that $\operatorname{supp} f_1 \subset \dot{B}_{r_1}$ and $\operatorname{supp} f_2 \subset X \backslash B_{r_2}$, we find $\mathcal{A}_j^\delta(f) = \mathcal{A}_j^\delta(f_1) + \mathcal{A}_j^\delta(f_2)$. Again, part (i) and Proposition 10.16(i) yield $\mathcal{A}_j^\delta(f) = 0$ in B_{r_1,r_2}. Finally, part (iii) is a consequence of (i) and Proposition 10.16(iv). Hence the lemma. □

For the rest of the section, we assume that $\mathcal{O}$ is a nonempty open K-invariant subset of X. Let us extend the mapping $\mathcal{A}_j^\delta(f)$ to the space $\mathcal{D}'_{\delta,j}(\mathcal{O})$.

For $f \in \mathcal{D}'_{\delta,j}(\mathcal{O})$, we define the distribution $\mathcal{A}_j^\delta(f) \in \mathcal{D}'_\natural(\mathcal{O})$ acting in $\mathcal{D}(\mathcal{O})$ by the formula

$$\langle \mathcal{A}_j^\delta(f), w \rangle = \langle \mathcal{A}_j^\delta(f\eta), w \rangle, \quad w \in \mathcal{D}(\mathcal{O}), \tag{10.169}$$

where $\eta \in \mathcal{D}_\natural(\mathcal{O})$ and $\eta = 1$ in some open set $\mathcal{O}_1 \subset \mathcal{O}$ containing the support of w. Notice that $f\eta \in \mathcal{E}'_{\delta,j}(X)$ and the right-hand side in (10.169) is independent of our choice of η (see Lemma 10.6). In addition, for each nonempty open K-invariant subset U of $\mathcal{O}$, one has

$$\mathcal{A}_j^\delta(f|_U) = \mathcal{A}_j^\delta(f)|_U. \tag{10.170}$$

Theorem 10.25. *Let $f \in \mathcal{D}'_{\delta,j}(\mathcal{O})$, $r \in (0, +\infty]$, $l \in \mathbb{Z}_+$, and $m = l + s(\delta) + 2\alpha_X + 3$. Then the following results are true.*

(i) *If $T \in \mathcal{E}'_\natural(X)$ and $\mathcal{O}_T \neq \varnothing$ (see (10.9)) then (10.164) holds with $u = T$. In particular,*

$$\mathcal{A}_j^\delta\big(p(L)f\big) = p(L)\big(\mathcal{A}_j^\delta(f)\big)$$

for each polynomial p.
(ii) *Assume that $B_r \subset \mathcal{O}$. Then $f = 0$ in B_r if and only if $\mathcal{A}_j^\delta(f) = 0$ in B_r.*
(iii) $\operatorname{supp} \mathcal{A}_j^\delta(f) \subset \operatorname{supp} f$.
(iv) *If $f \in (\mathcal{D}'_{\delta,j} \cap C^m)(\mathcal{O})$, then $\mathcal{A}_j^\delta(f) \in (\mathcal{D}'_\natural \cap C^l)(\mathcal{O})$. In addition, if $B_r \subset \mathcal{O}$, then (10.168) is satisfied for all $x \in B_r$.*
(v) *The mapping $\mathcal{A}_j^\delta$ is continuous from $\mathcal{D}'_{\delta,j}(\mathcal{O})$ into $\mathcal{D}'_\natural(\mathcal{O})$ and from $C^m_{\delta,j}(\mathcal{O})$ into $C^l_\natural(\mathcal{O})$.*
(vi) $\mathcal{A}_j^\delta(\Phi_{\mu,\nu,\delta,j}) = \Phi_{\mu,\nu,\mathrm{triv}}$ *for all $\mu \in \mathbb{C}$, $\nu \in \mathbb{Z}_+$.*

(vii) *If* $f \in C^{s(\delta)}(\mathcal{O}\backslash\{o\})$ *and* $f(x) = \psi(t)Y_j^\delta(kM)$ *for* $x = ka_t o \in \mathcal{O}\backslash\{o\}$, *then*

$$\mathcal{A}_j^\delta(f)(x) = \sum_{\nu=0}^{s(\delta)} a_\nu^\delta(t)\psi^{(\nu)}(t), \quad x \in \mathcal{O}\backslash\{o\}, \tag{10.171}$$

where the functions $a_\nu^\delta \in \mathrm{RA}(0, +\infty)$ *are independent of* f *and* j. *In addition,* $a_{s(\delta)}^\delta(t) \neq 0$ *for each* $t > 0$.

(viii) *If* $\mathcal{O} = B_r$, *then*

$$\mathcal{A}_j^\delta(f) = \mathfrak{B}_{\mathrm{triv}}\big(\mathfrak{A}_{\delta,j}(f)\big). \tag{10.172}$$

Proof. Assertions (i) and (ii) are easy consequences of Proposition 10.16(ii), Lemma 10.6, and the definition of $\mathcal{A}_j^\delta$ on $\mathcal{D}'_{\delta,j}(\mathcal{O})$. Next, since supp f is K-invariant, part (iii) is obvious from (10.170) and Lemma 10.6(ii). Next, using (10.170), Proposition 10.16(ii), and Corollary 10.8, we see that (iv) is true. The argument in (v) is quite parallel to the proof of Theorem 10.21(iv) (see Proposition 10.16). Turning to (vi), it can be supposed that $\mathcal{O} = X$ in view if (10.170). The required equality then follows by Theorem 10.21(vii) and (iv). To show (vii) and (viii), first observe that (10.171) and (10.172) are fulfilled with $f = \Phi_{\mu,0,\delta,j}$ for each $\mu \in \mathbb{C}$ (see (vi), [73, 2.8 (22)–2.8 (24)], and Theorem 10.21(vii)). Applying now (v) and Proposition 10.11, we obtain parts (vii) and (viii) in the general case. This completes the proof. □

The previous theorem shows that $\mathcal{A}_j^\delta$ is an analogue of the mapping $\mathcal{A}_j^k$ from the Euclidean case (see Sect. 9.4). We note also that Theorems 10.22 and 10.24 yield the inequality

$$\int_{B_R} \big|\big(L + \rho_X^2\big)^l \big(\mathcal{A}_j^\delta(f)\big)(x)\big|\, \mathrm{d}x \leqslant c \sum_{i=0}^{q} \int_{B_R} \big|\big(L + \rho_X^2\big)^{l+i} f(x)\big|\, \mathrm{d}x,$$

where $R \in (0, +\infty)$, $l \in \mathbb{Z}_+$, $q = 3 + [\alpha_X + s(\delta)/2]$, $f \in C_{\delta,j}^{2(q+l)}(\dot{B}_R)$, and the constant $c > 0$ is independent of l and f.

We conclude this section with the following result.

Theorem 10.26. *Let* $m \in \mathbb{Z}_+$, $f \in \mathcal{E}'_{\delta,j}(X)$, *and* $f \in C^\infty(B_\varepsilon)$ *for some* $\varepsilon > 0$. *Then the following assertions hold.*

(i) *If* $f \in L_m^{1,\mathrm{loc}}(X)$, *then* $\mathcal{B}_j^\delta(f) \in (L_{m+s(\delta)}^{1,\mathrm{loc}} \cap \mathcal{E}'_\natural)(X)$.

(ii) *If* $f \in C^m(X)$, *then* $\mathcal{B}_j^\delta(f) \in (C^{m+s(\delta)} \cap \mathcal{E}'_\natural)(X)$.

The proof starts with the following lemma.

Lemma 10.7. *Let* $m \in \mathbb{Z}_+$, $R > \varepsilon > 0$, $f \in \mathcal{D}'_{\delta,j}(B_R) \cap C^\infty(B_\varepsilon)$, *and let* q *is a polynomial of degree* d. *Assume that* $F \in \mathcal{D}'_{\delta,j}(B_R)$ *and* $q(L)F = f$. *Then the following statements are valid.*

(i) *If* $f \in L_m^{1,\mathrm{loc}}(B_R)$, *then* $F \in L_{m+2d}^{1,\mathrm{loc}}(B_R)$.

(ii) *If* $f \in C^m(B_R)$, *then* $F \in C^{m+2d}(B_R)$.

Proof. By assumption on f we infer that $F \in C^\infty(B_\varepsilon)$ since $p(L)$ is an elliptic differential operator. Using now [123, Chap. 3, the proof of Theorem 11.2] and [126, the proof of Corollary 3.1.6], one sees that (i) and (ii) hold. □

We now embark on the proof of Theorem 10.26. Let $d \in \mathbb{Z}_+$, $d > s(\delta)/2$, and assume that $f \in (\mathcal{E}'_{\delta,j} \cap L_m^{1,\mathrm{loc}})(X) \cap C^\infty(B_\varepsilon)$. By Corollary 10.4 and Lemma 10.7 there exists $F \in (\mathcal{E}'_{\delta,j} \cap L_{m+2d}^{1,\mathrm{loc}})(X) \cap C^\infty(B_\varepsilon)$ such that $q(L)F = f$ for some polynomial q of degree d. In view of Theorem 10.25(v), (vii),

$$\mathcal{A}_j^\delta(F) \in \big(\mathcal{E}'_\natural \cap L_{m+2d-s(\delta)}^{1,\mathrm{loc}}\big)(X) \cap C^\infty(B_\varepsilon).$$

Taking Proposition 10.16(iv) into account, we deduce from Lemma 10.7 that $\mathcal{B}_j^\delta(F) \in (\mathcal{E}'_\natural \cap L_{m+2d+s(\delta)}^{1,\mathrm{loc}})(X)$. Using now (10.165), one obtains

$$\mathcal{B}_j^\delta(f) = q(L)\mathcal{B}_j^\delta(F),$$

which establishes (i). The proof of (ii) is quite similar. □

10.9 Ideas and Methods of Sect. 9.5 Applied in Analogous Problems for G/K

Here we intend to study analogues of results in Sect. 9.5. Let $T \in \mathcal{E}'_{\natural\natural}(X)$, $T \neq 0$, and let

$$\mathcal{Z}_T = \big\{\lambda \in \mathcal{Z}\big(\mathring{T}\big) : \operatorname{Im}\lambda \geqslant 0,\ \lambda \notin (-\infty, 0)\big\}.$$

Throughout the section we suppose that $\mathcal{Z}_T \neq \varnothing$.

Let $\lambda \in \mathcal{Z}_T$ and $\eta \in \{0, \dots, n(\lambda, T)\}$. By an appeal to Proposition 6.6(ii) and Theorem 10.7(ii) we find that

$$\big|b^{\lambda,\eta}\big(\mathring{T}, z\big)\big| \leqslant \gamma_1(1 + |z|)^{\gamma_2} e^{r(T)|\operatorname{Im} z|}, \quad z \in \mathbb{C}, \tag{10.173}$$

where $\gamma_1, \gamma_2 > 0$ are independent of z. Because of (10.173) and Theorem 10.7, there exist $T_{\lambda,\eta} \in \mathcal{E}'_{\natural\natural}(X)$ and $T^{\lambda,\eta} \in \mathcal{E}'_{\natural\natural}(X)$ such that

$$r(T_{\lambda,\eta}) = r\big(T^{\lambda,\eta}\big) = r(T) \tag{10.174}$$

and

$$\mathring{T}_{\lambda,\eta}(z) = b^{\lambda,\eta}\big(\mathring{T}, z\big), \qquad \mathring{T}^{\lambda,\eta}(z)\big(z^2 - \lambda^2\big)^{\eta+1} = \mathring{T}(z), \quad z \in \mathbb{C}. \tag{10.175}$$

Proposition 10.17.

(i) $T_{\lambda,\eta} = \sum_{p=0}^{n(\lambda,T)} b_p^{\lambda,\eta}\big(\overset{\circ}{T}\big)T^{\lambda,n(\lambda,T)-p}$.

(ii) $\big(L+\lambda^2+|\rho|^2\big)^{n(\lambda,T)+1}T_{\lambda,\eta} = \sum_{p=0}^{n(\lambda,T)} b_p^{\lambda,\eta}\big(\overset{\circ}{T}\big)(-1)^{n(\lambda,T)+1-p}\big(L+\lambda^2+|\rho|^2\big)^p T$.

(iii) *If* $T \in \mathfrak{R}(X)$, *then*

$$\sum_{\lambda\in\mathcal{Z}_T} T_{\lambda,0} = \delta_0, \tag{10.176}$$

where the series converges unconditionally in the space $\mathcal{D}'(X)$.

(iv) *Assume that* $T = (L+c)Q$ *for some* $c \in \mathbb{C}$, $Q \in \mathcal{E}'_{\natural\natural}(X)$. *Then* $\mathcal{Z}_Q \subset \mathcal{Z}_T$ *and*

$T_{\lambda,0} = Q_{\lambda,0} - b_{n(\lambda,T)}^{\lambda,0}(\overset{\circ}{T})T$ *for all* $\lambda \in \mathcal{Z}_Q$.

Proof. Part (i) is clear from (10.175), (6.21), and (6.22). To prove (ii) we have only to combine (i), (10.175), and (10.73). As for (iii), assume that $f \in \mathcal{D}(X)$. Then using Proposition 10.9(ii), (10.36), and Theorem 10.5, we get

$$\langle T_{\lambda,0}, f\rangle = \frac{1}{|W|}\int_{\mathfrak{a}^*}\int_B \widetilde{f}(\zeta,b)\,\mathrm{d}b\,\overset{\circ}{T}_{\lambda,0}(\zeta)|\mathbf{c}(\zeta)|^{-2}\,\mathrm{d}\zeta.$$

This, together with (10.74) and (8.12), allows us to tell that the series $\sum_{\lambda\in\mathcal{Z}_T}\langle T_{\lambda,0}, f\rangle$ converges absolutely. Hence, the series $\sum_{\lambda\in\mathcal{Z}_T} T_{\lambda,0}$ converges in $\mathcal{D}'(X)$. To end the proof of (iii) it suffices to apply the argument in the proof of Proposition 9.11(iii). Finally, part (iv) follows from (10.175), (6.21), (6.22), and Proposition 6.9. □

Proposition 10.18. *For each* $\lambda \in \mathcal{Z}_T$, *the following assertions hold.*

(i) $(L+\lambda^2+|\rho|^2)T^{\lambda,0} = -T$.
In addition, if $n(\lambda,T) \geqslant 1$, *then*

$$\big(L+\lambda^2+|\rho|^2\big)T^{\lambda,\eta+1} = -T^{\lambda,\eta}$$

for all $\eta \in \{0,\dots,n(\lambda,T)-1\}$.

(ii) $(L+\lambda^2+|\rho|^2)T_{\lambda,n(\lambda,T)} = -b_{n(\lambda,T)}^{\lambda,n(\lambda,T)}(\overset{\circ}{T})T$.

(iii) *If* $\lambda \neq 0$ *and* $n(\lambda,T) \geqslant 1$, *then*

$$\big(L+\lambda^2+|\rho|^2\big)T_{\lambda,n(\lambda,T)-1} + 2\lambda n(\lambda,T)T_{\lambda,n(\lambda,T)} = -b_{n(\lambda,T)}^{\lambda,n(\lambda,T)-1}\big(\overset{\circ}{T}\big)T.$$

(iv) *If* $\lambda \neq 0$ *and* $n(\lambda,T) \geqslant 2$, *then*

$$\begin{aligned}\big(L+\lambda^2+|\rho|^2\big)T_{\lambda,\eta} &+ 2\lambda(\eta+1)T_{\lambda,\eta+1} + (\eta+2)(\eta+1)T_{\lambda,\eta+2}\\ &= -b_{n(\lambda,T)}^{\lambda,\eta}\big(\overset{\circ}{T}\big)T\end{aligned}$$

for all $\eta \in \{0,\dots,n(\lambda,T)-2\}$.

(v) *If* $0 \in \mathcal{Z}_T$ *and* $n(0,T) \geqslant 1$, *then*

$$\big(L + |\rho|^2\big)T_{0,\eta} + (2\eta+2)(2\eta+1)T_{0,\eta+1} = -b^{0,\eta}_{n(0,T)}\big(\overset{\circ}{T}\big)T$$

for all $\eta \in \{0,\dots,n(0,T)-1\}$.

Proof. Taking (10.175) with Proposition 6.8, we have the desired results. □

Assume now that $m \in \mathbb{N}$, $m \geqslant 2$, and $T_1,\dots,T_m$ are nonzero distributions in $\mathcal{E}'_{\natural\natural}(X)$ such that $\mathcal{Z}_{T_l} \neq \varnothing$ for all $l \in \{1,\dots,m\}$. For $l, p \in \{1,\dots,m\}$ and $\lambda_l \in \mathcal{Z}_{T_l}$, let us define the distribution $T_{\lambda_1,\dots,\lambda_m,p} \in \mathcal{E}'_{\natural\natural}(X)$ by

$$\widetilde{T_{\lambda_1,\dots,\lambda_m,p}} = \prod_{\substack{l=1,\\ l\neq p}}^{m} \widetilde{(T_l)_{\lambda_l,0}}$$

(see Theorem 10.7(ii)).

The analog of Proposition 9.16 goes as follows.

Proposition 10.19. *Let* $\bigcap_{l=1}^m \mathcal{Z}_{T_l} = \varnothing$ *and suppose that*

$$\mathcal{Z}(w_l) \cap \mathcal{Z}(w_l') = \varnothing \quad \textit{for all } l \in \{1,\dots,m\},$$

where $w_l(z) = \overset{\circ}{T}_l(\sqrt{z})$, $z \in \mathbb{C}$. *Let* $c_1,\dots,c_m \in \mathbb{C}$, $\sum_{l=1}^m c_l = 0$, *and* $\sum_{l=1}^m c_l\lambda_l^2 = 1$. *Then*

$$(T_1)_{\lambda_1,0} \times \cdots \times (T_m)_{\lambda_m,0} = -\sum_{l=1}^m c_l b_0^{\lambda_l,0}\big(\overset{\circ}{T}_l\big)T_l * T_{\lambda_1,\dots,\lambda_m,l}.$$

Proof. This result is proved similarly to Proposition 9.16 by using Proposition 10.18(ii). □

As in Sect. 9.5 for the case $m = 2$, we have the following.

Proposition 10.20. *Let* $\mathcal{Z}_{T_1} \cap \mathcal{Z}_{T_2} = \varnothing$ *and*

$$\nu = \nu(\lambda_1,\lambda_2) = n(\lambda_1,T_1) + n(\lambda_2,T_2) + 2, \quad \lambda_1 \in \mathcal{Z}_{T_1},\ \lambda_2 \in \mathcal{Z}_{T_2}.$$

Then

$$\begin{aligned}(T_1)_{\lambda_1,0} &\times (T_2)_{\lambda_2,0}\\ &= \big(\lambda_1^2-\lambda_2^2\big)^{-2\nu}\left(\sum_{p=0}^{\nu}\binom{2\nu}{\nu+p}\sum_{q=0}^{n(\lambda_1,T_1)} b_q^{\lambda_1,0}\big(\overset{\circ}{T}_1\big)\right.\\ &\quad\times\big(-L-\lambda_1^2-|\rho|^2\big)^{q+p+n(\lambda_2,T_2)+1}\end{aligned}$$

$$\times \big(L+\lambda_2^2+|\rho|^2\big)^{\nu-p}\big(T_1\times(T_2)_{\lambda_2,0}\big)+\sum_{p=1}^{\nu}\binom{2\nu}{\nu-p}\sum_{q=0}^{n(\lambda_2,T_2)} b_q^{\lambda_2,0}\big(\mathring{T}_2\big)$$

$$\times \big(L+\lambda_1^2+|\rho|^2\big)^{\nu-p}\big(-L-\lambda_2^2-|\rho|^2\big)^{q+p+n(\lambda_1,T_1)+1}\big(T_2\times(T_1)_{\lambda_1,0}\big)\Bigg).$$

This statement can be proved in the same way as Proposition 9.17 with attention to Propositions 10.18(i) and 9.13(i).

For the rest of the section, we assume that $\operatorname{rank} X=1$, $\delta\in\widehat{K}_M$, and $j\in\{1,\dots,d(\delta)\}$.

As before, let $T\in\mathcal{E}'_{\natural}(X)$, $T\neq 0$, $\lambda\in\mathcal{Z}_T$, and $\eta\in\{0,\dots,n(\lambda,T)\}$. Owing to Theorem 10.11(ii), there exists $T_{\lambda,\eta,\delta,j}\in\operatorname{conj}(\mathcal{E}'_{\delta,j}(X))$ such that

$$r(T_{\lambda,\eta,\delta,j})=r(T)$$

and

$$\mathcal{F}_j^{\delta}\big(\overline{T_{\lambda,\eta,\delta,j}}\big)(\zeta)=\overline{b^{\lambda,\eta}\big(\mathring{T},\overline{z}\big)},\quad z\in\mathbb{C}, \tag{10.177}$$

where $\zeta\in\mathfrak{a}^*_{\mathbb{C}}$ and $\zeta(H_0)/|H_0|=z$. Notice that if δ is trivial and $j=1$, then the distributions $T_{\lambda,\eta,\delta,j}$ and $T_{\lambda,\eta}$ coincide. It follows from (10.177), (6.21), (6.22), and Theorem 10.11 that

$$\operatorname{ord} T_{\lambda,\eta,\delta,j}\leqslant \operatorname{ord} T+2\alpha_X+3+s(\delta). \tag{10.178}$$

Formulae (10.177) and (6.23) give the condition of bi-orthogonality

$$\langle T_{\lambda,\eta,\delta,j},\Phi_{\mu,\nu,\delta,j}\rangle=\delta_{\lambda,\mu}\delta_{\eta,\nu} \tag{10.179}$$

for $\mu\in\mathcal{Z}_T$ and $\nu\in\{0,\dots,n(\mu,T)\}$.

Consider now the case where $T\in(\mathcal{E}'_{\natural}\cap C^m)(X)$ with $m=2\alpha_X+s(\delta)+3+l$ for some $l\in\mathbb{Z}_+$. Theorem 10.11 implies that $T_{\lambda,\eta,\delta,j}\in(\mathcal{E}'\cap C^l)(X)$ (see (6.21) and (6.22)). Moreover, if $\lambda\neq 0$, then, in view of Proposition 6.6(ii) and (10.97), (10.65), and (10.37), one has

$$\max_{x\in X}\big|(DT_{\lambda,\eta,\delta,j})(x)\big|\leqslant \gamma\sigma^{\lambda,\eta}\big(\mathring{T}\big), \tag{10.180}$$

where D is an arbitrary differential operator of order at most l, and the constant $\gamma>0$ is independent of λ,η.

Proposition 10.21. *Suppose that $r(T)>0$ and let $f\in C^m_{\delta,j}(\mathring{B}_{r(T)})$, where $m=$* $\operatorname{ord} T+4\alpha_X+6+s(\delta)$. *Then*

$$\langle T_{\lambda,\eta,\delta,j},f\rangle=\big\langle T_{\lambda,\eta},\mathcal{A}_j^{\delta}(f)\big\rangle \tag{10.181}$$

and

$$\langle T_{\lambda,\eta,\delta,j},f\rangle=\begin{cases}2\langle \Lambda(T)_{\lambda,\eta},\mathfrak{A}_{\delta,j}(f)\rangle & \text{if } \lambda\in\mathcal{Z}_T\backslash\{0\},\\ \langle \Lambda(T)_{0,2\eta},\mathfrak{A}_{\delta,j}(f)\rangle & \text{if } \lambda=0\in\mathcal{Z}_T.\end{cases} \tag{10.182}$$

Proof. It is enough to prove (10.181) and (10.182) for the case $f = \Phi_{z,0,\delta,j}$, $z \in \mathbb{C}$ (see Proposition 10.11, (10.99), (10.178), Proposition 8.5(iii), Corollary 8.1, and Theorems 10.21(iv) and 10.25(v)). However, this case can be treated directly, by using Theorem 10.21(vii), Theorem 10.25(vi), (6.19), and (10.177). □

Proposition 10.22. *Let $p \in \mathbb{N}$, $r(T) > 0$, and $f \in C^m_{\delta,j}(\mathring{B}_{r(T)})$, where $m = \operatorname{ord} T + 2\alpha_X + 8 + s(\delta) + 2p$. Suppose that*

$$\langle T, A^{\delta}_{j}(L^{\nu} f)\rangle = 0$$

for all $\nu \in \{0, \dots, p\}$. Let $\lambda \in \mathcal{Z}_T$, $|\lambda| > 1$, and $\eta \in \{0, \dots, n(\lambda, T)\}$. Then

$$|\langle T_{\lambda,\eta,\delta,j}, f\rangle| \leqslant \frac{\gamma \sigma^{\lambda,\eta}(\overset{\circ}{T})}{(|\lambda|-1)^{2p}} \sum_{i=0}^{l} \left\| \left(L + \rho_X^2\right)^{p+i} f \right\|_{L^1(B_{r(T)})},$$

where

$$l = [(\operatorname{ord} T)/2] + \left[\left(2\alpha_X + s(\delta)\right)/2\right] + 4, \tag{10.183}$$

and the constant $\gamma > 0$ is independent of λ, η, p, f.

Proof. It is not difficult to adapt the argument in the proof of Proposition 9.15 to show that

$$\left\langle \Lambda(T), \left(\mathfrak{A}_{\delta,j}(f)\right)^{(\mu)} \right\rangle = 0$$

for all $\mu \in \{0, \dots, 2p\}$ (see (10.172), Theorem 10.21(i), (iv), and (10.99)). To complete the proof we have only to combine (10.182), Theorem 10.21(iii), Proposition 8.13, (10.100), and Theorem 10.22. □

The analogues of Theorems 9.8 and 9.9 run as follows.

Theorem 10.27. *Let $r(T) > 0$ and $f \in C^m_{\delta,j}(\mathring{B}_{r(T)})$, where $m = \operatorname{ord} T + 4\alpha_X + 6 + s(\delta)$. Assume that*

$$\langle T_{\lambda,\eta,\delta,j}, f\rangle = 0 \quad \text{for all } \lambda \in \mathcal{Z}_T,\ \eta \in \{0, \dots, n(\lambda, T)\}.$$

Then $f = 0$.

Once (10.182), (10.99), and Theorem 10.21(ii), (iii) were established, the proof of this theorem is so close to that of Theorem 9.8 that we leave it for the reader.

Theorem 10.28. *Let $R > r(T)$, $f \in \mathcal{D}'(B_R)$, and let*

$$f \times T^{\lambda, n(\lambda,T)} = 0 \quad \text{for all } \lambda \in \mathcal{Z}_T. \tag{10.184}$$

Then $f = 0$. The same is valid if (10.184) is replaced by

$$f \times T_{\lambda,\eta} = 0 \quad \text{for all } \lambda \in \mathcal{Z}_T, \eta \in \{0, \dots, n(\lambda, T)\}.$$

Proof. The proof of this result is similar to that of Theorem 9.9; we just have to use Theorem 10.21(i), (ii). □

For the remainder of the section, we suppose that $r(T) > 0$. Consider now some analogies of results in Sect. 9.5 for a series of the form

$$\sum_{\lambda \in \mathcal{Z}_T} \sum_{\eta=0}^{n(\lambda,T)} c_{\lambda,\eta} \Phi_{\lambda,\eta,\delta,j}, \tag{10.185}$$

where $c_{\lambda,\eta} \in \mathbb{C}$, $\delta \in \widehat{K}_M$, $j \in \{1, \dots, d(\delta)\}$.

Proposition 10.23.

(i) *Assume that*

$$\sup_{\lambda \in \mathcal{Z}_T} \frac{\operatorname{Im} \lambda + n(\lambda, T)}{\log(2 + |\lambda|)} < +\infty, \tag{10.186}$$

and let

$$|c_{\lambda,\eta}| \leqslant (2 + |\lambda|)^{\gamma} \quad \text{for all } \lambda \in \mathcal{Z}_T,\ \eta \in \{0, \dots, n(\lambda, T)\}, \tag{10.187}$$

where $\gamma > 0$ *is independent of* λ, η. *Then series* (10.185) *converges in* $\mathcal{D}'(X)$.

(ii) *Let* $R > 0$, $q \in \mathbb{Z}_+$, $\lambda \in \mathcal{Z}_T \backslash \{0\}$, $\eta \in \{0, \dots, n(\lambda, T)\}$, *and*

$$B_{\lambda,\eta}(R, q) = |\lambda|^{q - s(\delta)} \left(R + \frac{n(\lambda, T) + q}{|\lambda|} \right)^{\eta}. \tag{10.188}$$

Suppose that

$$\sum_{\lambda \in \mathcal{Z}_T \backslash \{0\}} \sum_{\eta=0}^{n(\lambda,T)} |c_{\lambda,\eta}| B_{\lambda,\eta}(R, q) \mathrm{e}^{R \operatorname{Im} \lambda} < +\infty.$$

Then series (10.185) *converges in* $C^q(\mathring{B}_R)$. *In particular, if* (10.186) *holds and*

$$\max_{0 \leqslant \eta \leqslant n(\lambda,T)} |c_{\lambda,\eta}| = O\big(|\lambda|^{-\gamma}\big) \quad \text{as } \lambda \to \infty \tag{10.189}$$

for each fixed $\gamma > 0$, *then series* (10.185) *converges in* $\mathcal{E}(X)$.

(iii) *Assume that* (10.186) *holds and let*

$$\max_{0 \leqslant \eta \leqslant n(\lambda,T)} |c_{\lambda,\eta}| \leqslant \frac{M_q}{(2 + |\lambda|)^{2q}}, \quad q = 1, 2, \dots, \tag{10.190}$$

where the constants $M_q > 0$ *are independent of* λ, *and*

$$\sum_{\nu=1}^{\infty} \frac{1}{\inf_{q \geqslant \nu} M_q^{1/2q}} = +\infty. \tag{10.191}$$

Then series (10.185) *converges in* $\mathcal{E}(X)$ *to* $f \in \mathrm{QA}(X)$.

(iv) *Let* $\alpha > 0$, *let*

$$\operatorname{Im}\lambda + n(\lambda, T) = o\big(|\lambda|^{1/\alpha}\big) \quad \text{as } \lambda \to \infty, \tag{10.192}$$

and suppose that

$$|c_{\lambda,\eta}| \leqslant \gamma_1 \exp\big(-\gamma_2|\lambda|^{1/\alpha}\big), \tag{10.193}$$

where the constants $\gamma_1, \gamma_2 > 0$ *are independent of* λ, η. *Then series* (10.185) *converges in* $\mathcal{E}(X)$ *to* $f \in G^\alpha(X)$.

Proof. We can essentially use the same arguments as in the proof of Propositions 9.18–9.20 taking into account Proposition 10.7 and Theorem 10.11. □

The following is the analogue of Theorem 9.10.

Theorem 10.29. *Let* $p, q \in \mathbb{N}$, $q \geqslant \operatorname{ord} T + 4\alpha_X + 6 + s(\delta)$, *and let* $f \in C^m_{\delta,j}(\mathring{B}_{r(T)})$, *where* $m = \operatorname{ord} T + 4\alpha_X + 8 + s(\delta) + 2p$. *Suppose that*

$$\big\langle T, \mathcal{A}^\delta_j\big(L^\nu f\big)\big\rangle = 0 \tag{10.194}$$

for each $\nu \in \{0, \dots, p\}$ *and*

$$\sum_{\lambda \in \mathcal{Z}_T \setminus \{0\}} \sum_{\eta=0}^{n(\lambda,T)} \frac{\sigma^{\lambda,\eta}(\mathring{T})}{(1+|\lambda|)^{2p}} B_{\lambda,\eta}(r(T), q) e^{r(T)\operatorname{Im}\lambda} < +\infty$$

(*see* (10.188)). *Then*

$$f = \sum_{\lambda \in \mathcal{Z}_T} \sum_{\eta=0}^{n(\lambda,T)} c_{\lambda,\eta} \Phi_{\lambda,\eta,\delta,j}, \tag{10.195}$$

where $c_{\lambda,\eta} = \langle T_{\lambda,\eta,\delta,j}, f\rangle$, *and the series converges in* $C^q(\mathring{B}_{r(T)})$. *In addition, if* $|\lambda| > 1$, *then*

$$|c_{\lambda,\eta}| \leqslant \frac{\gamma \sigma^{\lambda,\eta}(\mathring{T})}{(|\lambda|-1)^{2p}} \sum_{i=0}^{l} \big\| \big(L + \rho_X^2\big)^{p+i} f \big\|_{L^1(B_{r(T)})} \tag{10.196}$$

where l *is defined by* (10.183), *and the constant* $\gamma > 0$ *is independent of* λ, η, p, f.

Proof. Inequality (10.196) is evident from Proposition 10.22. Once we have Proposition 10.23, Theorem 10.27, (10.179), and (10.178), we can finish the proof by arguing in the same way as in the proof of the Theorem 9.10. □

Corollary 10.9. *If* $T \in \mathfrak{M}(X)$, *then the following statements are equivalent.*

(i) $f \in C^\infty_{\delta,j}(\mathring{B}_{r(T)})$, *and* (10.194) *is fulfilled for all* $\nu \in \mathbb{Z}_+$.
(ii) *Relation* (10.195) *is valid with* $c_{\lambda,\eta} = \langle T_{\lambda,\eta,\delta,j}, f\rangle$, *and the series converges in* $\mathcal{E}(\mathring{B}_{r(T)})$.

The proof is clear from Theorem 10.25(vi), Proposition 10.7(ii), and Theorem 10.29.

Corollary 10.10. *If $T \in \mathfrak{M}(X)$, then the following statements are equivalent.*

(i) $f \in (C^{\infty}_{\delta,j} \cap \mathrm{QA})(\mathring{B}_{r(T)})$, *and* (10.194) *holds for each* $\nu \in \mathbb{Z}_+$.
(ii) *Conditions* (10.190) *and* (10.191) *are satisfied with* $c_{\lambda,\eta} = \langle T_{\lambda,\eta,\delta,j}, f\rangle$, *and the series in* (10.195) *converges to* f *in* $\mathcal{E}(\mathring{B}_{r(T)})$.

Proof. The proof of the implication (i)→(ii) can be found in Chap. 15 (see Theorems 15.19, 15.15(iii), and 15.9 and Proposition 15.9(iii)). The converse result follows by Proposition 10.23(iii), Theorem 10.25(vi), and Proposition 10.7(ii). □

In analogy with (9.124), for $f \in C^{\infty}_{\delta,j}(\mathring{B}_{r(T)})$, $\lambda \in \mathcal{Z}_T$, and $\eta \in \{0, \dots, n(\lambda, T)\}$, we define

$$\mu_{\lambda,\eta}(f) = \begin{cases} \inf\limits_{p\in\mathbb{N}} (|\lambda| - 1)^{-2p} \sum\limits_{i=0}^{l} \|(L + \rho_X^2)^{p+i} f\|_{L^1(B_{r(T)})} & \text{if } |\lambda| > 1, \\ 0 & \text{if } |\lambda| \leqslant 1, \end{cases}$$

where l is defined by (10.183). Then we have the following version of the previous theorem, which is proved in the same way.

Theorem 10.30. *Let $f \in C^{\infty}_{\delta,j}(\mathring{B}_{r(T)})$ and suppose that* (10.194) *is satisfied for each $\nu \in \mathbb{Z}_+$. Let $q \in \mathbb{N}$, $q \geqslant \operatorname{ord} T + 4\alpha_X + 6 + s(\delta)$, and*

$$\sum_{\lambda\in\mathcal{Z}_T\setminus\{0\}} \sum_{\eta=0}^{n(\lambda,T)} \sigma^{\lambda,\eta}\big(\mathring{T}\big) B_{\lambda,\eta}(r(T), q)\mu_{\lambda,\eta}(f)\mathrm{e}^{r(T)\operatorname{Im}\lambda} < +\infty.$$

Then relation (10.195) *holds with $c_{\lambda,\eta} = \langle T_{\lambda,\eta,\delta,j}, f\rangle$, and the series converges in $C^q(\mathring{B}_{r(T)})$.*

One corollary of this theorem is worth recording.

Corollary 10.11. *Let $\alpha > 0$ and $T \in \mathfrak{G}_{\alpha}(X)$. Then the following statements are equivalent.*

(i) $f \in (C^{\infty}_{\delta,j} \cap G^{\alpha})(\mathring{B}_{r(T)})$, *and* (10.194) *holds for all* $\nu \in \mathbb{Z}_+$.
(ii) *Condition* (10.193) *is satisfied with* $c_{\lambda,\eta} = \langle T_{\lambda,\eta,\delta,j}, f\rangle$, *and the series in* (10.195) *converges to* f *in* $\mathcal{E}(\mathring{B}_{r(T)})$.

Proof. The argument is quite parallel to the proof of Corollary 8.10, while we now use Theorem 10.30 and Proposition 10.23(iv). □

We end the consideration by reporting an analog of Proposition 10.23 for a series of the form

$$\sum_{\lambda \in \mathcal{Z}_T} \sum_{\eta=0}^{n(\lambda,T)} c_{\lambda,\eta} \Psi_{\lambda,\eta,\delta,j}, \tag{10.197}$$

which can be useful in Parts III and IV.

Proposition 10.24.

(i) *If conditions* (10.186) *and* (10.187) *hold, then series* (10.197) *converges in* $\mathcal{D}'(X\backslash\{o\})$.
(ii) *If* (10.186) *and* (10.189) *are valid, then series* (10.197) *converges in* $\mathcal{E}(X\backslash\{o\})$.
(iii) *If* (10.186), (10.190), *and* (10.191) *are fulfilled, then series* (10.197) *converges in* $\mathcal{E}(X\backslash\{o\})$ *to* $f \in \mathrm{QA}(X\backslash\{o\})$.
(iv) *If* (10.192) *and* (10.193) *are satisfied for some* $\alpha > 0$, *then series* (10.197) *converges in* $\mathcal{E}(X\backslash\{o\})$ *to* $f \in G^{\alpha}(X\backslash\{o\})$.

The proof is identical to that of Proposition 9.21. We just have to use Proposition 10.8.

Chapter 11
The Case of Compact Symmetric Spaces

Having in the last chapter dealt with transmutation operators on symmetric spaces of noncompact type, we shall now study the case of compact symmetric spaces U/K of rank one.

Some new features arise when we pass to the compact case. Firstly, we use substantially the realizations of symmetric spaces of Chap. 3, and the theory of spherical harmonics developed in Chap. 4. All the necessary material of these chapters is summarized in Sect. 11.1. Secondly, the set of spherical functions in this case is discrete. Accordingly, we need continuous analogues of the spherical transform and corresponding Paley–Wiener-type theorems. The method we adopt to define these analogues is based on the following argument for $\mathbb{S}^1$. Consider a smooth function $f : \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{C}$ and suppose that f has support in $[-r, r] + 2\pi\mathbb{Z}$, where $0 < r < \pi$. We denote the space of such functions by $C_r^\infty(\mathbb{S}^1)$. The Fourier transform of f is the Fourier coefficient map $n \to \widehat{f}(n)$ on $\mathbb{Z}$, where

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\mathrm{e}^{\mathrm{i}t}\right)\mathrm{e}^{-\mathrm{i}nt}\,\mathrm{d}t,$$

and it extends to a holomorphic function on $\mathbb{C}$, defined by the same formula with n replaced by $\lambda \in \mathbb{C}$. By the classical Paley–Wiener theorem for $\mathbb{R}^1$ this holomorphic extension has at most exponential growth of type r, and every holomorphic function on $\mathbb{C}$ of this type arises in this fashion from a unique function $f \in C_r^\infty(\mathbb{S}^1)$. Analogously, the Fourier coefficients of a smooth K-invariant function on U/K are given by integration of the function against the spherical functions. For functions with support in a ball, we describe the size of the support by means of the exponential type of a holomorphic extension of the Fourier coefficients (see Theorem 11.2 in Sect. 11.3). The proof of Theorem 11.2 requires a detailed discussion of local eigenfunctions of the Laplace–Beltrami operator. Various properties of these eigenfunctions are presented in Sect. 11.2. Next, we construct and investigate transmutation operators on U/K. In contrast to the noncompact case, we treat here two cases. Section 11.4 considers the case of a ball in U/K. The operators $\mathfrak{A}_{k,m,j}$ de-

V.V. Volchkov, V.V. Volchkov, *Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group,* Springer Monographs in Mathematics,

fined in Sect. 11.4 are closely related to the Jacobi polynomials expansion. Finally, Sect. 11.5 is devoted to the study of analogues of $\mathfrak{A}_{k,m,j}$ in exterior of a ball.

11.1 Compact Symmetric Spaces of Rank One from the Point of View of Realizations

Let $\mathcal{X}$ be a rank one symmetric space of compact type. We shall assume everywhere that the diameter of $\mathcal{X}$ is equal to $\pi/2$ and that $\mathcal{X}$ is realized in the same manner as in Chap. 3. By $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ we understand $\overline{\mathbb{R}^n}$ with the metric

$$g_{ij}(x) = \frac{\delta_{i,j}}{(1+|x|^2)^2}.$$

According to (3.7), (3.27), (3.39), and (3.50), $\mathcal{X} = \mathbb{R}^{a_\mathcal{X}} \cup \mathrm{Ant}\,\{0\}$, where $a_\mathcal{X} = \dim_\mathbb{R} \mathcal{X}$,

$$\mathrm{Ant}\,\{0\} = \begin{cases} \infty, & \mathcal{X} = \overline{\mathbb{R}^n}, \\ \mathbb{P}^{n-1}_\mathbb{K}, & \mathcal{X} = \mathbb{P}^n_\mathbb{K} \quad (\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{Q}), \\ S^8, & \mathcal{X} = \mathbb{P}^2_{\mathbb{C}a}. \end{cases}$$

The set $\mathrm{Ant}\,\{0\}$ is the antipodal manifold of the point $0 \in \mathbb{R}^{a_\mathcal{X}}$. The distance d on $\mathcal{X}$ is defined by

$$d(0,p) = \begin{cases} \arctan |p|, & p \in \mathbb{R}^{a_\mathcal{X}}, \\ \pi/2, & p \in \mathrm{Ant}\,\{0\}, \end{cases} \tag{11.1}$$

and the condition of invariance under the isometry group $I(\mathcal{X})$. Because of (11.1), the geodesic ball $B_r = \{p \in \mathcal{X} \colon d(0,p) < r\}$ $(0 < r \leqslant \pi/2)$ is the open Euclidean ball in $\mathbb{R}^{a_\mathcal{X}}$ of radius $\tan r$ centered at 0. The set $B_{r_1,r_2} = \{p \in \mathcal{X} \colon r_1 < d(0,p) < r_2\}$ $(0 \leqslant r_1 < \pi/2, 0 < r_2 \leqslant \pi/2)$ coincides with the annular region $\{p \in \mathbb{R}^{a_\mathcal{X}} \colon \tan r_1 < |p| < \tan r_2\}$.

Put $\alpha_\mathcal{X} = a_\mathcal{X}/2 - 1$, $\gamma_\mathcal{X} = \alpha_\mathcal{X} + \beta_\mathcal{X}$, where

$$\beta_\mathcal{X} = \begin{cases} n/2-1, & \mathcal{X} = \overline{\mathbb{R}^n}, \\ -1/2, & \mathcal{X} = \mathbb{P}^n_\mathbb{R}, \\ 0, & \mathcal{X} = \mathbb{P}^n_\mathbb{C}, \\ 1, & \mathcal{X} = \mathbb{P}^n_\mathbb{Q}, \\ 3, & \mathcal{X} = \mathbb{P}^2_{\mathbb{C}a}. \end{cases}$$

Let $\mathfrak{X} = \mathcal{X} \backslash \mathrm{Ant}\,\{0\}$. The Riemannian measure on $\mathfrak{X}$ has the form

$$\mathrm{d}\mu(p) = \frac{\mathrm{d}p}{(1+|p|^2)^{\gamma_\mathcal{X}+2}}, \tag{11.2}$$

where $\mathrm{d}p$ is the Lebesgue measure on $\mathbb{R}^{a_{\mathcal{X}}}$. The area $A_{\mathcal{X}}(r)$ of the sphere $S_r = \{p \in \mathcal{X} : d(0, p) = r\}$ $(0 < r < \pi/2)$ is calculated by

$$A_{\mathcal{X}}(r) = b_{\mathcal{X}}(\sin r)^{2\alpha_{\mathcal{X}}+1}(\cos r)^{2\beta_{\mathcal{X}}+1},$$

where

$$b_{\mathcal{X}} = \int_{\mathbb{S}^{a_{\mathcal{X}}-1}} \mathrm{d}\omega(\sigma) = \frac{2\pi^{\alpha_{\mathcal{X}}+1}}{\Gamma(\alpha_{\mathcal{X}}+1)}. \tag{11.3}$$

For the radial part L_0 of the Laplace–Beltrami operator L, on $\mathcal{X}$ we have the equalities

$$L_0 = \frac{\partial^2}{\partial r^2} + \frac{A'_{\mathcal{X}}(r)}{A_{\mathcal{X}}(r)}\frac{\partial}{\partial r} = \frac{\partial^2}{\partial r^2} + \big((2\alpha_{\mathcal{X}}+1)\cot r - (2\beta_{\mathcal{X}}+1)\tan r\big)\frac{\partial}{\partial r}.$$

Spherical functions on $\mathcal{X}$ are given by

$$\varphi_j(p) = R_j^{(\alpha_{\mathcal{X}},\beta_{\mathcal{X}})}\big(\cos\big(2d(0,p)\big)\big) = \varphi_{2j+1+\gamma_{\mathcal{X}},\alpha_{\mathcal{X}},\beta_{\mathcal{X}}}\big(d(0,p)\big), \quad j \in \mathbb{Z}_+ \tag{11.4}$$

(see (7.48)). The function φ_j satisfies the equation

$$L\varphi_j = -4j(j+\rho_{\mathcal{X}})\varphi_j$$

with $\rho_{\mathcal{X}} = \gamma_{\mathcal{X}} + 1$. The orthogonal decomposition

$$L^2(\mathcal{X}) = \bigoplus_{j=0}^{\infty} E_j \tag{11.5}$$

holds, where

$$E_j = \big\{f \in C^\infty(\mathcal{X}) : \ Lf = -4j(j+\rho_{\mathcal{X}})f\big\} \tag{11.6}$$

(see Besse [10], Chap. 8).

Take $k \in \mathbb{Z}_+$ and $m \in \big\{0, \dots, M_{\mathcal{X}}(k)\big\}$, where

$$M_{\mathcal{X}}(k) = \begin{cases} 0, & \mathcal{X} = \overline{\mathbb{R}^n},\ \mathbb{P}^n_{\mathbb{R}}, \\ [k/2], & \mathcal{X} = \mathbb{P}^n_{\mathbb{C}},\ \mathbb{P}^n_{\mathbb{Q}},\ \mathbb{P}^2_{\mathbb{C}a}. \end{cases}$$

Preserving the notation from Chap. 4, we set

$$\mathcal{H}^{k,m}_{\mathcal{X}} = \begin{cases} \mathcal{H}^{n,k}_1, & \mathcal{X} = \overline{\mathbb{R}^n},\ \mathbb{P}^n_{\mathbb{R}}, \\ \mathcal{H}^{n,k,m}_3, & \mathcal{X} = \mathbb{P}^n_{\mathbb{C}}, \\ \mathcal{H}^{n,k,m}_5, & \mathcal{X} = \mathbb{P}^n_{\mathbb{Q}}, \\ \mathcal{H}^{k,m}_6, & \mathcal{X} = \mathbb{P}^2_{\mathbb{C}a}, \end{cases}$$

$$d^{k,m}_{\mathcal{X}} = \dim \mathcal{H}^{k,m}_{\mathcal{X}}.$$

The space $\mathcal{H}^{k,m}_{\mathcal{X}}$ is an invariant subspace of the quasi-regular representation $T_{\mathcal{X}}(\tau)$ of the group $K_{\mathcal{X}}$ on $L^2(\mathbb{S}^{a_{\mathcal{X}}-1})$, where

$$K_{\mathcal{X}} = \begin{cases} \mathrm{O}(n), & \mathcal{X} = \overline{\mathbb{R}^n},\ \mathbb{P}^n_{\mathbb{R}}, \\ \mathrm{O}_{\mathbb{C}}(n), & \mathcal{X} = \mathbb{P}^n_{\mathbb{C}}, \\ \mathrm{O}_{\mathbb{Q}}(n), & \mathcal{X} = \mathbb{P}^n_{\mathbb{Q}}, \\ \mathrm{O}_{\mathbb{C}a}(2), & \mathcal{X} = \mathbb{P}^2_{\mathbb{C}a}. \end{cases}$$

Moreover, as we have seen in Chap. 4, $T_{\mathcal{X}}(\tau)$ is the orthogonal direct sum of the pairwise nonequivalent irreducible unitary representations

$$T^{k,m}_{\mathcal{X}}(\tau) = T_{\mathcal{X}}(\tau)|_{\mathcal{H}^{k,m}_{\mathcal{X}}},$$

i.e.,

$$T_{\mathcal{X}}(\tau) = \bigoplus_{k=0}^{\infty} \bigoplus_{m=0}^{M_{\mathcal{X}}(k)} T^{k,m}_{\mathcal{X}}(\tau). \tag{11.7}$$

Consider a nonempty open set $\mathcal{O} \subset \mathcal{X}$ such that

$$\tau\mathcal{O} = \mathcal{O} \quad \text{for all } \tau \in K_{\mathcal{X}}.$$

(For the action of $K_{\mathcal{X}}$ on $\mathrm{Ant}\{0\}$, see Chap. 3). Every point $p \in \mathcal{O} \setminus (\{0\} \cup \mathrm{Ant}\{0\})$ can be represented as

$$p = \varrho\sigma \quad \text{with } \varrho = |p|, \sigma = \frac{p}{|p|}. \tag{11.8}$$

To any function $f \in L^{1,\mathrm{loc}}(\mathcal{O})$ there corresponds the Fourier series

$$f(p) \sim \sum_{k=0}^{\infty} \sum_{m=0}^{M_{\mathcal{X}}(k)} \sum_{j=1}^{d^{k,m}_{\mathcal{X}}} f_{k,m,j}(\varrho) Y^{k,m}_j(\sigma), \tag{11.9}$$

where $\{Y^{k,m}_j\}$, $j \in \{1, \dots, d^{k,m}_{\mathcal{X}}\}$, is a fixed orthonormal basis in $\mathcal{H}^{k,m}_{\mathcal{X}}$, $Y^{0,0}_1(\sigma) = 1/\sqrt{b_{\mathcal{X}}}$, and

$$f_{k,m,j}(\varrho) = \int_{\mathbb{S}^{a_{\mathcal{X}}-1}} f(\varrho\sigma)\overline{Y^{k,m}_j(\sigma)}\, \mathrm{d}\omega(\sigma). \tag{11.10}$$

The function $f_{k,m,j}$ is locally summable on the set $\{t > 0 : S_{\arctan t} \subset \mathcal{O}\}$ by the Fubini theorem. Analogously, the function

$$f^{k,m,j}(p) = f_{k,m,j}(\varrho) Y^{k,m}_j(\sigma)$$

is in the class $L^{1,\mathrm{loc}}(\mathcal{O})$.

Let $\{t_{i,j}^{k,m}(\tau)\}$, $i, j \in \{1, \dots, d_{\mathcal{X}}^{k,m}\}$, be the matrix of the representation $T_{\mathcal{X}}^{k,m}(\tau)$ in the basis $\{Y_j^{k,m}\}$, that is,

$$T_{\mathcal{X}}^{k,m}(\tau)Y_j^{k,m} = \sum_{i=1}^{d_{\mathcal{X}}^{k,m}} t_{i,j}^{k,m}(\tau)Y_i^{k,m}. \tag{11.11}$$

Then

$$t_{i,j}^{k,m}(\tau) = \int_{\mathbb{S}^{a_{\mathcal{X}}-1}} Y_j^{k,m}\left(\tau^{-1}\sigma\right)\overline{Y_i^{k,m}(\sigma)}\,\mathrm{d}\omega(\sigma).$$

In particular, we see that $t_{i,j}^{k,m}$ are real-analytic functions on the group $K_{\mathcal{X}}$.

Denote by $\mathrm{d}\tau$ the Haar measure on $K_{\mathcal{X}}$ normalized by the condition

$$\int_{K_{\mathcal{X}}} \mathrm{d}\tau = 1.$$

Using (11.9), (11.11), and the orthogonality relations for $t_{i,j}^{k,m}(\tau)$ on $K_{\mathcal{X}}$, we derive

$$f^{k,m,i,j}(p) = d_{\mathcal{X}}^{k,m}\int_{K_{\mathcal{X}}} f\left(\tau^{-1}p\right)\overline{t_{i,j}^{k,m}(\tau)}\,\mathrm{d}\tau, \tag{11.12}$$

where

$$f^{k,m,i,j}(p) = f_{k,m,j}(\varrho)Y_i^{k,m}(\sigma).$$

We conclude from (11.12) that if $f \in C^s(\mathcal{O})$ for some $s \in \mathbb{Z}_+ \cup \{\infty\}$, then $f^{k,m,i,j}$ coincides almost everywhere with a function in the class $C^s(\mathcal{O})$. In addition, formula (11.12) gives

$$\begin{aligned}\int_{\mathcal{O}} f^{k,m,i,j}(p)\psi(p)\,\mathrm{d}\mu(p) &= d_{\mathcal{X}}^{k,m}\int_{\mathcal{O}} f(p)\int_{K_{\mathcal{X}}}\psi\left(\tau^{-1}p\right)t_{j,i}^{k,m}(\tau)\,\mathrm{d}\tau\,\mathrm{d}\mu(p)\\ &= \int_{\mathcal{O}} f(p)\overline{\left(\overline{\psi}\right)_{k,m,i}(\varrho)}\,\overline{Y_j^{k,m}(\sigma)}\,\mathrm{d}\mu(p)\end{aligned}$$

for each function $\psi \in \mathcal{D}(\mathcal{O})$. Now we can extend the mapping $f \to f^{k,m,i,j}$ and decomposition (11.9) to distributions $f \in \mathcal{D}'(\mathcal{O})$ as follows:

$$\begin{aligned}\left\langle f^{k,m,i,j}, \psi\right\rangle &= \left\langle f, d_{\mathcal{X}}^{k,m}\int_{K_{\mathcal{X}}}\psi\left(\tau^{-1}p\right)t_{j,i}^{k,m}(\tau)\,\mathrm{d}\tau\right\rangle\\ &= \left\langle f, \overline{\left(\overline{\psi}\right)_{k,m,i}(\varrho)}\,\overline{Y_j^{k,m}(\sigma)}\right\rangle, \quad \psi \in \mathcal{D}(\mathcal{O}),\end{aligned}$$

$$f \sim \sum_{k=0}^{\infty}\sum_{m=0}^{M_{\mathcal{X}}(k)}\sum_{j=1}^{d_{\mathcal{X}}^{k,m}} f^{k,m,j}, \tag{11.13}$$

where

$$f^{k,m,j} = f^{k,m,j,j}.$$

One can prove that series (11.13) converges in $\mathcal{D}'(\mathcal{O})$ (respectively $\mathcal{E}(\mathcal{O})$) for $f \in \mathcal{D}'(\mathcal{O})$ (respectively $f \in \mathcal{E}(\mathcal{O})$).

The mapping $f \to f^{k,m,i,j}$ is a continuous operator from $\mathcal{D}'(\mathcal{O})$ into $\mathcal{D}'(\mathcal{O})$. For $f \in \mathcal{D}'(\mathcal{O})$ and $T \in \mathcal{D}'(\mathcal{X})$ such that the set $\mathcal{U} = \{g0\colon g \in I(\mathcal{X}), g \operatorname{supp} T \subset \mathcal{O}\}$ is nonempty, we have

$$(f \times T)^{k,m,i,j} = f^{k,m,i,j} \times T \quad \text{in } \mathcal{U} \tag{11.14}$$

(see (1.56) and the proof of (9.10)). By substituting $T = P(L)\delta_0$ in (11.14), where P is an algebraic polynomial on $\mathbb{R}^1$, and δ_0 is the Dirac measure supported at 0, we obtain

$$\big(P(L)f\big)^{k,m,i,j} = P(L)f^{k,m,i,j}.$$

Let $\mathfrak{W}(\mathcal{O})$ be a certain class of distributions on $\mathcal{O}$. We write

$$\mathfrak{W}_{k,m,j}(\mathcal{O}) = \big\{f \in \mathfrak{W}(\mathcal{O})\colon f = f^{k,m,j}\big\}.$$

Note that $\mathfrak{W}_{0,0,1}(\mathcal{O}) = \mathfrak{W}_{\natural}(\mathcal{O})$, where $\mathfrak{W}_{\natural}(\mathcal{O})$ is the set of all $f \in \mathfrak{W}(\mathcal{O})$ for which

$$\langle f, \psi\rangle = \langle f, \psi \circ \tau\rangle, \quad \psi \in \mathcal{D}(\mathcal{O}), \tau \in K_{\mathcal{X}}. \tag{11.15}$$

It is not difficult to verify that the support of a distribution $f \in \mathcal{D}'_{k,m,j}(\mathcal{O})$ is $K_{\mathcal{X}}$-invariant. For $f \in \mathcal{D}'_{k,m,j}(\mathcal{O})$, we define $r(f) = \inf\{r \geqslant 0\colon \operatorname{supp} f \subset \mathring{B}_r\}$, where $\mathring{B}_r = \{p \in \mathcal{X}\colon d(0, p) \leqslant r\}$.

We now discuss the action of the operator L on $C^2_{k,m,j}(\mathcal{O})$. Put

$$\mathcal{N}_{\mathcal{X}}(k) = \begin{cases} (k+1)/2, & \mathcal{X} = \mathbb{P}^n_{\mathbb{R}}, \\ k, & \mathcal{X} \neq \mathbb{P}^n_{\mathbb{R}}. \end{cases}$$

For an open set $E \subset (0, +\infty)$, $E \neq \varnothing$, we introduce the differential operator $D(\alpha, \beta)\colon C^1(E) \to C(E)$ by the rule

$$\big(D(\alpha, \beta)\varphi\big)(\varrho) = \frac{(1+\varrho^2)^{\beta+1}}{\varrho^\alpha}\frac{\mathrm{d}}{\mathrm{d}\varrho}\bigg(\frac{\varrho^\alpha}{(1+\varrho^2)^\beta}\varphi(\varrho)\bigg), \quad \varphi \in C^1(E). \tag{11.16}$$

A simple calculation shows that

$$\begin{aligned}
&\big(D(\gamma, \delta)D(\alpha, \beta)\varphi\big)(\varrho)\\
&\quad = \big(1+\varrho^2\big)^2\varphi''(\varrho) + \frac{1+\varrho^2}{\varrho}\big(\alpha + \gamma + (\alpha + \gamma - 2\beta - 2\delta + 2)\varrho^2\big)\varphi'(\varrho)\\
&\qquad + \frac{\varphi(\varrho)}{\varrho^2}\big(\big(1+\varrho^2\big)\big(-\alpha + (\alpha - 2\beta)\varrho^2\big)\\
&\qquad + \big(\alpha + (\alpha - 2\beta)\varrho^2\big)\big(\gamma + (\gamma - 2\delta)\varrho^2\big)\big)
\end{aligned} \tag{11.17}$$

when $\varphi \in C^2(E)$.

Proposition 11.1. *For $f \in C^2_{k,m,j}(\mathcal{O})$, we have the following relations:*

$$\big(D\big(k+1+2\alpha_{\mathcal{X}}, \mathcal{N}_{\mathcal{X}}(k+1)+\gamma_{\mathcal{X}}-m\big)D\big(-k, m+1-\mathcal{N}_{\mathcal{X}}(k+1)\big)f_{k,m,j}\big)(\varrho)$$
$$\times Y_j^{k,m}(\sigma) = \big(L + 4\big(\mathcal{N}_{\mathcal{X}}(k+1)-m-1\big)\big(\mathcal{N}_{\mathcal{X}}(k+1)+\gamma_{\mathcal{X}}-m\big)\,\mathrm{Id}\big)f(p),$$
$$\big(D\big(1-k, m+1-\mathcal{N}_{\mathcal{X}}(k)\big)D\big(k+2\alpha_{\mathcal{X}}, \mathcal{N}_{\mathcal{X}}(k)+\gamma_{\mathcal{X}}-m\big)f_{k,m,j}\big)(\varrho)Y_j^{k,m}(\sigma)$$
$$= \big(L + 4\big(\mathcal{N}_{\mathcal{X}}(k)-m-1\big)\big(\mathcal{N}_{\mathcal{X}}(k)+\gamma_{\mathcal{X}}-m\big)\,\mathrm{Id}\big)f(p).$$

Furthermore, if $\mathcal{X} \neq \mathbb{P}^n_{\mathbb{R}}$, then

$$\big(D(k+1+2\alpha_{\mathcal{X}}, \alpha_{\mathcal{X}}+m+1)D(-k, \beta_{\mathcal{X}}-m\big)f_{k,m,j})(\varrho)Y_j^{k,m}(\sigma)$$
$$= \big(L + 4(m-\beta_{\mathcal{X}})(\alpha_{\mathcal{X}}+m+1)\,\mathrm{Id}\big)f(p)$$

and

$$\big(D(1-k, \beta_{\mathcal{X}}-m+1)D(k+2\alpha_{\mathcal{X}}, \alpha_{\mathcal{X}}+m)f_{k,m,j}\big)(\varrho)Y_j^{k,m}(\sigma)$$
$$= \big(L + 4(m-\beta_{\mathcal{X}}-1)(\alpha_{\mathcal{X}}+m)\,\mathrm{Id}\big)f(p).$$

Proof. Applying the formulae for the operator L from Chap. 3, we find

$$(Lf)(p) = (L_{k,m}f_{k,m,j})(\varrho)Y_j^{k,m}(\sigma), \tag{11.18}$$

where

$$L_{k,m} = \big(1+\varrho^2\big)^2\frac{\mathrm{d}^2}{\mathrm{d}\varrho^2} + \frac{1+\varrho^2}{\varrho}\big(2\alpha_{\mathcal{X}}+1+(1-2\beta_{\mathcal{X}})\varrho^2\big)\frac{\mathrm{d}}{\mathrm{d}\varrho}$$
$$+ \frac{1+\varrho^2}{\varrho^2}\big(-k(k+2\alpha_{\mathcal{X}}) + \big(-k(k-2\beta_{\mathcal{X}})+\lambda_{\mathcal{X},k,m}\big)\varrho^2\big)\,\mathrm{Id} \tag{11.19}$$

with

$$\lambda_{\mathcal{X},k,m} = 4\big(\mathcal{N}_{\mathcal{X}}(k)-m\big)\big(k+m-\beta_{\mathcal{X}}-\mathcal{N}_{\mathcal{X}}(k)\big).$$

Combining (11.17) and (11.18), we deduce the desired statement. □

In the final of this section we present a result on the smoothness of some functions in $\mathcal{X}$.

Lemma 11.1.

(i) *The function*

$$f(p) = \begin{cases} \big(1+\varrho^2\big)^{-1}, & p \in \mathfrak{X}, \\ 0, & p \in \mathrm{Ant}\{0\}, \end{cases}$$

belongs to $C^\infty(\mathcal{X})$.

(ii) *Let* $\mathcal{X} \neq \mathbb{P}^n_{\mathbb{R}}$. *Then there is* $g \in C^\infty(\mathcal{X})$ *such that*

$$g(p) = \varrho^k\big(1+\varrho^2\big)^{m-k} Y_j^{k,m}(\sigma) \quad \textit{for } p \in \mathfrak{X}\backslash\{0\}.$$

Proof. (i) In dependence on $\mathcal{X}$ the function f can be regarded as the mapping $F : \mathcal{X} \to \mathbb{R}^1$ defined as follows (see Chap. 3):

- $\mathcal{X} = \overline{\mathbb{R}^n}$:

$$F(x) = \begin{cases} \big(1+|x|^2\big)^{-1}, & x \in \mathbb{R}^n, \\ 0, & x = \infty. \end{cases}$$

- $\mathcal{X} = \mathbb{P}^n_{\mathbb{R}}, \mathbb{P}^n_{\mathbb{C}}$:

$$F\big([(\omega_0, \dots, \omega_n)]\big) = \frac{|\omega_0|^2}{|\omega_0|^2 + \dots + |\omega_n|^2}.$$

- $\mathcal{X} = \mathbb{P}^n_{\mathbb{Q}}$:

$$F\big([(\omega_0, \dots, \omega_{2n+1})]\big) = \frac{|\omega_0|^2 + |\omega_{n+1}|^2}{|\omega_0|^2 + \dots + |\omega_{2n+1}|^2}.$$

- $\mathcal{X} = \mathbb{P}^2_{\mathbb{C}a}$:

$$F\big((\xi_1, \xi_2, \xi_3, a_1, a_2, a_3)\big) = a_3.$$

This makes (i) clear.

(ii) The space $L^2_{k,m,j}(\mathcal{X})$ is infinite-dimensional and has an orthogonal basis $\{e_i\}_{i=0}^\infty$ consisting of functions in $\bigcup_{i=0}^\infty E_i$ (see (11.5) and (11.6)). Next, as we already know, $\varphi_{\lambda,\alpha,\beta}(t)$ is the solution of (7.41) which satisfies (7.43). All this, together with Proposition 11.1, allows us to conclude that for some $l \in \mathbb{Z}_+$, $c \in \mathbb{C}\backslash\{0\}$ and $h \in \{e_i\}_{i=0}^\infty$,

$$h(p) = c\varrho^k\big(1+\varrho^2\big)^{m-k} R_l^{(\alpha,\beta)}\left(\frac{1-\varrho^2}{1+\varrho^2}\right) Y_j^{k,m}(\sigma), \quad p \in \mathfrak{X}\backslash\{0\},$$

where $\alpha = k + \alpha_{\mathcal{X}}$ and $\beta = k - 2m + \beta_{\mathcal{X}}$. Since $R_l^{(\alpha,\beta)}(\pm 1) \neq 0$ (see Erdélyi (ed.) [73, 10.8 (13)]), we complete the proof using (i). □

11.2 Continuous Family of Eigenfunctions of the Laplace–Beltrami Operator

As we know, spherical functions on $\mathcal{X}$ are, up to normalization, Jacobi polynomials of the variable $\cos(2d(0, p))$ (see (11.4)). However, the set of radial eigenfunctions of the Laplace–Beltrami operator on $\mathcal{X}$ is a continuous family if we allow local eigenfunctions. Here we study this family and its generalizations.

Let $\eta, k \in \mathbb{Z}_+$, $m \in \{0, \dots, M_{\mathcal{X}}(k)\}$, $z \in \mathbb{C}$, and let $\varrho \in (0, +\infty)$. Put

$$\nu_{\mathcal{X}}(z) = \frac{\rho_{\mathcal{X}} + z}{2},$$

$$a = \nu_{\mathcal{X}}(z) + \mathcal{N}_{\mathcal{X}}(k+1) - m - 1, \qquad b = \nu_{\mathcal{X}}(-z) + \mathcal{N}_{\mathcal{X}}(k+1) - m - 1, \tag{11.20}$$

$$c = k + \alpha_{\mathcal{X}} + 1, \tag{11.21}$$

$$x = \frac{\varrho^2}{1+\varrho^2}, \qquad y = \varrho^k \left(1+\varrho^2\right)^{m+1-\mathcal{N}_{\mathcal{X}}(k+1)}. \tag{11.22}$$

For $\lambda \in \mathbb{C}$, $j \in \{1, \dots, d_{\mathcal{X}}^{k,m}\}$ and $p \in \mathfrak{X}$, we define

$$\Phi_{\lambda,\eta,k,m,j}(p) = \begin{cases} \sqrt{b_{\mathcal{X}}}\, \Phi_{\lambda,\eta}^{k,m}(\varrho) Y_j^{k,m}(\sigma) & \text{if } p = \varrho\sigma \in \mathfrak{X}\backslash\{0\}, \\ \delta_{0,\eta}\delta_{0,k} & \text{if } p = 0, \end{cases} \tag{11.23}$$

where

$$\Phi_{\lambda,\eta}^{k,m}(\varrho) = \left(\frac{\partial}{\partial z}\right)^{\varkappa} \left(yF(a,b;c;x)\right)\Big|_{z=\lambda} \tag{11.24}$$

with

$$\varkappa = \begin{cases} \eta & \text{if } \lambda \neq 0, \\ 2\eta & \text{if } \lambda = 0. \end{cases} \tag{11.25}$$

It is not hard to make sure that $\Phi_{\lambda,\eta,k,m,j} \in \mathrm{RA}(\mathfrak{X})$.

Next, in the case where $c \notin \mathbb{N}$, set

$$\Psi_{\lambda,\eta}^{k,m}(\varrho) = \left(\frac{\partial}{\partial z}\right)^{\varkappa} \left(yx^{1-c}F(a-c+1, b-c+1; 2-c; x)\right)\Big|_{z=\lambda}, \quad \lambda \in \mathbb{C}. \tag{11.26}$$

If $c \in \mathbb{N}$, we introduce $\Psi_{\lambda,\eta}^{k,m}(\varrho)$ by the formula

$$\Psi_{\lambda,\eta}^{k,m}(\varrho) = \begin{cases} \left(\frac{\partial}{\partial z}\right)^{\varkappa} \left(y\mathcal{F}(a,b,c,x)\right)\Big|_{z=\lambda}, & \lambda \in \mathbb{C}\backslash E_{\mathcal{X},k,m}, \\ \left(\frac{\partial}{\partial z}\right)^{\varkappa} \left(yF(a,b;a+b+1-c;1-x)\right)\Big|_{z=\lambda}, & \lambda \in E_{\mathcal{X},k,m}, \end{cases} \tag{11.27}$$

where

$$\mathcal{F}(a,b,c,x) = \frac{\partial}{\partial \gamma} F(a,b;\gamma;x)\Big|_{\gamma=c} - \frac{\Gamma(c)}{(a-c+1)_{c-1}(b-c+1)_{c-1}} \times \frac{\partial}{\partial \gamma}\left(x^{1-\gamma}\frac{F(a-\gamma+1, b-\gamma+1; 2-\gamma; x)}{\Gamma(2-\gamma)}\right)\Big|_{\gamma=c}$$

and

$$E_{\mathcal{X},k,m} = \left\{z \in \mathbb{C} : (a-c+1)_{c-1}(b-c+1)_{c-1} = 0\right\}.$$

Finally, for $p \in \mathfrak{X}\backslash\{0\}$, we define

$$\Psi_{\lambda,\eta,k,m,j}(p) = \Psi_{\lambda,\eta}^{k,m}(\varrho)Y_j^{k,m}(\sigma). \tag{11.28}$$

Consider the basic properties of the functions $\Phi_{\lambda,\eta,k,m,j}$ and $\Psi_{\lambda,\eta,k,m,j}$.

Proposition 11.2.

(i) *For all $k \in \mathbb{Z}_+$ and $m \in \{0,\dots,M_{\mathcal{X}}(k)\}$, we have the equality*

$$D\big(-k, m+1-\mathcal{N}_{\mathcal{X}}(k+1)\big)\Phi_{z,0}^{k,m} = \frac{2ab}{c}\Phi_{z,0}^{k+1,m}. \tag{11.29}$$

(ii) *If $m \leqslant M_{\mathcal{X}}(k+1)-1$, then*

$$D(-k,\beta_{\mathcal{X}}-m)\Phi_{z,0}^{k,m} = \frac{2(\nu_{\mathcal{X}}(z)+m-\beta_{\mathcal{X}})(\nu_{\mathcal{X}}(-z)+m-\beta_{\mathcal{X}})}{c}\Phi_{z,0}^{k+1,m+1}. \tag{11.30}$$

(iii) *If $k \geqslant 1$ and $m \leqslant M_{\mathcal{X}}(k-1)$, then*

$$D(k+2\alpha_{\mathcal{X}}, \mathcal{N}_{\mathcal{X}}(k)+\gamma_{\mathcal{X}}-m)\Phi_{z,0}^{k,m} = 2(c-1)\Phi_{z,0}^{k-1,m}. \tag{11.31}$$

(iv) *If $m \geqslant 1$, then*

$$D(k+2\alpha_{\mathcal{X}}, \alpha_{\mathcal{X}}+m)\Phi_{z,0}^{k,m} = 2(c-1)\Phi_{z,0}^{k-1,m-1}. \tag{11.32}$$

Proof. From Erdélyi (ed.) [73], Chap. 2, Sect. 2.8, one has

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^l F(\alpha,\beta;\gamma;t) = \frac{(\alpha)_l(\beta)_l}{(\gamma)_l}F(\alpha+l,\beta+l;\gamma+l;t), \tag{11.33}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^l \big(t^{\gamma-1}F(\alpha,\beta;\gamma;t)\big) = (\gamma-l)_l t^{\gamma-l-1}F(\alpha,\beta;\gamma-l;t), \tag{11.34}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^l \big((1-t)^{\alpha+\beta-\gamma}F(\alpha,\beta;\gamma;t)\big) = \frac{(\gamma-\alpha)_l(\gamma-\beta)_l}{(\gamma)_l}(1-t)^{\alpha+\beta-\gamma-l}F(\alpha,\beta;\gamma+l;t), \tag{11.35}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^l \big(t^{\gamma-1}(1-t)^{\alpha+\beta-\gamma}F(\alpha,\beta;\gamma;t)\big) = (\gamma-l)_l t^{\gamma-l-1}(1-t)^{\alpha+\beta-\gamma-l}F(\alpha-l,\beta-l;\gamma-l;t). \tag{11.36}$$

Applying (11.33)–(11.36) with $l=1$ and bearing in mind that

$$F(\alpha,\beta;\gamma;t) = (1-t)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta;\gamma;t),$$

we derive (11.29)–(11.32) by a direct calculation. □

We set $c_1(k,m,z) = 2(1-c)$ if $c \notin \mathbb{N}$,

$$c_1(k,m,z) = \frac{2(-1)^{c-1}(\Gamma(c))^2}{(a-c+1)_{c-1}(b-c+1)_{c-1}} \quad \text{if } c \in \mathbb{N}, z \in \mathbb{C} \setminus E_{\mathcal{X},k,m},$$

and

$$c_1(k,m,z) = \frac{(-2)\Gamma(a+b+1-c)\Gamma(c)}{\Gamma(a)\Gamma(b)} \quad \text{if } c \in \mathbb{N}, z \in E_{\mathcal{X},k,m}.$$

Theorem 11.1.

(i) *If* $q \in \mathbb{Z}_+$ *and* $\mu \in \mathbb{C}\backslash\{0\}$*, then in* $\mathfrak{X}$ *the following equalities are valid:*

$$L^q \Phi_{0,\eta,k,m,j} = \sum_{l=\max\{0,\eta-q\}}^{\eta} (-1)^{\eta-l}\binom{2\eta}{2l}\binom{q}{\eta-l}(2\eta-2l)!\rho_{\mathcal{X}}^{2(q-\eta+l)}\Phi_{0,l,k,m,j}, \tag{11.37}$$

$$\big(L-\rho_{\mathcal{X}}^2\big)^q \Phi_{0,\eta,k,m,j} = \begin{cases} (-1)^q(-2\eta)_{2q}\Phi_{0,\eta-q,k,m,j} & \text{if } q \leqslant \eta, \\ 0 & \text{if } q > \eta, \end{cases} \tag{11.38}$$

$$L^q \Phi_{\mu,\eta,k,m,j} = \sum_{\nu=\max\{0,\eta-2q\}}^{\eta} \sum_{\frac{\eta-\nu}{2}\leqslant l\leqslant q} (-1)^l\binom{\eta}{\nu}\binom{q}{l}\frac{(2l)!\rho_{\mathcal{X}}^{2(q-l)}\mu^{2l-\eta+\nu}}{(2l-\eta+\nu)!}\Phi_{\mu,\nu,k,m,j}, \tag{11.39}$$

$$\big(L-\rho_{\mathcal{X}}^2\big)^q \Phi_{\mu,\eta,k,m,j} = (-1)^q \sum_{l=\max\{0,\eta-2q\}}^{\eta} \binom{\eta}{l}\frac{(2q)!\mu^{2q-\eta+l}}{(2q-\eta+l)!}\Phi_{\mu,l,k,m,j}, \tag{11.40}$$

$$\big(L+\mu^2-\rho_{\mathcal{X}}^2\big)^q \Phi_{\mu,\eta,k,m,j} = \sum_{l=\max\{0,\eta-2q\}}^{\eta} (-1)^{q+l-\eta}\frac{\eta!2^{2q-\eta+l}(-q)_{2q-\eta+l}}{l!(2q-\eta+l)!}\mu^{2q-\eta+l}\Phi_{\mu,l,k,m,j}. \tag{11.41}$$

In particular, for $\lambda \in \mathbb{C}$,

$$\big(L+\lambda^2-\rho_{\mathcal{X}}^2\big)^{\eta+1}\Phi_{\lambda,\eta,k,m,j} = 0. \tag{11.42}$$

(ii) *In* (11.37)–(11.42) *the functions* $\Phi_{\lambda,\eta,k,m,j}$ *may be replaced by the corresponding functions* $\Psi_{\lambda,\eta,k,m,j}$*. In this case the relations remain true in* $\mathfrak{X}\backslash\{0\}$*. Furthermore,* $\Psi_{\lambda,0,0,0,1} \in L^{1,\mathrm{loc}}(\mathfrak{X})$ *and*

$$\big(L+\lambda^2-\rho_{\mathcal{X}}^2\big)\Psi_{\lambda,0,0,0,1} = \sqrt{b_{\mathcal{X}}}\,c_1(0,0,\lambda)\delta_0, \quad \lambda \in \mathbb{C}.$$

Proof. In view of (11.18), the equation

$$\big(L+z^2-\rho_{\mathcal{X}}^2\big)\big(u(\varrho)Y_j^{k,m}(\sigma)\big) = 0$$

reduces to

$$0 = u''(\varrho) + \left(\frac{2\alpha_{\mathcal{X}} + 1 + (1 - 2\beta_{\mathcal{X}})\varrho^2}{\varrho(1+\varrho^2)}\right)u'(\varrho)$$
$$+ \left(\frac{-k(k+2\alpha_{\mathcal{X}}) + (\lambda_{\mathcal{X},k,m} - k(k-2\beta_{\mathcal{X}}))\varrho^2}{\varrho^2(1+\varrho^2)} - \frac{\rho_{\mathcal{X}}^2 - z^2}{(1+\varrho^2)^2}\right)u(\varrho). \tag{11.43}$$

By substituting $u(\varrho) = yv(x)$ into (11.43) a hypergeometric differential equation is obtained with parameters a, b, c. This implies (11.42) for $\eta = 0$ and the same relation for the function $\Psi_{\lambda,0,k,m,j}$ in $\mathfrak{X}\backslash\{0\}$ (see Nikiforov and Uvarov [161], Chap. 4, Sect. 17.4). The rest of the proof now duplicates Theorem 10.4 (see Proposition 11.1). □

Proposition 11.3. *Let $\varrho \in (0, +\infty)$ and $z \in \mathbb{C}$. Then*

$$\Phi_{z,0}^{k,m}(\varrho)\frac{\mathrm{d}}{\mathrm{d}\varrho}\Psi_{z,0}^{k,m}(\varrho) - \Psi_{z,0}^{k,m}(\varrho)\frac{\mathrm{d}}{\mathrm{d}\varrho}\Phi_{z,0}^{k,m}(\varrho) = c_1(k,m,z)\frac{\left(1+\varrho^2\right)^{\gamma_{\mathcal{X}}}}{\varrho^{2\alpha_{\mathcal{X}}+1}}. \tag{11.44}$$

Proof. Denote by $f(\varrho)$ the expression on the left-hand side of (11.44). Since the functions $\Phi_{z,0}^{k,m}(\varrho)$ and $\Psi_{z,0}^{k,m}(\varrho)$ satisfy (11.43), by means of the Liouville–Ostrogradsky formula we find

$$f(\varrho) = \frac{c_2(1+\varrho^2)^{\gamma_{\mathcal{X}}}}{\varrho^{2\alpha_{\mathcal{X}}+1}},$$

where $c_2 = \lim_{\varrho\to 0}\varrho^{2\alpha_{\mathcal{X}}+1}f(\varrho)$. It remains to compute the constant c_2. First, suppose that $c \notin \mathbb{N}$. Then from the expansion of the hypergeometric function into a power series about the origin we obtain $c_2 = 2(1-c)$. Next, let $c \in \mathbb{N}$ and $z \in \mathbb{C}\backslash E_{\mathcal{X},k,m}$. In this case we have (see [161, 17.4 (32)])

$$\mathcal{F}(a,b,c,x) = \sum_{l=1}^{c-1}\frac{(-1)^{l-1}(c-l)_l(l-1)!}{(a-l)_l(b-l)_l}x^{-l}$$
$$+ \sum_{l=0}^{\infty}\frac{(a)_l(b)_l}{l!(c)_l}x^l\big(\log x + \psi(a+l) - \psi(a-c+1) + \psi(b+l)$$
$$- \psi(b-c+1) + \psi(c) - \psi(c+l) - \psi(l+1)\big), \tag{11.45}$$

where ψ is the logarithmic derivative of the gamma function (the first sum in (11.45) is set to be equal to zero for $c = 1$). Hence,

$$c_2 = \frac{2(-1)^{c-1}(\Gamma(c))^2}{(a-c+1)_{c-1}(b-c+1)_{c-1}}.$$

Finally, if $c \in \mathbb{N}$ and $z \in E_{\mathcal{X},k,m}$, we conclude from the relation [73, 2.10 (14)]

$$\frac{F(a,b;a+b+1-c;1-x)}{\Gamma(a+b+1-c)} = \frac{\Gamma(c-1)}{\Gamma(a)\Gamma(b)}\sum_{l=0}^{c-2}\frac{(a+1-c)_l(b+1-c)_l}{(2-c)_l l!}x^{l+1-c} + \frac{(-1)^{c-1}}{\Gamma(a+1-c)\Gamma(b+1-c)} \times \sum_{l=0}^{\infty}\frac{(a)_l(b)_l}{(l+c-1)!l!}\big(\psi(l+1)+\psi(l+c)-\psi(a+l)-\psi(b+l)-\log x\big)x^l$$

that

$$c_2 = \frac{(-2)\Gamma(a+b+1-c)\Gamma(c)}{\Gamma(a)\Gamma(b)}.$$

This finishes the proof. □

Remark 11.1. Let $\{\mu_1,\dots,\mu_r\}$ be a set of complex numbers such that the numbers $\{\mu_1^2,\dots,\mu_r^2\}$ are distinct, and let $\mathcal{O}$ be a nonempty open subset in $\mathfrak{X}$. Then Theorem 11.1, Proposition 11.3, and the proof of Proposition 10.4 show that, for the functions $\Phi_{\mu_i,\nu,k,m,j}$ and $\Psi_{\mu_i,\nu,k,m,j}$, an exact analogue of statements (i) and (ii) in Proposition 10.4 holds.

Next, we present useful integral representations for functions (11.23), (11.24). Denote

$$Q_{\mathcal{X},k,m}(t,\theta) = \frac{2^{c-1/2}\Gamma(c)\sqrt{b_{\mathcal{X}}}}{\sqrt{\pi}\Gamma(c-1/2)}(\sin\theta)^{-k-2\alpha_{\mathcal{X}}}(\cos\theta)^{-\beta_{\mathcal{X}}-1/2} \times(\cos t-\cos\theta)^{c-3/2}v_{\mathcal{X},k,m}\left(\frac{\cos\theta-\cos t}{2\cos\theta}\right), \quad (11.46)$$

where

$$v_{\mathcal{X},k,m}(z) = F\left(\frac{1}{2}+\beta_{\mathcal{X}}+k-2m, \frac{1}{2}-\beta_{\mathcal{X}}-2\mathcal{N}_{\mathcal{X}}(k)+k+2m; c-\frac{1}{2}; z\right).$$

Proposition 11.4. *For $\theta \in (0,\pi/2)$, one has*

$$\Phi_{\lambda,0}^{k,m}(\tan\theta) = \frac{1}{\sqrt{b_{\mathcal{X}}}}\int_0^{\theta}\cos(\lambda t)Q_{\mathcal{X},k,m}(t,\theta)\,\mathrm{d}t. \quad (11.47)$$

The proof of (11.47) follows from formula (7.47).

Set

$$e_{\mathcal{X},\lambda,\xi}(p) = \left(\frac{1+|p|^2}{1-2\mathrm{i}\langle p,\xi\rangle_{\mathbb{R}}-F_{\mathcal{X}}(p,\xi)}\right)^{\nu_{\mathcal{X}}(\lambda)}, \quad p\in B_{\pi/4},\ \xi\in\mathbb{S}^{a_{\mathcal{X}}-1},$$

where

$$F_{\mathcal{X}}(p,\xi)=\begin{cases}|p|^2, & \mathcal{X}=\overline{\mathbb{R}^n},\\ |\langle p,\xi\rangle_{\mathbb{K}}|^2, & \mathcal{X}=\mathbb{P}^n_{\mathbb{K}}\,(\mathbb{K}=\mathbb{R},\mathbb{C},\mathbb{Q}),\\ \Phi_{\mathbb{C}a}(p,\xi), & \mathcal{X}=\mathbb{P}^2_{\mathbb{C}a}\end{cases}$$

(see Sect. 1.1).

Proposition 11.5. *The integral representation*

$$\Phi_{\lambda,0,k,m,j}(p)=\sqrt{b_{\mathcal{X}}}\,c_3(\lambda)\int_{\mathbb{S}^{a_{\mathcal{X}}-1}}e_{\mathcal{X},\lambda,\xi}(p)Y_j^{k,m}(\xi)\,\mathrm{d}\omega_{\mathrm{norm}}(\xi),\qquad p\in B_{\pi/4},\tag{11.48}$$

holds, where

$$c_3(\lambda)=\frac{(-\mathrm{i})^k\big(a_{\mathcal{X}}/2\big)_k}{\big(\nu_{\mathcal{X}}(\lambda)\big)_{k-m}\big(\nu_{\mathcal{X}}(\lambda)-\beta_{\mathcal{X}}\big)_m}\quad \textit{if } \mathcal{X}\neq\mathbb{P}^n_{\mathbb{R}}$$

and

$$c_3(\lambda)=\frac{(-2\mathrm{i})^k(n/2)_k}{\big(2\nu_{\mathcal{X}}(\lambda)\big)_k}\quad \textit{if } \mathcal{X}=\mathbb{P}^n_{\mathbb{R}}.$$

Proof. Identity (7.52) gives

$$\Phi^{k,m}_{\lambda,\eta}(\varrho)=\left(\frac{\partial}{\partial z}\right)^{\varkappa}\Big(\varrho^k\big(1+\varrho^2\big)^{\nu_{\mathcal{X}}(z)}F\big(a,c-b;c;-\varrho^2\big)\Big)\Big|_{z=\lambda}.$$

Therefore, (11.48) follows from relations (5.9), (5.24), (5.40), and (5.42) by analytic continuation with respect to ϱ. □

We now turn to the asymptotic behavior of the functions $\Phi^{k,m}_{\lambda,0}(\varrho)$ and $\Psi^{k,m}_{\lambda,0}(\varrho)$. For brevity, we put

$$c_4=\begin{cases}-2^{c+1/2}\Gamma(c)/\sqrt{\pi} & \text{if } c\in\mathbb{N},\\ 2^{3/2-c}\Gamma(2-c)/\sqrt{\pi} & \text{if } c\notin\mathbb{N},\end{cases}$$

$$c_5=\begin{cases}1 & \text{if } c\in\mathbb{N},\\ 0 & \text{if } c\notin\mathbb{N},\end{cases}\qquad c_6=\begin{cases}c-1/2 & \text{if } c\in\mathbb{N},\\ 3/2-c & \text{if } c\notin\mathbb{N}.\end{cases}$$

Fix $r, R\in(0,\pi/2)$, $r<R$.

Proposition 11.6. *Let* $\varrho\in[\tan r,\tan R]$. *Then for every* $\varepsilon\in(0,\pi)$, *as* $\lambda\to\infty$, $|\arg\lambda|\leqslant\pi-\varepsilon$, *the asymptotic expansions*

$$\Phi^{k,m}_{\lambda,0}(\varrho)=\frac{2^{c-1/2}\Gamma(c)}{\sqrt{\pi}}\frac{(1+\varrho^2)^{\rho_{\mathcal{X}}/2}}{\varrho^{\alpha_{\mathcal{X}}+1/2}}\frac{\cos(\lambda\arctan\varrho-\pi(2c-1)/4)}{\lambda^{c-1/2}}+O\left(\frac{\mathrm{e}^{(\arctan\varrho)|\mathrm{Im}\,\lambda|}}{|\lambda|^{c+1/2}}\right)\tag{11.49}$$

and

$$\Psi^{k,m}_{\lambda,0}(\varrho) = c_4 \frac{(1+\varrho^2)^{\rho\varkappa/2}}{\varrho^{\alpha\varkappa+1/2}} (\log\lambda)^{c_5} \frac{\cos(\lambda \arctan\varrho - c_6\pi/2)}{\lambda^{c_6}} + O\left(\frac{e^{(\arctan\varrho)|\operatorname{Im}\lambda|}}{|\lambda|^{c_6-c_5+1}}\right) \tag{11.50}$$

hold, where the constants in O depend only on $\mathcal{X}$, r, R, k, m, and ε.

Proof. Applying Proposition 7.8, we obtain (11.50) with $c \notin \mathbb{N}$ and (11.49). Let us prove (11.50) when $c \in \mathbb{N}$. In this case we have [161, 17.4 (30)]

$$\begin{aligned}\mathcal{F}(a,b,c,x) = {} & \frac{(-1)^c\Gamma(a-c+1)\Gamma(b-c+1)\Gamma(c)}{\Gamma(a+b-c+1)} \\ & \times F(a,b;a+b-c+1;1-x) \\ & - \big(\psi(a-c+1)+\psi(b-c+1)-\psi(c)\big)F(a,b;c;x).\end{aligned}$$

Hence (see Erdélyi (ed.) [73, 1.2 (5) and 1.7 (11)]),

$$\begin{aligned}\mathcal{F}(a,b,c,x) = {} & \frac{(-1)^c\pi\Gamma(c)}{\Gamma(a+b-c+1)} \frac{\Gamma(a-c+1)}{\Gamma(c-b)\sin(\pi(c-b))} \\ & \times F(a,b;a+b-c+1;1-x) \\ & - \big(\psi(a-c+1)+\psi(c-b)+\pi\cot\big(\pi(c-b)\big)-\psi(c)\big) \\ & \times F(a,b;c;x).\end{aligned}$$

Next, the inequality

$$|\sin z| \geqslant \frac{1}{2\pi e} e^{|\operatorname{Im} z|}, \quad z \in \mathcal{D}, \tag{11.51}$$

is valid, where $\mathcal{D} = \{z \in \mathbb{C} : |z-\pi l| \geqslant 1/2 \text{ for all } l \in \mathbb{Z}\}$ (see Berenstein, Gay, and Yger [28]). Taking (11.51) into account, we conclude from (7.61) and the asymptotic behavior of Γ and ψ [73, 1.18 (4) and 1.18 (7)] that

$$\Psi^{k,m}_{\lambda,0}(\varrho) = \frac{-2^{c+1/2}\Gamma(c)}{\sqrt{\pi}} \frac{(1+\varrho^2)^{\rho\varkappa/2}}{\varrho^{\alpha\varkappa+1/2}} \log\lambda \frac{\cos\big(\lambda\arctan\varrho - \pi(2c-1)/4\big)}{\lambda^{c-1/2}} + O\left(\frac{e^{(\arctan\varrho)|\operatorname{Im}\lambda|}}{|\lambda|^{c-1/2}}\right) \tag{11.52}$$

as $\lambda \to \infty$, $|\arg\lambda| \leqslant \pi - \varepsilon$, $\lambda \notin \mathcal{D}$. Since the function $\Psi^{k,m}_{\lambda,0}(\varrho)$ is holomorphic with respect to the variable λ for $\lambda \notin E_{\mathcal{X},k,m}$, development (11.52) is valid as $\lambda \to \infty$, $|\arg\lambda| \leqslant \pi - \varepsilon$, by the maximum-modulus principle. Thus, (11.50) holds for $c \in \mathbb{N}$. Thereby the proposition is established. □

Corollary 11.1. *Let* $\varrho \in [\tan r, \tan R]$, $\operatorname{Re}\lambda \geqslant 0$, *and* $\theta > 2$. *Assume that* $\eta \in \mathbb{Z}_+$ *and* $\eta\theta < (\arctan\varrho)|\lambda|$. *Then*

$$\Phi^{k,m}_{\lambda,\eta}(\varrho) = \frac{2^{c-1/2}\Gamma(c)(1+\varrho^2)^{\rho_{\mathcal{X}}/2}}{\sqrt{\pi}\,\varrho^{\alpha_{\mathcal{X}}+1/2}}(\arctan\varrho)^{\eta} \times \left(\frac{\cos(\lambda\arctan\varrho - \pi(2c-2\eta-1)/4)}{\lambda^{c-1/2}} + O\left((1+\eta)\frac{e^{(\arctan\varrho)|\operatorname{Im}\lambda|}}{|\lambda|^{c+1/2}}\right)\right),$$

where the constant in O *depends only on* $\mathcal{X}, r, R, \theta, k, m$.

Proof. Combine (11.49) with Proposition 6.13. □

We set

$$\mathring{B}_{r,R} = \{p \in \mathcal{X} : r \leqslant d(0,p) \leqslant R\}.$$

Proposition 11.7. *Let* $\lambda \in \mathbb{C}$, $\eta, q \in \mathbb{Z}_+$, $\varepsilon \in (0,1)$, $\eta < \varepsilon R|\lambda|$. *Then for every differential operator* D *of order* q *on* $\mathfrak{X}$, *we have the following estimates:*

$$\|\mathrm{D}\Phi_{\lambda,\eta,k,m,j}\|_{C(\mathring{B}_{r,R})} \leqslant \gamma_1\sqrt{1+\eta}(1+|\lambda|)^{q-c+1/2}R^{\eta}e^{R|\operatorname{Im}\lambda|}, \tag{11.53}$$

$$\|\mathrm{D}\Phi_{\lambda,\eta,k,m,j}\|_{C(\mathring{B}_{R})} \leqslant \gamma_2\sqrt{1+\eta}(1+|\lambda|)^{q-k}R^{\eta}e^{R|\operatorname{Im}\lambda|}, \tag{11.54}$$

where $\gamma_1, \gamma_2 > 0$ *do not depend on* λ, η. *In addition, if* $\max\{|z| : z \in E_{\mathcal{X},k,m}\} < |\lambda| - 1$, *then*

$$\|\mathrm{D}\Psi_{\lambda,\eta,k,m,j}\|_{C(\mathring{B}_{r,R})} \leqslant \gamma_3\sqrt{1+\eta}(1+|\lambda|)^{q-c_6+1/2}\big(\log(2+|\lambda|)\big)^{c_5}R^{\eta}e^{R|\operatorname{Im}\lambda|}, \tag{11.55}$$

where $\gamma_3 > 0$ *does not depend on* λ, η.

Proof. Estimates (11.53) and (11.55) with $\eta = 0$ can be obtained from Proposition 11.6 and (11.33) by induction on $|q|$. Using now Proposition 6.11, we deduce (11.53) and (11.55) for an arbitrary $\eta \in \mathbb{Z}_+$. Let us prove (11.54). Set

$$f(p,\xi) = \frac{1+|p|^2}{1-2i\langle p,\xi\rangle_{\mathbb{R}} - F_{\mathcal{X}}(p,\xi)}, \quad p \in B_{\pi/4}, \xi \in \mathbb{S}^{a_{\mathcal{X}}-1}. \tag{11.56}$$

By (11.56) and (1.24), $1 \leqslant |f(p,\xi)| \leqslant (1+|p|^2)/(1-|p|^2)$. In addition,

$$|\arg f(p,\xi)| = \left|\arctan\frac{2\langle p,\xi\rangle_{\mathbb{R}}}{1-F_{\mathcal{X}}(p,\xi)}\right| \leqslant \arctan\frac{2|p|}{1-|p|^2} = 2\arctan|p|.$$

Hence,

$$|e_{\mathcal{X},\lambda,\xi}(p)| \leqslant \left(\frac{1+|p|^2}{1-|p|^2}\right)^{\rho_{\mathcal{X}}/2} e^{(\arctan|p|)|\operatorname{Im}\lambda|}, \tag{11.57}$$

provided that $\operatorname{Re}\lambda \leqslant 0$. Since $\Phi_{\lambda,0,k,m,j}(p)$ is an even function in λ, (11.57), (11.48), and (11.53) give

$$|\mathrm{D}\Phi_{\lambda,0,k,m,j}(p)| \leqslant \gamma_4(1+|\lambda|)^{q-k}\mathrm{e}^{R|\operatorname{Im}\lambda|}, \quad p \in \dot{B}_R,$$

where $\gamma_4 > 0$ is independent of p, λ. Then appealing to Proposition 6.11, we arrive at (11.54). □

To close this section we consider expansions in the functions $\Phi^{k,m}_{\lambda,0}(\varrho)$. According to (11.24) and (7.42),

$$\Phi^{k,m}_{\lambda,0}(\varrho) = \varrho^k\big(1+\varrho^2\big)^{m+1-\mathcal{N}_{\mathcal{X}}(k+1)}\varphi_{\lambda,\alpha(k),\beta(k,m)}(\arctan\varrho),$$

where

$$\alpha(k) = \alpha_{\mathcal{X}} + k,\ \beta(k,m) = \beta_{\mathcal{X}} + 2\mathcal{N}_{\mathcal{X}}(k+1) - k - 2m - 2. \tag{11.58}$$

In particular,

$$\mathcal{N}_{\alpha(k),\beta(k,m)}(r) = \big\{\lambda > 0 : \Phi^{k,m}_{\lambda,0}(\tan r) = 0\big\}$$

(see (7.63)). For $\lambda, \mu \in \mathcal{N}_{\alpha(k),\beta(k,m)}(r)$, we set

$$\xi(\lambda,\mu) = \int_0^{\tan r} \frac{\varrho^{2\alpha_{\mathcal{X}}+1}}{(1+\varrho^2)^{\rho_{\mathcal{X}}+1}}\Phi^{k,m}_{\lambda,0}(\varrho)\Phi^{k,m}_{\mu,0}(\varrho)\,\mathrm{d}\varrho.$$

By Lemma 7.7,

$$\xi(\lambda,\mu) = 0 \quad \text{if} \quad \lambda \neq \mu, \tag{11.59}$$

and $\xi(\lambda,\lambda)\lambda^{2\alpha(k)+2} > c_7$, where $c_7 > 0$ does not depend on λ. Next, the proof of Lemma 7.6 shows that if $v \in L^1[0,\tan r]$ and

$$\int_0^{\tan r} \frac{\varrho^{2\alpha_{\mathcal{X}}+1}}{(1+\varrho^2)^{\rho_{\mathcal{X}}+1}} v(\varrho)\Phi^{k,m}_{\lambda,0}(\varrho)\,\mathrm{d}\varrho = 0$$

for all $\lambda \in \mathcal{N}_{\alpha(k),\beta(k,m)}(r)$, then $v = 0$. Hence, as in Sect. 7.3, we obtain the following:

Proposition 11.8. *Let $r \in (0,\pi/2)$, $\zeta \in \mathbb{N}$, $\zeta > \alpha(k) + (3 - c(\alpha(k)))/2$, where $c(\alpha(k))$ is given by* (7.57). *Let $L_{k,m}$ be the operator defined by* (11.19) *and assume that a function u satisfies the following conditions:*

(1) $L^s_{k,m}u \in C^2[0,\tan r]$ *if* $s = 0,1,\dots,\zeta-1$, *and* $L^\zeta_{k,m}u \in C[0,\tan r]$.
(2) $(L^s_{k,m}u)(\tan r) = 0$, $s = 0,1,\dots,\zeta-1$.

For $\lambda \in \mathcal{N}_{\alpha(k),\beta(k,m)}(r)$, put

$$c_\lambda(u) = \big(\xi(\lambda,\lambda)\big)^{-1}\int_0^{\tan r} \frac{\varrho^{2\alpha_{\mathcal{X}}+1}}{(1+\varrho^2)^{\rho_{\mathcal{X}}+1}} u(\varrho)\Phi^{k,m}_{\lambda,0}(\varrho)\,\mathrm{d}\varrho.$$

Then $c_\lambda(u) = O(\lambda^{-2\zeta - c(\alpha(k)) + 2\alpha(k) + 2})$ *as* $\lambda \to +\infty$ *and*

$$u(\varrho) = \sum_{\lambda \in \mathcal{N}_{\alpha(k),\beta(k,m)}(r)} c_\lambda(u) \Phi^{k,m}_{\lambda,0}(\varrho), \quad \varrho \in [0, \tan r], \tag{11.60}$$

where the series in (11.60) *converges to* u *in the space* $C^s[0, \tan r]$ *with* $s < \zeta - \alpha(k) - (3 - c(\alpha(k)))/2$.

For $l \in \mathbb{Z}_+$, put

$$\lambda_l = 2l + \alpha + \beta + 1, \tag{11.61}$$

$$\mu_l = \frac{2\lambda_l}{b_{\mathcal{X}} l!} \frac{\Gamma(\alpha + l + 1)\Gamma(\lambda_l - l)}{\Gamma(\beta + l + 1)(\Gamma(\alpha + 1))^2}, \tag{11.62}$$

where $\alpha = \alpha(k)$, $\beta = \beta(k, m)$. Note that the function

$$\Phi_{\lambda_l,0,k,m,j}(p) = \sqrt{b_{\mathcal{X}}} \varrho^k \big(1 + \varrho^2\big)^{m+1-\mathcal{N}_{\mathcal{X}}(k+1)} R_l^{(\alpha,\beta)}\left(\frac{1 - \varrho^2}{1 + \varrho^2}\right) Y_j^{k,m}(\sigma)$$

belongs to $C^\infty(\mathcal{X})$ if $\mathcal{X} \neq \mathbb{P}^n_{\mathbb{R}}$ (see Lemma 11.1), and

$$\mu_l = O\big(l^{2\alpha+1}\big) \quad \text{as } l \to +\infty. \tag{11.63}$$

Proposition 11.9. *The system of functions* $\{\Phi_{\lambda_l,0,k,m,j}\}_{l=0}^\infty$ *forms an orthogonal basis in* $L^2_{k,m,j}(\mathcal{X})$. *In addition,*

$$\int_{\mathcal{X}} |\Phi_{\lambda_l,0,k,m,j}(p)|^2 \, \mathrm{d}\mu(p) = \frac{1}{\mu_l}. \tag{11.64}$$

Proof. The mapping

$$g \to G(p) = \varrho^k \big(1 + \varrho^2\big)^{m+1-\mathcal{N}_{\mathcal{X}}(k+1)} g\left(\frac{1 - \varrho^2}{1 + \varrho^2}\right) Y_j^{k,m}(\sigma)$$

is an isomorphism of the space $L^2((-1, 1), 2^{-\alpha-\beta-2}(1 - t)^\alpha (1 + t)^\beta \, \mathrm{d}t)$ onto the space $L^2_{k,m,j}(\mathcal{X})$, since

$$\begin{aligned} \int_{\mathcal{X}} |G(p)|^2 \, \mathrm{d}\mu(p) &= \int_{\mathbb{R}^{a_{\mathcal{X}}}} \varrho^{2k} \big(1 + \varrho^2\big)^{2m - 2\mathcal{N}_{\mathcal{X}}(k+1) - \gamma_{\mathcal{X}}} \\ &\quad \times \left| g\left(\frac{1 - \varrho^2}{1 + \varrho^2}\right)\right|^2 \big|Y_j^{k,m}(\sigma)\big|^2 \, \mathrm{d}p \\ &= \int_0^\infty \varrho^{2\alpha+1} \big(1 + \varrho^2\big)^{-\alpha-\beta-2} \left| g\left(\frac{1 - \varrho^2}{1 + \varrho^2}\right)\right|^2 \mathrm{d}\varrho \\ &= 2^{-\alpha-\beta-2} \int_{-1}^1 (1 - t)^\alpha (1 + t)^\beta |g(t)|^2 \, \mathrm{d}t. \end{aligned}$$

Bearing in mind that the polynomials $R_l^{(\alpha,\beta)}$, $l \in \mathbb{Z}_+$, form an orthogonal basis in $L^2((-1,1),(1-t)^\alpha(1+t)^\beta dt)$ and

$$\int_{-1}^{1}(1-t)^\alpha(1+t)^\beta\big(R_l^{(\alpha,\beta)}(t)\big)^2\,\mathrm{d}t=\frac{2^{\alpha+\beta+2}}{\mu_l b_{\mathcal{X}}}$$

(see [73, 10.8 (3) and 10.8 (4)]), we obtain the desired statement. □

Remark 11.2. Equality (11.64) shows that the Fourier coefficients c_l of a function $f\in L^2_{k,m,j}(\mathcal{X})$ with respect to the system $\{\Phi_{\lambda_l,0,k,m,j}\}_{l=0}^{\infty}$ are calculated by the formula

$$c_l=\mu_l\int_{\mathbb{R}^a\mathcal{X}} f(p)\overline{\Phi_{\lambda_l,0,k,m,j}}(p)\,\mathrm{d}\mu(p). \tag{11.65}$$

11.3 Analytic Extension of the Discrete Fourier–Jacobi Transform

Let $k\in\mathbb{Z}_+$, $m\in\{0,\dots,M_{\mathcal{X}}(k)\}$, and $j\in\{1,\dots,d_{\mathcal{X}}^{k,m}\}$. For $f\in\mathcal{E}'_{k,m,j}(\mathfrak{X})$, we put

$$\mathcal{F}_j^{k,m}(f)(\lambda)=\big\langle f,\overline{\Phi_{\overline{\lambda},0,k,m,j}}\big\rangle=\sqrt{b_{\mathcal{X}}}\Big\langle f,\Phi_{\lambda,0}^{k,m}(\varrho)\overline{Y_j^{k,m}(\sigma)}\Big\rangle,\quad \lambda\in\mathbb{C}, \tag{11.66}$$

where the distribution f acts with respect to the variable $p=\varrho\sigma$ (see (11.8)). By (11.24) and (11.25), $\mathcal{F}_j^{k,m}(f)$ is an even entire function of λ. It is the analytic continuation of the corresponding discrete Fourier–Jacobi transform (see (11.65)).

Consider some properties of the transform $\mathcal{F}_j^{k,m}$. If $f\in\mathcal{E}'_\natural(\mathfrak{X})$, we shall often write $\widetilde{f}(\lambda)$ instead of $\mathcal{F}_1^{0,0}(f)(\lambda)$, that is,

$$\widetilde{f}(\lambda)=\langle f,\Phi_{\lambda,0,0,0,1}\rangle.$$

Proposition 11.10. *Let $T\in\mathcal{E}'_\natural(\mathfrak{X})$ and $R\in(r(T),\pi/2]$. Suppose that $f\in\mathcal{D}'(B_R)$ and*

$$Lf=\big(\rho_{\mathcal{X}}^2-\lambda^2\big)f \tag{11.67}$$

for some $\lambda\in\mathbb{C}$. Then

$$f\times T=\widetilde{T}(\lambda)f \tag{11.68}$$

in the ball $B_{R-r(T)}$.

Proof. Since L is an elliptic operator, $f\in\mathrm{RA}(B_R)$ (see Hörmander [126], Chap. 8.6). Fix $g\in I(\mathcal{X})$ such that $g\dot{B}_{r(T)}\subset B_R$. For $p\in B_{r(T)+\varepsilon_0}$, where $\varepsilon_0=\sup\{\varepsilon>0: g\dot{B}_{r(T)}\subset B_{R-\varepsilon}\}$, we set

$$f_g(p)=\int_{K_{\mathcal{X}}} f(g\tau p)\,\mathrm{d}\tau.$$

The definition of f_g shows that

$$f_g \in \mathrm{RA}_\natural(B_{r(T)+\varepsilon_0}) \quad \text{and} \quad f_g(0) = f(g0). \tag{11.69}$$

In addition, in view of (11.67),

$$(Lf_g)(p) = \left(\rho_{\mathcal{X}}^2 - \lambda^2\right) f_g(p), \quad p \in B_{r(T)+\varepsilon_0}. \tag{11.70}$$

From (11.69), (11.70), and (11.18) we get

$$f_g(p) = f(g0)\Phi_{\lambda,0}^{0,0}(\varrho).$$

Now, according to (11.15),

$$\widetilde{T}(\lambda) f(g0) = \langle T, f_g\rangle = \left\langle T, \int_{K_{\mathcal{X}}} f(g\tau p)\,\mathrm{d}\tau \right\rangle = \langle T, f(gp)\rangle = (f \times T)(g0),$$

which proves (11.68). □

Proposition 11.11. *Assume that $f \in \mathcal{E}'_{k,m,j}(\mathfrak{X})$, $T \in \mathcal{E}'_\natural(\mathfrak{X})$ and let $r(f) + r(T) < \pi/2$. Then*

$$\mathcal{F}_j^{k,m}(f \times T)(\lambda) = \mathcal{F}_j^{k,m}(f)(\lambda)\widetilde{T}(\lambda). \tag{11.71}$$

In particular,

$$\mathcal{F}_j^{k,m}\big(P(L)f\big)(\lambda) = P\left(\rho_{\mathcal{X}}^2 - \lambda^2\right)\mathcal{F}_j^{k,m}(f)(\lambda) \tag{11.72}$$

for each polynomial P.

Proof. Owing to (11.42) and (11.68),

$$\left\langle f \times T, \overline{\Phi_{\overline{\lambda},0,k,m,j}}\right\rangle = \left\langle f, \overline{\Phi_{\overline{\lambda},0,k,m,j}} \times T\right\rangle = \widetilde{T}(\lambda)\left\langle f, \overline{\Phi_{\overline{\lambda},0,k,m,j}}\right\rangle.$$

This, together with (11.14), gives (11.71). By substituting $T = P(L)\delta_0$ into (11.71), we obtain (11.72). □

Proposition 11.12. *The transform $\mathcal{F}_j^{k,m}$ is injective on $\mathcal{E}'_{k,m,j}(\mathfrak{X})$.*

Proof. Let $f \in \mathcal{E}'_{k,m,j}(\mathfrak{X})$ and $\mathcal{F}_j^{k,m}(f) = 0$. Take $\psi \in \mathcal{D}_\natural(\mathfrak{X})$ for which $r(\psi) < \pi/2 - r(f)$. By (11.71), (11.66), (11.2), and (11.47) we have

$$\begin{aligned} 0 &= \mathcal{F}_j^{k,m}(f \times \psi)(\lambda) \\ &= \frac{1}{b_{\mathcal{X}}}\int_0^{\pi/2} \cos(\lambda t)\int_t^{\pi/2} A_{\mathcal{X}}(\theta)(f \times \psi)_{k,m,j}(\tan\theta)Q_{\mathcal{X},k,m}(t,\theta)\,\mathrm{d}\theta\,\mathrm{d}t. \end{aligned} \tag{11.73}$$

From (11.73) we infer that

$$\begin{aligned} 0 = \int_0^x &\left(1 - y^2\right)^{-k/2} y^{\beta_{\mathcal{X}}+1/2}(f \times \psi)_{k,m,j}\left(\sqrt{1-y^2}/y\right) \\ &\times (x-y)^{k+\alpha_{\mathcal{X}}-1/2} v_{\mathcal{X},k,m}\big((y-x)/2y\big)\,\mathrm{d}y \end{aligned}$$

for $x \in (0,1)$. Then, thanks to Corollary 6.1, $f \times \psi = 0$. Since ψ can be chosen arbitrary, we obtain $f = 0$. Hence the proposition. □

Proposition 11.13.

(i) *Assume that* $f \in (C^s \cap \mathcal{E}'_{k,m,j})(\mathfrak{X})$ *for some* $s \in \mathbb{Z}_+$. *Then*

$$\left|\mathcal{F}_j^{k,m}(f)(\lambda)\right| \leqslant c \frac{e^{r(f)|\mathrm{Im}\,\lambda|}}{(1+|\lambda|)^{s+k}}, \quad \lambda \in \mathbb{C}, \tag{11.74}$$

where the constant c does not depend on λ.

(ii) *If* $f \in \mathcal{E}'_{k,m,j}(\mathfrak{X})$ *and for some* $s \in \mathbb{Z}_+$,

$$\mathcal{F}_j^{k,m}(f)(\lambda_l) = O\big(l^{-s-k-2\alpha_{\mathcal{X}}-3}\big) \quad \text{as } l \to +\infty,$$

where λ_l *is given by* (11.61), *then* $f \in C^s(\mathfrak{X})$.

Proof. (i) By (11.2) and the definition of $\mathcal{F}_j^{k,m}$,

$$\mathcal{F}_j^{k,m}(f)(\lambda) = \sqrt{b_{\mathcal{X}}} \int_0^\infty \frac{\varrho^{2\alpha_{\mathcal{X}}+1}}{(1+\varrho^2)^{\gamma_{\mathcal{X}}+2}} f_{k,m,j}(\varrho) \Phi_{\lambda,0}^{k,m}(\varrho)\, d\varrho.$$

Integrating by parts using Proposition 11.2, we find

$$\begin{aligned} \mathcal{F}_j^{k,m}(f)(\lambda) = \kappa \int_0^{\tan r(f)} & \frac{\varrho^{2\alpha_{\mathcal{X}}+1}}{(1+\varrho^2)^{\gamma_{\mathcal{X}}+2}} \big(D_2^{s-2[s/2]}(D_1 D_2)^{[s/2]} f_{k,m,j}\big)(\varrho) \\ & \times \Phi_{\lambda,0}^{k+s-2[s/2],m}(\varrho)\, d\varrho, \end{aligned} \tag{11.75}$$

where

$$D_1 = D\big(k+1+2\alpha_{\mathcal{X}}, \mathcal{N}_{\mathcal{X}}(k+1)+\gamma_{\mathcal{X}}-m\big), \qquad D_2 = D\big(-k, m+1-\mathcal{N}_{\mathcal{X}}(k+1)\big),$$

$$\kappa = \frac{\sqrt{b_{\mathcal{X}}}}{((\rho_{\mathcal{X}} + 2(\mathcal{N}_{\mathcal{X}}(k+1)-m-1))^2 - \lambda^2)^{[s/2]}} \left(\frac{-1}{2\big(k+\alpha_{\mathcal{X}}+1\big)}\right)^{s-2[s/2]}.$$

Combining (11.75) with (11.54), we deduce (11.74).

(ii) The required assertion follows easily from Proposition 11.9, (11.65), (11.66), (11.63), and (11.54). □

We now state and prove an analogue of Theorem 9.2 for the transform $\mathcal{F}_j^{k,m}$.

Theorem 11.2.

(i) *Let* $f \in \mathcal{E}'_{k,m,j}(\mathfrak{X})$ *and* supp $f \subset \mathring{B}_r$. *Then*

$$\left|\mathcal{F}_j^{k,m}(f)(\lambda)\right| \leqslant c_1(1+|\lambda|)^{c_2} e^{r|\mathrm{Im}\,\lambda|} \quad \text{for all } \lambda \in \mathbb{C}, \tag{11.76}$$

where $c_1, c_2 > 0$ *do not depend on* λ*. Conversely, for every even entire function* $w(\lambda)$ *satisfying the estimate of the form* (11.76) *with some* $r \in [0, \pi/2)$*, there is a distribution* $f \in \mathcal{E}'_{k,m,j}(\mathfrak{X})$ *such that*

$$\operatorname{supp} f \subset \mathring{B}_r \quad \textit{and} \quad \mathcal{F}_j^{k,m}(f) = w. \tag{11.77}$$

(ii) *If* $f \in \mathcal{D}_{k,m,j}(\mathfrak{X})$ *and* supp $f \subset \mathring{B}_r$*, then for each* $N \in \mathbb{Z}_+$*, there exists a constant* $c_N > 0$ *such that*

$$\left|\mathcal{F}_j^{k,m}(f)(\lambda)\right| \leqslant c_N(1+|\lambda|)^{-N} e^{r|\operatorname{Im}\lambda|} \quad \textit{for all } \lambda \in \mathbb{C}. \tag{11.78}$$

Conversely, for every even entire function $w(\lambda)$ *satisfying the estimate of the form* (11.78) *with some* $r \in [0, \pi/2)$ *and all* $N \in \mathbb{Z}_+$*, there is a function* $f \in \mathcal{D}_{k,m,j}(\mathfrak{X})$ *such that* (11.77) *holds.*

Proof. (i) By (11.54) and the definition of ord f, for any $\varepsilon > 0$, there exists $\varkappa_\varepsilon > 0$ such that

$$\left|\mathcal{F}_j^{k,m}(f)(\lambda)\right| \leqslant \varkappa_\varepsilon e^{(r+\varepsilon)|\operatorname{Im}\lambda|}(1+|\lambda|)^{\operatorname{ord} f - k}, \quad \lambda \in \mathbb{C}. \tag{11.79}$$

Using (11.79) and the Phragmén–Lindelöf principle, we derive (11.76) (see, for example, the proof of Lemma 4.3 in Stein and Weiss [203], Chap. 3). Let us prove the converse statement. We shall distinguish two cases.

(a) The number of zeroes of the function w is finite. In this situation w is an even polynomial by virtue of Proposition 6.1(iv). We write w in the form

$$w(\lambda) = c\left(\lambda^2 - z_1^2\right)\cdots\left(\lambda^2 - z_l^2\right).$$

Consider the differential operator $Y_j^{k,m}(\partial)$ associated with the polynomial $Y_j^{k,m}(p) = \varrho^k Y_j^{k,m}(\sigma)$. By means of relations (4.2), (5.10), (5.19), and (11.3) we have $Y_j^{k,m}(\partial)^*\delta_0 \in \mathcal{E}'_{k,m,j}(\mathfrak{X})$ and

$$\mathcal{F}_j^{k,m}\left(Y_j^{k,m}(\partial)^*\delta_0\right)(\lambda) = \frac{2^k\left(\alpha_{\mathcal{X}}+1\right)_k}{\sqrt{b_{\mathcal{X}}}}, \tag{11.80}$$

where $Y_j^{k,m}(\partial)^*$ is the adjoint to the operator $Y_j^{k,m}(\partial)$. Hence, it follows from (11.72) that conditions (11.77) hold for the distribution $f = cP_1(L)Y_j^{k,m}(\partial)^*\delta_0$ with

$$P_1(t) = \frac{\sqrt{b_{\mathcal{X}}}}{2^k\left(\alpha_{\mathcal{X}}+1\right)_k}\left(-t+\rho_{\mathcal{X}}^2 - z_1^2\right)\cdots\left(-t+\rho_{\mathcal{X}}^2 - z_l^2\right).$$

(b) The function w has infinitely many zeroes. Let c_2 be the constant from estimate (11.76) for the function w, and let $s = 2\alpha_{\mathcal{X}}$. Pick a natural number $l \geqslant (s + c_2 + 6)/2$ and introduce the entire even function

$$W(\lambda) = \frac{w(\lambda)}{(\lambda^2 - z_1^2)\cdots(\lambda^2 - z_l^2)},$$

where $z_1, \dots, z_l \in \mathcal{Z}(w)$. By hypothesis on w and the Paley–Wiener theorem for the Fourier-cosine transform, there is an even function $\varphi \in C^{s+2}(\mathbb{R}^1)$ such that $\operatorname{supp} \varphi \subset [-r, r]$ and

$$W(\lambda) = \int_0^r \varphi(t) \cos(\lambda t)\, \mathrm{d}t, \quad \lambda \in \mathbb{C}.$$

We now find $h \in L_\natural^{1,\mathrm{loc}}(\mathfrak{X})$ for which

$$\operatorname{supp} h \subset \mathring{B}_r \quad \text{and} \quad \widetilde{h}(\lambda) = W(\lambda). \tag{11.81}$$

Let $\delta \leqslant y \leqslant x < 1$, where $\delta = (\cos r)/2$. Put

$$\begin{aligned}
h_1(x, y) &= v_{\mathcal{X},0,0}\left(\frac{y-x}{2y}\right),\\
h_2(x, y) &= \frac{1}{2^s} \int_{-1}^1 \left(1 - t^2\right)^{(s-1)/2} h_1\left(\frac{x+y-(x-y)t}{2}, y\right) \mathrm{d}t,\\
K(x, y) &= \frac{(\frac{\partial}{\partial x})^{s+1}((x-y)^s h_2(x, y))}{s!\, h_2(x, x)}
\end{aligned}$$

and define

$$\begin{aligned}
K_1(x, y) &= K(x, y),\\
K_{q+1}(x, y) &= \int_y^x K(x, t) K_q(t, y)\, \mathrm{d}t, \quad q \in \mathbb{N},\\
R(x, y) &= \sum_{q=1}^{\infty} (-1)^{q-1} K_q(x, y).
\end{aligned}$$

Also set

$$\psi_1(x) = v(x) - \int_\delta^x R(x, y) v(y)\, \mathrm{d}y, \tag{11.82}$$

where

$$v(x) = \frac{1}{s!\, h_2(x, x)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{s+1} \left(\int_\delta^x \varphi(\arccos t)(x-t)^{(s-1)/2}\, \mathrm{d}t\right). \tag{11.83}$$

As is well known, $\psi_1 \in C[\delta, 1)$ and

$$\psi_1(x) + \int_\delta^x K(x, y) \psi_1(y)\, \mathrm{d}y = v(x) \tag{11.84}$$

(see Vladimirov [221], Chap. 4, Sect. 17.3). We claim that (11.81) is valid for the function $h(p) = H(\varrho)$ with

$$H(\tan\theta) = \begin{cases} \dfrac{\sqrt{\pi}\,\Gamma\left(\alpha_{\mathcal{X}} + 1/2\right)}{2^{\alpha_{\mathcal{X}}+1/2}\Gamma\left(\alpha_{\mathcal{X}}+1\right) b_{\mathcal{X}}} \dfrac{\psi_1(\cos\theta)}{(\cos\theta)^{\beta_{\mathcal{X}}+1/2}}, & 0 < \theta < \arccos\delta,\\ 0, & \arccos\delta \leqslant \theta < \pi/2. \end{cases}$$

Since supp $\varphi \subset [-r, r]$, then $\psi_1 = 0$ on $[\delta, \cos r]$ and supp $h \subset \mathring{B}_r$. In addition, it is not difficult to verify that

$$|v(x)| \leqslant \begin{cases} c_3(1-x)^{-s/2}, & s \in \mathbb{N}, \\ c_4 \log \dfrac{1}{1-x}, & s = 0, \end{cases} \tag{11.85}$$

where $c_3, c_4 > 0$ are independent of x. By (11.85) and (11.82), $h \in L_{\natural}^{1,\mathrm{loc}}(\mathfrak{X})$. Next, we write

$$\psi_2(x) = \int_\delta^x \psi_1(y)(x-y)^{(s-1)/2} h_1(x, y)\, \mathrm{d}y. \tag{11.86}$$

We have

$$\int_\delta^x \psi_1(y) \int_y^x \big((t-y)(x-t)\big)^{(s-1)/2} h_1(t, y)\, \mathrm{d}t\, \mathrm{d}y = \int_\delta^x \psi_2(t)(x-t)^{(s-1)/2}\, \mathrm{d}t,$$

whence

$$\int_\delta^x \psi_1(y)(x-y)^s h_2(x, y)\, \mathrm{d}y = \int_\delta^x \psi_2(t)(x-t)^{(s-1)/2}\, \mathrm{d}t. \tag{11.87}$$

Differentiating (11.87) $s+1$ times with respect to x, we obtain

$$\psi_1(x) + \int_\delta^x \psi_1(y) K(x, y)\, \mathrm{d}y = \frac{1}{s!\, h_2(x, x)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{s+1} \left(\int_\delta^x \psi_2(t)(x-t)^{(s-1)/2}\, \mathrm{d}t\right). \tag{11.88}$$

Relations (11.83), (11.84), and (11.88) yield

$$\int_\delta^x \big(\varphi(\arccos t) - \psi_2(t)\big)(x-t)^{(s-1)/2}\, \mathrm{d}t = P_2(x) \tag{11.89}$$

for some polynomial P_2. Taking the equality

$$\varphi(\arccos t) = \psi_2(t) = 0, \qquad \delta \leqslant t \leqslant \cos r,$$

into account, we conclude that $P_2 \equiv 0$. Then, as before, (11.89) and (11.86) imply

$$\varphi(\arccos x) = \int_\delta^x \psi_1(y)(x-y)^{(s-1)/2} h_1(x, y)\, \mathrm{d}y. \tag{11.90}$$

By (11.66), (11.2), (11.47), and (11.90),

$$\begin{aligned} \widetilde{h}(\lambda) &= \frac{2^{\alpha_{\mathcal{X}}+1/2}\Gamma(\alpha_{\mathcal{X}}+1) b_{\mathcal{X}}}{\sqrt{\pi}\,\Gamma(\alpha_{\mathcal{X}}+1/2)} \int_0^{\pi/2} \cos(\lambda t) \int_t^{\pi/2} (\sin\theta)(\cos\theta)^{\beta_{\mathcal{X}}+1/2} H(\tan\theta) \\ &\quad \times (\cos t - \cos\theta)^{\alpha_{\mathcal{X}}-1/2} h_1(\cos t, \cos\theta)\, \mathrm{d}\theta\, \mathrm{d}t \\ &= \int_0^r \cos(\lambda t)\varphi(t)\, \mathrm{d}t \\ &= W(\lambda), \end{aligned}$$

as contended. Finally, from (11.81), (11.80), and Proposition 11.11 we see that the distribution $f = Y_j^{k,m}(\partial)^*\delta_0 \times P_1(L)h$ satisfies (11.77). Thus, assertion (i) is proved.

Part (ii) is a straightforward consequence of (i) and Proposition 11.13. □

Remark 11.3. Let $\mathcal{S}'(\mathbb{Z}_+)$ (respectively, $\mathcal{S}(\mathbb{Z}_+)$) be the space of slowly increasing (respectively, rapidly decreasing) sequences (see Besse [10], Chap. 8, Sect. D). Then by the technique developed in the proof of Theorem 11.2 we infer that for $\mathcal{X} \neq \mathbb{P}^n_{\mathbb{R}}$, the mapping

$$f \to \left\{\left\langle f, \overline{\Phi_{\lambda_l,0,k,m,j}}\right\rangle\right\}_{l=0}^{\infty}$$

(see (11.61)) defines a bijection between:

(i) $\mathcal{D}'_{k,m,j}(\mathcal{X})$ and $\mathcal{S}'(\mathbb{Z}_+)$;
(ii) $C^{\infty}_{k,m,j}(\mathcal{X})$ and $\mathcal{S}(\mathbb{Z}_+)$.

This is the analog of a Paley–Wiener-type theorem for compact Riemannian manifolds [10, Propositions 8.27 and 8.29].

Remark 11.4. The proofs of Theorem 11.2(i) and Propositions 11.13 and 11.11 show that the constant c_2 in (11.76) is associated with ord f as follows:

(i) The estimate

$$\mathcal{F}_j^{k,m}(f)(\lambda) = O\left((1+|\lambda|)^{\operatorname{ord} f - k} e^{r(f)|\operatorname{Im}\lambda|}\right), \quad \lambda \in \mathbb{C}, \tag{11.91}$$

holds.

(ii) Inequality (11.76) implies that ord $f \leqslant \max\left\{0, c_2 + k + 2\alpha_{\mathcal{X}} + 5\right\}$.

Let $T \in \operatorname{conj}(\mathcal{E}'_{k,m,j}(\mathfrak{X})) = \{\overline{f} : f \in \mathcal{E}'_{k,m,j}(\mathfrak{X})\}$. Utilizing Theorems 6.3 and 11.2, we introduce the distribution $\Lambda^{k,m,j}(T) \in \mathcal{E}'_{\natural}(-\pi/2, \pi/2)$ according to the rule

$$\widehat{\Lambda^{k,m,j}(T)}(\lambda) = \overline{\mathcal{F}_j^{k,m}(\overline{T})(\overline{\lambda})} = \langle T, \Phi_{\lambda,0,k,m,j}\rangle, \quad \lambda \in \mathbb{C}. \tag{11.92}$$

The correspondence $\Lambda^{k,m,j} : T \to \Lambda^{k,m,j}(T)$ is a bijection of conj $(\mathcal{E}'_{k,m,j}(\mathfrak{X}))$ onto $\mathcal{E}'_{\natural}(-\pi/2, \pi/2)$ and $r(\Lambda^{k,m,j}(T)) = r(T)$. From (11.91) and Theorem 6.3 we have

$$\operatorname{ord} \Lambda^{k,m,j}(T) \leqslant \max\{0, \operatorname{ord} T - k + 1\}.$$

Note also that $\Lambda^{0,0,1}$, or shortly Λ, acts from $\mathcal{E}'_{\natural}(\mathfrak{X})$ onto $\mathcal{E}'_{\natural}(-\pi/2, \pi/2)$. Now we can define

$$\mathfrak{W}(\mathcal{X}) = \left\{T \in \mathcal{E}'_{\natural}(\mathfrak{X}) : \Lambda(T) \in \mathfrak{W}\left(\mathbb{R}^1\right)\right\},$$

where $\mathfrak{W}(\mathbb{R}^1)$ denotes one of the classes $\mathfrak{M}(\mathbb{R}^1)$, $\mathfrak{N}(\mathbb{R}^1)$, $\operatorname{Inv}_+(\mathbb{R}^1)$, $\mathfrak{R}(\mathbb{R}^1)$ (see Sect. 8.1).

In closing, we give the inversion formula for the transform $\mathcal{F}_j^{k,m}$.

Proposition 11.14. *Suppose that* $f \in (\mathcal{E}'_{k,m,j} \cap C^s)(\mathfrak{X})$ *with some* $s \geqslant 2\alpha_{\mathcal{X}} + 3$. *Then*

$$\mu_l \mathcal{F}_j^{k,m}(f)(\lambda_l) = O\big(l^{k+2\alpha_{\mathcal{X}}+1-s}\big) \quad \text{as } l \to +\infty \tag{11.93}$$

and

$$f = \sum_{l=0}^{\infty} \mu_l \mathcal{F}_j^{k,m}(f)(\lambda_l) \Phi_{\lambda_l,0,k,m,j} \tag{11.94}$$

in $\mathfrak{X}$, *where* λ_l *and* μ_l *are given by* (11.61) *and* (11.62).

Proof. Estimate (11.93) is immediate from (11.63) and (11.74). To prove (11.94) it suffices to use Proposition 11.9 and (11.54). □

11.4 The Transmutation Operators $\mathfrak{A}_{k,m,j}$ Associated with the Jacobi Polynomials Expansion

In this section we construct and investigate transmutation operators closely connected with expansion (11.94).

Let $k \in \mathbb{Z}_+$, $m \in \{0, \ldots, M_{\mathcal{X}}(k)\}$, and $j \in \{1, \ldots, d_{\mathcal{X}}^{k,m}\}$. For $f \in \mathcal{E}'_{k,m,j}(\mathfrak{X})$, we put

$$\mathfrak{A}_{k,m,j}(f)(t) = \sum_{l=0}^{\infty} \mu_l \mathcal{F}_j^{k,m}(f)(\lambda_l) \cos(\lambda_l t), \quad t \in (-\pi/2, \pi/2), \tag{11.95}$$

where μ_l and λ_l are given by (11.62) and (11.61). In view of (11.63) and (11.91), $\mathfrak{A}_{k,m,j}(f)$ is well defined by (11.95) as a distribution in $\mathcal{D}'_\natural(-\pi/2, \pi/2)$. We shall now see that the mapping $f \to \mathfrak{A}_{k,m,j}(f)$ is an analogue of operators (9.53) and (10.101) for the spaces $\mathcal{X}$.

Lemma 11.2.

(i) *Let* $f \in \mathcal{E}'_{k,m,j}(\mathfrak{X})$, $T \in \mathcal{E}'_\natural(\mathfrak{X})$, *and suppose that* $r(f) + r(T) < \pi/2$. *Then*

$$\mathfrak{A}_{k,m,j}(f \times T) = \mathfrak{A}_{k,m,j}(f) * \Lambda(T) \tag{11.96}$$

on the interval $(-\pi/2 + r(T), \pi/2 - r(T))$.

(ii) *Let* $f \in (\mathcal{E}'_{k,m,j} \cap C^{2\alpha_{\mathcal{X}}+k+4+N})(\mathfrak{X})$ *with some* $N \in \mathbb{Z}_+$. *Then* $\mathfrak{A}_{k,m,j}(f)$ *is in* $C_\natural^N(-\pi/2, \pi/2)$, *and for* $\theta \in (0, \pi/2)$,

$$f_{k,m,j}(\tan\theta) = \int_0^\theta \mathfrak{A}_{k,m,j}(f)(t) Q_{\mathcal{X},k,m}(t,\theta)\, \mathrm{d}t, \tag{11.97}$$

where $Q_{\mathcal{X},k,m}(t,\theta)$ *is given by* (11.46).

(iii) *Let* $f \in \mathcal{E}'_{k,m,j}(\mathfrak{X})$ *and* $r \in (0, \pi/2]$. *Then* $f = 0$ *in* B_r *if and only if* $\mathfrak{A}_{k,m,j}(f) = 0$ *on* $(-r, r)$.

The proof is an immediate extension of that of Lemma 9.2 (see Propositions 11.4 and 11.14).

Let us extend the operator $\mathfrak{A}_{k,m,j}$ to the space $\mathcal{D}'_{k,m,j}(B_R)$, $R \in (0, \pi/2]$. For $f \in \mathcal{D}'_{k,m,j}(B_R)$, we set

$$\big\langle \mathfrak{A}_{k,m,j}(f), \psi \big\rangle = \big\langle \mathfrak{A}_{k,m,j}(f\eta), \psi \big\rangle, \quad \psi \in \mathcal{D}(-R, R), \tag{11.98}$$

where $\eta \in \mathcal{D}_\natural(B_R)$ and $\eta = 1$ in $B_{r_0(\psi)+\varepsilon}$ for some $\varepsilon \in \big(0, R - r_0(\psi)\big)$. By virtue of Lemma 11.2(iii), $\mathfrak{A}_{k,m,j}(f)$ is well defined by (11.98) as a distribution in $\mathcal{D}'_\natural(-R, R)$, and

$$\mathfrak{A}_{k,m,j}\big(f|_{B_r}\big) = \mathfrak{A}_{k,m,j}(f)|_{(-r,r)}$$

for each $r \in (0, R]$.

Theorem 11.3. *For $R \in (0, \pi/2]$, $N \in \mathbb{Z}_+$, and $\nu = 2\alpha_{\mathcal{X}} + k + 3 + N$, the following are true.*

(i) *Let $f \in \mathcal{D}'_{k,m,j}(B_R)$ and $r \in (0, R]$. Then $f = 0$ in B_r if and only if $\mathfrak{A}_{k,m,j}(f) = 0$ on $(-r, r)$.*

(ii) *If $f \in C^\nu_{k,m,j}(B_R)$, then $\mathfrak{A}_{k,m,j}(f) \in C^N_\natural(-R, R)$, and (11.97) is valid for $\theta \in (0, R)$. Furthermore,*

$$\mathfrak{A}_{k,m,j}(f)(0) = \frac{1}{\sqrt{b_{\mathcal{X}}}} \lim_{\varrho \to 0} f_{k,m,j}(\varrho)\varrho^{-k}. \tag{11.99}$$

(iii) *The mapping $\mathfrak{A}_{k,m,j}$ is continuous from $\mathcal{D}'_{k,m,j}(B_R)$ into $\mathcal{D}'_\natural(-R, R)$ and from $C^\nu_{k,m,j}(B_R)$ into $C^N_\natural(-R, R)$.*

(iv) *Let $f \in \mathcal{D}'_{k,m,j}(B_R)$ and* ord $f = N$. *Then* ord $\mathfrak{A}_{k,m,j}(f) \leqslant \nu$.

(v) *Suppose that $f \in C^\nu_{k,m,j}(B_R)$ has all derivatives of order $\leqslant \nu$ vanishing at* 0. *Then*

$$\mathfrak{A}_{k,m,j}(f)^{(s)}(0) = 0, \quad s = 0, \dots, N.$$

(vi) *For $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{Z}_+$, one has*

$$\mathfrak{A}_{k,m,j}(\Phi_{\lambda,\mu,k,m,j}) = u_{\lambda,\mu}, \tag{11.100}$$

where $u_{\lambda,\mu}$ is given by (9.60).

(vii) *If $T \in \operatorname{conj}(\mathcal{E}'_{k,m,j}(\mathfrak{X}))$, $r(T) < R$, and $f \in C^s_{k,m,j}(B_R)$ with $s = \max\{\operatorname{ord} T + 2\alpha_{\mathcal{X}} + 4, 2\alpha_{\mathcal{X}} + k + 3\}$, then*

$$\langle T, f\rangle = \big\langle \Lambda^{k,m,j}(T), \mathfrak{A}_{k,m,j}(f)\big\rangle.$$

(viii) *Let $f \in \mathcal{D}'_{k,m,j}(B_R)$, $T \in \mathcal{E}'_\natural(\mathfrak{X})$, and $r(T) < R$. Then (11.96) holds on the interval $(r(T) - R, R - r(T))$. In particular,*

$$\mathfrak{A}_{k,m,j}\big(P(L)f\big) = P\bigg(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \rho^2_{\mathcal{X}}\bigg)\mathfrak{A}_{k,m,j}(f) \tag{11.101}$$

for every polynomial P.

Proof. Using (11.94), (11.95), (11.92), and Lemma 11.2 and repeating the argument of Theorem 9.3, we obtain (i)–(vii). In (viii) we can assume that $f \in \mathcal{D}_{k,m,j}(B_R)$. In this case

$$f \times T = \sum_{l=0}^{\infty} \mu_l \mathcal{F}_j^{k,m}(f)(\lambda_l)\widetilde{T}(\lambda_l)\Phi_{\lambda_l,0,k,m,j},$$

where the series converges in $C^\infty(\mathfrak{X})$ (see Propositions 11.14 and 11.10). Hence, by (11.92) and (11.100),

$$\mathfrak{A}_{k,m,j}(f \times T)(t) = \sum_{l=0}^{\infty} \mu_l \mathcal{F}_j^{k,m}(f)(\lambda_l)\widehat{\Lambda(T)}(\lambda_l)\cos(\lambda_l t). \tag{11.102}$$

Comparing $\mathfrak{A}_{k,m,j}(f) * \Lambda(T)$ with (11.102), we infer that (11.96) holds on the interval $(r(T) - R, R - r(T))$. By substituting $T = P(L)\delta_0$ in (11.96) we derive (11.101). $\square$

Remark 11.5. Let $r \in (0, \pi/2)$ and $f \in C^\nu_{k,m,j}(\mathring{B}_r)$. We set

$$\mathfrak{A}_{k,m,j}(f) = \mathfrak{A}_{k,m,j}(f_1)|_{[-r,r]},$$

where $f_1 \in C^\nu_{k,m,j}(\mathfrak{X})$ is selected so that $f_1|_{\mathring{B}_r} = f$. Then $\mathfrak{A}_{k,m,j}(f) \in C^N_\natural[-r, r]$. Theorem 11.3(i), (ii) ensures the correctness of this definition.

Theorem 11.4. *Let $r \in (0, \pi/2)$. Then there exists a constant $c > 0$ such that*

$$\int_{-r}^{r} \left|\mathfrak{A}_{k,m,j}(f)^{(M)}(t)\right| \mathrm{d}t \leqslant c \sum_{i=0}^{[(k+2\alpha_{\mathcal{X}}+5)/2]} \int_{B_r} \left|\left(L - \rho_{\mathcal{X}}^2\right)^{[(M+1)/2]+i} f(p)\right| \mathrm{d}\mu(p)$$

for all $M \in \mathbb{Z}_+$ and $f \in C^s_{k,m,j}(\mathring{B}_r)$, where $s = 2([(M+1)/2]+[(k+2\alpha_{\mathcal{X}}+5)/2])$.

The proof of this theorem reproduces the proof of Theorem 9.4 (see Theorem 11.3, Proposition 11.13, Remark 11.4, and (11.59)).

We construct now an analog of operator (9.72). Let $F \in \mathcal{E}'_\natural(-\pi/2, \pi/2)$. For $w \in \mathcal{D}(\mathfrak{X})$, we put

$$\begin{aligned}\left\langle \mathfrak{B}_{k,m,j}(F), w\right\rangle &= \frac{1}{\pi}\int_0^\infty \widehat{F}(\lambda)\mathcal{F}_j^{k,m}\left(\overline{(\overline{w})_{k,m,j}(\varrho)Y_j^{k,m}(\sigma)}\right)(\lambda)\,\mathrm{d}\lambda \\ &= \frac{1}{\pi}\int_0^\infty \widehat{F}(\lambda)\langle w, \Phi_{\lambda,0,k,m,j}\rangle\,\mathrm{d}\lambda.\end{aligned}$$

It is not hard to prove that $\mathfrak{B}_{k,m,j}(F) \in \mathcal{D}'_{k,m,j}(\mathfrak{X})$.

Lemma 11.3.

(i) *Let* $F \in \mathcal{E}'_\natural(-\pi/2, \pi/2)$, $T \in \mathcal{E}'_\natural(\mathfrak{X})$, *and* $r(F) + r(T) < \pi/2$. *Then*

$$\mathfrak{B}_{k,m,j}(F) \times T = \mathfrak{B}_{k,m,j}\big(F * \Lambda(T)\big) \tag{11.103}$$

in the ball $B_{\pi/2-r(T)}$.

(ii) *If* $F \in (\mathcal{E}'_\natural \cap C^s)(-\pi/2, \pi/2)$ *for some* $s \geqslant 2$, *then* $\mathfrak{B}_{k,m,j}(F) \in C^{s+k-2}_{k,m,j}(\mathfrak{X})$ *and*

$$\mathfrak{B}_{k,m,j}(F)(p) = \int_0^{\arctan\varrho} F(t) Q_{\mathcal{X},k,m}(t, \arctan\varrho)\, \mathrm{d}t\, Y_j^{k,m}(\sigma). \tag{11.104}$$

(iii) *Let* $F \in \mathcal{E}'_\natural(-\pi/2, \pi/2)$ *and* $r \in (0, \pi/2]$. *Then* $F = 0$ *on* $(-r, r)$ *if and only if* $\mathfrak{B}_{k,m,j}(F) = 0$ *in* B_r.

The proof of this statement is the same as that of Lemma 9.3.

Owing to Lemma 11.3(iii), we can extend the mapping $\mathfrak{B}_{k,m,j}$ to the space $\mathcal{D}'_\natural(-R, R)$, $R \in (0, \pi/2]$. Namely, for $F \in \mathcal{D}'_\natural(-R, R)$, define

$$\big\langle \mathfrak{B}_{k,m,j}(F), w\big\rangle = \big\langle \mathfrak{B}_{k,m,j}(F\eta), w\big\rangle, \quad w \in \mathcal{D}(B_R),$$

where $\eta \in \mathcal{D}_\natural(-R, R)$ and $\eta = 1$ on $(-r_0(w) - \varepsilon, r_0(w) + \varepsilon)$ for some $\varepsilon \in (0, R - r_0(w))$ ($r_0(w) = \inf\{r > 0 : \operatorname{supp} w \subset B_r\}$). Then $\mathfrak{B}_{k,m,j}(F) \in \mathcal{D}'_{k,m,j}(B_R)$, and

$$\mathfrak{B}_{k,m,j}(F|_{(-r,r)}) = \mathfrak{B}_{k,m,j}(F)|_{B_r}$$

for every $r \in (0, R]$.

Theorem 11.5. *For* $R \in (0, \pi/2]$ *and* $s \in \{2, 3, \ldots\}$, *the following statements hold.*

(i) *Let* $F \in \mathcal{D}'_\natural(-R, R)$, $r \in (0, R]$. *Then* $F = 0$ *on* $(-r, r)$ *if and only if* $\mathfrak{B}_{k,m,j}(F) = 0$ *in* B_r.

(ii) *If* $F \in C^s_\natural(-R, R)$, *then* $\mathfrak{B}_{k,m,j}(F) \in C^{s+k-2}_{k,m,j}(B_R)$, *and* (11.104) *holds for* $p \in B_R\backslash\{0\}$. *In addition,*

$$\lim_{p\to 0} \frac{\mathfrak{B}_{k,m,j}(F)(p)(Y_j^{k,m}(p/|p|))^{-1}}{|p|^k} = \sqrt{b_{\mathcal{X}}}\,F(0).$$

(iii) *The mapping* $\mathfrak{B}_{k,m,j}$ *is continuous from* $\mathcal{D}'_\natural(-R, R)$ *into* $\mathcal{D}'_{k,m,j}(B_R)$ *and from* $C^s_\natural(-R, R)$ *into* $C^{s+k-2}_{k,m,j}(B_R)$.

(iv) *If* $F \in \mathcal{D}'_\natural(-R, R)$, *then* $\operatorname{ord} \mathfrak{B}_{k,m,j}(F) \leqslant \max\{0, \operatorname{ord} F - k + 3\}$.

(v) *Assume that* $F \in C^s_\natural(-R, R)$ *and* $F^{(\nu)}(0) = 0$ *for* $\nu = 0, \ldots, s$. *Then* $\mathfrak{B}_{k,m,j}(F)$ *has all derivatives of order* $\leqslant s + k - 2$ *vanishing at* 0.

(vi) *For* $F \in \mathcal{D}'_\natural(-R, R)$, *we have*

$$\mathfrak{A}_{k,m,j}\big(\mathfrak{B}_{k,m,j}(F)\big) = F.$$

(vii) *Suppose that* $T \in \operatorname{conj}(\mathcal{E}'_{k,m,j}(\mathfrak{X}))$, $r(T) < R$ *and* F *belongs to* $C^l_\natural(-R,R)$, $l = \max\{2, \operatorname{ord} T - k + 2\}$. *Then*

$$\langle T, \mathfrak{B}_{k,m,j}(F)\rangle = \langle \Lambda^{k,m,j}(T), F\rangle.$$

(viii) *Let* $F \in \mathcal{D}'_\natural(-R,R)$, $T \in \mathcal{E}'_\natural(\mathfrak{X})$, *and* $r(T) < R$. *Then* (11.103) *is valid in* $B_{R-r(T)}$. *In particular,*

$$P(L)\mathfrak{B}_{k,m,j}(F) = \mathfrak{B}_{k,m,j}\left(P\left(\rho^2_{\mathcal{X}} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\right)F\right)$$

for each polynomial P.

Proof. Lemma 11.3 and the proof of Theorem 9.5 show that assertions (i)–(vii) hold. To prove (viii) it suffices to use (vi) and Theorem 11.3(i), (viii). □

An immediate consequence of Theorems 11.3 and 11.5 is the following:

Corollary 11.2. *Let* $R \in (0, \pi/2]$. *Then the transform* $f \to \mathfrak{A}_{k,m,j}(f)$ *defines a homeomorphism between:*

(i) $\mathcal{D}'_{k,mj}(B_R)$ *and* $\mathcal{D}'_\natural(-R,R)$;
(ii) $C^\infty_{k,mj}(B_R)$ *and* $C^\infty_\natural(-R,R)$.

In addition,

$$\mathfrak{A}^{-1}_{k,m,j} = \mathfrak{B}_{k,m,j}. \tag{11.105}$$

Remark 11.6. If $F \in C^s_\natural[-r,r]$, $s \geqslant 2$, $r \in (0,\pi/2)$, put $\mathfrak{B}_{k,m,j}(F) = \mathfrak{B}_{k,m,j}(F_1)|_{\mathring{B}_r}$, where $F_1 \in C^s_\natural(-\pi/2,\pi/2)$ and $F_1|_{[-r,r]} = F$. By Theorem 11.5(i), (ii), $\mathfrak{B}_{k,m,j}(F)$ does not depend on the choice of F_1 and $\mathfrak{B}_{k,m,j}(F) \in C^{s+k-2}_{k,m,j}(\mathring{B}_r)$.

Theorem 11.6. *Let* $r \in (0,\pi/2)$. *Then there is a constant* $c > 0$ *such that for all* $N \in \mathbb{Z}_+$ *and* $F \in C^{2N+2}_\natural[-r,r]$,

$$\int_{B_r} \left|\left(L - \rho^2_{\mathcal{X}}\right)^N \mathfrak{B}_{k,m,j}(F)(p)\right| \mathrm{d}\mu(p) \leqslant c\int_{-r}^{r}\left(\left|F^{(2N)}(t)\right| + \left|F^{(2N+2)}(t)\right|\right)\mathrm{d}t.$$

The proof of Theorem 11.6 is performed as that of Theorem 9.6 with attention to Theorem 11.5(vii), (viii).

Now define the map $\mathcal{A}^{k,m}_j : \mathcal{D}'_{k,m,j}(B_R) \to \mathcal{D}'_\natural(B_R)$, $R \in (0,\pi/2]$, by

$$\mathcal{A}^{k,m}_j = \mathfrak{A}^{-1}_{0,0,1}\mathfrak{A}_{k,m,j}. \tag{11.106}$$

Then

$$\mathcal{A}^{k,m}_j(\Phi_{\lambda,\mu,k,m,j}) = \Phi_{\lambda,\mu,0,0,1} \tag{11.107}$$

and

$$\mathcal{A}^{k,m}_j(f \times T) = \mathcal{A}^{k,m}_j(f) \times T \quad \text{in } B_{R-r(T)} \tag{11.108}$$

if $f \in \mathcal{D}'_{k,m,j}(B_R)$, $T \in \mathcal{E}'_{\natural}(\mathfrak{X})$, and $r(T) < R$ (see (11.100), (11.96), (11.103), and (11.105)).

Proposition 11.15. *Let $D(\cdot,\cdot)$ be the differential operator given by* (11.16)*, and let*

$$c_k = \left(\sqrt{b_{\mathcal{X}}}\,2^k \prod_{l=0}^{k-1}\bigl(k-l+\alpha_{\mathcal{X}}\bigr)\right)^{-1}, \quad k \in \mathbb{N}.$$

Then for $f \in C^{2\alpha_{\mathcal{X}}+k+6}_{k,m,j}(B_R)$, $R \in (0,\pi/2]$, we have:

$$\mathcal{A}^{k,m}_j(f) = f \quad \text{if } k = 0,$$
$$\mathcal{A}^{k,m}_j(f) = c_k D\bigl(1+2\alpha_{\mathcal{X}}, \mathcal{N}_{\mathcal{X}}(1)+\gamma_{\mathcal{X}}\bigr)\cdots D\bigl(k+2\alpha_{\mathcal{X}}, \mathcal{N}_{\mathcal{X}}(k)+\gamma_{\mathcal{X}}\bigr)(f_{k,m,j})$$

if $k \geqslant 1$, $m = 0$, and

$$\begin{aligned}\mathcal{A}^{k,m}_j(f) = {} & c_k D\bigl(1+2\alpha_{\mathcal{X}}, \mathcal{N}_{\mathcal{X}}(1)+\gamma_{\mathcal{X}}\bigr)\cdots D\bigl(k-m+2\alpha_{\mathcal{X}}, \mathcal{N}_{\mathcal{X}}(k-m)+\gamma_{\mathcal{X}}\bigr)\\ & \times D(k-m+1+2\alpha_{\mathcal{X}}, \alpha_{\mathcal{X}}+1)\cdots D(k+2\alpha_{\mathcal{X}}, \alpha_{\mathcal{X}}+m)\bigl(f_{k,m,j}\bigr)\end{aligned}$$

if $m \geqslant 1$.

The proof of Proposition 11.15 is almost identical to that of Lemma 9.4. However, we now use (11.94), (11.107), (11.31), and (11.32).

Remark 11.7. Let $\mathcal{O}$ be an arbitrary nonempty open $K_{\mathcal{X}}$-invariant subset of $\mathfrak{X}$. Because of Proposition 11.15, we can extend the operator $\mathcal{A}^{k,m}_j$ to the space $\mathcal{D}'_{k,m,j}(\mathcal{O})$ just as we did in Sect. 9.4. In this case Theorem 9.7 has an obvious analogue for the operator $\mathcal{A}^{k,m}_j$, which we leave for the reader to state and prove.

11.5 Analogues of $\mathfrak{A}_{k,m,j}$ in Exterior of a Ball. The Zaraisky Theorem

The purpose of this section is to construct the analog of the operator $\mathfrak{A}_{k,m,j}$ for $B_{\pi/2-R,\infty} = \{p \in \mathcal{X} : d(0,p) > \pi/2 - R\}$, $0 < R \leqslant \pi/2$. All the spaces $\mathcal{X}$ considered here will be assumed to be simply connected. For the notation below, see Sects. 11.2 and 11.3.

Let $\mathcal{D}'(\mathbb{T})$ be the space of distributions on the unit circle $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Denote by $\langle u, \psi\rangle_{\mathbb{T}}$ the value of $u \in \mathcal{D}'(\mathbb{T})$ on an element $\psi \in \mathcal{D}(\mathbb{T})$. We shall freely identify $\mathcal{D}'(\mathbb{T})$ with the space of 2π-periodic distributions on $\mathbb{R}^1$.

Put

$$\Pi_\nu = \bigl\{u \in \mathcal{D}'_{\natural}\bigl(\mathbb{R}^1\bigr) : u(t-\pi) = (-1)^\nu u(t)\bigr\} \quad \text{if } \nu = 0, 1$$

and

$$\Pi_\nu = \left\{u \in \Pi_{2\{\nu/2\}} : \big\langle u(t), \cos\big((\nu - 2\mu)t\big)\big\rangle_{\mathbb{T}} = 0, \mu = 1, \dots, [\nu/2]\right\}$$

if $\nu = 2, 3, \dots$, where $\{\nu/2\} = \nu/2 - [\nu/2]$. Suppose that $\nu \in \{0, 1\}$, $r \in [0, \pi/2)$. Then, clearly, the following conditions are equivalent:

(a) $u \in \Pi_\nu$ and $\operatorname{supp} u \subset [-r, r] + \pi\mathbb{Z}$;
(b) u has the form

$$u = \left(\sum_{l=-\infty}^{\infty} (-1)^{\nu l} \delta_{\pi l}\right) * V \quad \text{for some } V \in \mathcal{D}'_{\natural}(\mathbb{R}^1) \text{ with } \operatorname{supp} V \subset [-r, r]. \tag{11.109}$$

Furthermore,

$$\big\langle u(t), \cos\big((\nu + 2l)t\big)\big\rangle_{\mathbb{T}} = 2\widehat{V}(\nu + 2l), \quad l \in \mathbb{Z}, \tag{11.110}$$

provided that u is given by (11.109). Here $\delta_{\pi l}$ is the Dirac measure supported at πl.

Take $f \in \mathcal{D}'_{k,m,j}(\mathcal{X})$ and define (see (11.61), (11.62), and (11.58))

$$\Lambda_{k,m,j}(f)(t) = (-1)^{[(\alpha+\beta)/2]+1} \sum_{l=0}^{\infty} \frac{2\lambda_l^{1-\varepsilon}(\alpha+\beta+l)!}{\Gamma(\alpha+1)\Gamma(\beta+1)l!} \big\langle f, \overline{\Phi_{\lambda_l,0,k,m,j}}\big\rangle \cos(\lambda_l t), \tag{11.111}$$

where

$$\varepsilon = 2\{(\alpha+\beta+1)/2\}.$$

Utilizing Remark 11.3, it is easy to check that $\Lambda_{k,m,j}$ is a homeomorphism of $\mathcal{D}'_{k,m,j}(\mathcal{X})$ (respectively, $C^\infty_{k,m,j}(\mathcal{X})$) onto $\Pi_{\alpha+\beta+1}$ (respectively, onto $\Pi_{\alpha+\beta+1} \cap C^\infty(\mathbb{R}^1)$). For $f \in \mathcal{E}'_{k,m,j}(\mathfrak{X})$,

$$\Lambda_{k,m,j}(f)(t) = \frac{(-1)^{[(\alpha+\beta)/2]+1}}{2^{\alpha+\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)} \sum_{l=0}^{\infty} \omega(\lambda_l) \mathcal{F}_j^{k,m}(f)(\lambda_l) \cos(\lambda_l t) \tag{11.112}$$

with

$$\omega(\lambda) = \begin{cases} \displaystyle\prod_{\mu=1}^{[(\alpha+\beta+1)/2]} \big(\lambda^2 - (\alpha+\beta+1-2\mu)^2\big) & \text{if } \alpha+\beta \geqslant 1, \\ 1 & \text{otherwise.} \end{cases}$$

We also note the equality

$$\Lambda_{k,m,j}(f)\left(t + \frac{\pi}{2}\right) = (-1)^{\varepsilon} \Lambda_{k,m,j}(f)\left(\frac{\pi}{2} - t\right). \tag{11.113}$$

Lemma 11.4. *Let* $r \in [0, \pi/2)$ *and* $f \in \mathcal{D}'_{k,m,j}(\mathcal{X})$. *Then* $\operatorname{supp} f \subset \mathring{B}_r$ *if and only if*

$$\operatorname{supp} \Lambda_{k,m,j}(f) \subset [-r, r] + \pi\mathbb{Z}. \tag{11.114}$$

Proof. Assume that $\operatorname{supp} f \subset \mathring{B}_r$. According to Theorems 11.2 and 6.3, there exists $V \in \mathcal{D}'_{\natural}(\mathbb{R}^1)$ with $\operatorname{supp} V \subset [-r, r]$ such that

$$\widehat{V}(\lambda) = \frac{(-1)^{[(\alpha+\beta)/2]+1}\pi}{2^{\alpha+\beta}\Gamma(\alpha+1)\Gamma(\beta+1)}\omega(\lambda)\mathcal{F}_j^{k,m}(f)(\lambda), \quad \lambda \in \mathbb{C}.$$

If we define u by (11.109), where $\nu = \varepsilon$, we have

$$\big\langle u(t), \cos\big((\varepsilon + 2l)t\big)\big\rangle_{\mathbb{T}} = \big\langle \Lambda_{k,m,j}(f)(t), \cos\big((\varepsilon + 2l)t\big)\big\rangle_{\mathbb{T}}, \quad l \in \mathbb{Z}_+$$

because of (11.110) and (11.112). This gives $u = \Lambda_{k,m,j}(f)$. Hence the “only if” part of the lemma. Conversely, let (11.114) be satisfied. Then the distribution $u = \Lambda_{k,m,j}(f)$ has the form (11.109), where $\nu = \varepsilon$, and $\widehat{V}/\omega$ is an even entire function of exponential type at most r, since

$$\begin{aligned}\widehat{V}(\alpha+\beta+1-2\mu) &= \frac{1}{2}\big\langle \Lambda_{k,m,j}(f)(t), \cos\big((\alpha+\beta+1-2\mu)t\big)\big\rangle_{\mathbb{T}} \\ &= 0, \quad \mu = 1, \dots, [(\alpha+\beta+1)/2]\end{aligned}$$

in the case $\alpha + \beta \geqslant 1$. By Theorem 11.2,

$$\widehat{V}(\lambda) = \frac{(-1)^{[(\alpha+\beta)/2]+1}\pi}{2^{\alpha+\beta}\Gamma(\alpha+1)\Gamma(\beta+1)}\omega(\lambda)\mathcal{F}_j^{k,m}(f_0)(\lambda), \quad \lambda \in \mathbb{C}, \tag{11.115}$$

for some $f_0 \in \mathcal{D}'_{k,m,j}(\mathcal{X})$ supported in $\mathring{B}_r$. Relations (11.115) and (11.111) and equality (11.110) with $\nu = \varepsilon$ imply

$$\mathcal{F}_j^{k,m}(f_0)(\lambda_l) = \big\langle f, \overline{\Phi_{\lambda_l,0,k,m,j}}\big\rangle, \quad l \in \mathbb{Z}_+,$$

which means $f = f_0$. Thus, $\operatorname{supp} f \subset \mathring{B}_r$, and Lemma 11.4 is proved. □

Let $0 < R \leqslant \pi/2$. Introduce the mapping $\mathfrak{C}_{k,m,j} : \mathcal{D}'_{k,m,j}(B_{\pi/2-R,\infty}) \to \mathcal{D}'_{\natural}(-R, R)$ as follows. If $f \in \mathcal{D}'_{k,m,j}(B_{\pi/2-R,\infty})$, we set

$$\mathfrak{C}_{k,m,j}(f) = \frac{1}{\sqrt{b_{\mathcal{X}}}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\varepsilon}\Lambda_{k,m,j}(f\eta)\left(t + \frac{\pi}{2}\right) \quad \text{on } (-R', R') \text{ for all } R' \in (0, R), \tag{11.116}$$

where $\eta \in \mathcal{D}_{\natural}(B_{\pi/2-R,\infty})$ is selected so that $\eta = 1$ in $B_{\pi/2-R',\infty}$. Lemma 11.4 and (11.113) assure the correctness of definition (11.116). In addition, $\mathfrak{C}_{k,m,j}(f) \in C_{\natural}^{\infty}(-R, R)$ if $f \in C_{k,m,j}^{\infty}(B_{\pi/2-R,\infty})$.

With the help of (11.116), (11.111), (11.64), and the identity

$$R_l^{(\alpha,\beta)}(-1) = (-1)^l \frac{\Gamma(\alpha+1)\Gamma(\beta+l+1)}{\Gamma(\alpha+l+1)\Gamma(\beta+1)}$$

(see Erdélyi (ed.) [73, 10.8 (3) and 10.8 (13)]), one finds

$$\mathfrak{C}_{k,m,j}(\Phi_{\lambda_l,0,k,m,j})(t) = \sqrt{b_{\mathcal{X}}}\, R_l^{(\alpha,\beta)}(-1)\cos(\lambda_l t). \tag{11.117}$$

We shall use (11.117) in order to establish the analog of (11.99) for the operator $\mathfrak{C}_{k,m,j}$.

Lemma 11.5. *Let* $g \in C^\infty[0,1]$. *Then the function*

$$G(p) = \varrho^k \left(1+\varrho^2\right)^{m-k} g\left(\frac{1}{1+\varrho^2}\right) Y_j^{k,m}(\sigma)$$

is in $C^\infty(\mathcal{X})$, *and*

$$\mathfrak{C}_{k,m,j}(G)(0) = g(0). \tag{11.118}$$

Proof. The first statement of the lemma is a consequence of Lemma 11.1. To prove (11.118) we expand g on $[0,1]$ into a uniformly convergent series:

$$g(t) = \sum_{l=0}^{\infty} c_l R_l^{(\alpha,\beta)}(2t-1), \tag{11.119}$$

where $c_l \in \mathbb{C}$ and $c_l = O(l^{-N})$ as $l \to +\infty$ for any $N \in \mathbb{N}$ (see Suetin [206], Chap. 7, Theorem 7.6). Then

$$G = \sum_{l=0}^{\infty} \frac{c_l}{\sqrt{b_{\mathcal{X}}}} \Phi_{\lambda_l,0,k,m,j}. \tag{11.120}$$

Combining (11.120), (11.117), and (11.119), we obtain (11.118). □

For $\lambda \in \mathbb{C}$ and $p \in \mathfrak{X}\backslash\{0\}$, we define

$$\Theta_{\lambda,\eta,k,m,j}(p) = \left(\frac{\partial}{\partial z}\right)^{\varkappa} \left(yF(a,b;a+b+1-c;1-x)\right)\Big|_{z=\lambda} Y_j^{k,m}(\sigma),$$

where a, b, c, x, y, and $\varkappa$ are given in (11.20)–(11.22) and (11.25). By Lemma 11.1 the function $\Theta_{\lambda,\eta,k,m,j}$ admits continuous extension to Ant$\{0\}$ and belongs to $C^\infty(\mathcal{X}\backslash\{0\})$. In addition, the proof of Theorem 11.1 and [73, 2.9 (5)] show that

$$\left(L+\lambda^2-\rho_{\mathcal{X}}^2\right)\Theta_{\lambda,0,k,m,j} = 0. \tag{11.121}$$

We are now able to state and prove the Zaraisky theorem (unpublished), which is similar to Theorem 11.3.

Theorem 11.7.

(i) *The mapping* $\mathfrak{C}_{k,m,j}$ *is a homeomorphism of* $\mathcal{D}'_{k,m,j}(B_{\pi/2-R,\infty})$ *onto* $\mathcal{D}'_{\natural}(-R,R)$ *(respectively,* $C^{\infty}_{k,m,j}(B_{\pi/2-R,\infty})$ *onto* $C^{\infty}_{\natural}(-R,R)$*).*

(ii) *The equality*

$$\mathfrak{C}_{k,m,j}(f|_{B_{\pi/2-R',\infty}}) = \mathfrak{C}_{k,m,j}(f)|_{(-R',R')}$$

holds for all $R' \in (0,R]$.

(iii) *Let* $f \in \mathcal{D}'_{k,m,j}(B_{\pi/2-R,\infty})$, $T \in \mathcal{E}'_{\natural}(\mathfrak{X})$, *and* $r(T) < R$. *Then*

$$\mathfrak{C}_{k,m,j}(f \times T) = \mathfrak{C}_{k,m,j}(f) * \Lambda(T) \tag{11.122}$$

on $(r(T)-R, R-r(T))$. *In particular,*

$$\mathfrak{C}_{k,m,j}\big(P(L)f\big) = P\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \rho^2_{\mathcal{X}}\right)\mathfrak{C}_{k,mj}(f) \tag{11.123}$$

for each polynomial P.

(iv) *For* $\lambda \in \mathbb{C}$ *and* $\mu \in \mathbb{Z}_+$, *one has*

$$\mathfrak{C}_{k,m,j}(\Theta_{\lambda,\mu,k,m,j}) = u_{\lambda,\mu}, \tag{11.124}$$

where $u_{\lambda,\mu}$ *is the function on the right of* (9.60).

Proof. (i) We treat the case of $\mathcal{D}'$. (In C^{∞}-category the proof can be carried out along the same lines.) The injectivity of $\mathfrak{C}_{k,m,j}$ follows from Lemma 11.4. We claim that $\mathfrak{C}_{k,m,j}$ is surjective. Suppose that $\alpha+\beta \geqslant 1$. As the functions $\cos((\alpha+\beta+1-2\nu)t)$, $\nu = 1,\dots,[(\alpha+\beta+1)/2]$, are linearly independent on every interval, there exist $g_\nu \in \Pi_\varepsilon$ with the supports in $(R-\pi/2, \pi/2-R)+\pi\mathbb{Z}$ satisfying

$$\big\langle g_\mu(t), \cos\big((\alpha+\beta+1-2\nu)t\big)\big\rangle_{\mathbb{T}} = \delta_{\mu,\nu}, \quad \mu,\nu = 1,\dots,[(\alpha+\beta+1)/2].$$

Fix $R' \in (0,R)$ and pick $\psi \in \mathcal{D}_{\natural}(-R,R)$ which equals 1 on $(-R',R')$. If $g \in \mathcal{D}'_{\natural}(-R,R)$, put

$$h = \left(\sum_{-\infty}^{\infty}(-1)^{\varepsilon l}\delta_{\pi/2+\pi l}\right) * \left(\psi\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{-\varepsilon} g\right),$$

where $(\frac{\mathrm{d}}{\mathrm{d}t})^0 = \mathrm{Id}$ and $(\frac{\mathrm{d}}{\mathrm{d}t})^{-1}$ is the inverse operator to $\frac{\mathrm{d}}{\mathrm{d}t}: \mathcal{D}'_{\mathrm{odd}}(-R,R) \to \mathcal{D}'_{\natural}(-R,R)$. Now define

$$f = \left(\Lambda^{-1}_{k,m,j}\left(h - \sum_{\nu=1}^{[(\alpha+\beta+1)/2]}\big\langle h(t), \cos\big((\alpha+\beta+1-2\nu)t\big)\big\rangle_{\mathbb{T}} g_\nu\right)\right)\Bigg|_{B_{\pi/2-R',\infty}} \tag{11.125}$$

on $B_{\pi/2-R',\infty}$ for all $R' \in (0, R)$. By equation (11.125) and Lemma 11.4, $f \in \mathcal{D}'_{k,m,j}(B_{\pi/2-R,\infty})$ and $\mathfrak{C}_{k,m,j}(f) = g$. For $\alpha + \beta = 0$, the above argument is applicable with obvious modifications, which finishes the proof of the claim. So, $\mathfrak{C}_{k,m,j}$ is a bijection between $\mathcal{D}'_{k,m,j}(B_{\pi/2-R,\infty})$ and $\mathcal{D}'_{\natural}(-R, R)$. Finally, it is not hard to make sure that the mappings

$$f \to \mathfrak{C}_{k,m,j}(f)|_{(-R',R')} \quad \text{and} \quad g \to \mathfrak{C}^{-1}_{k,m,j}(g)\big|_{B_{\pi/2-R',\infty}}$$

are continuous. In view of the arbitrariness of $R' \in (0, R)$, we arrive at the desired conclusion.

(ii) This is a straightforward consequence of the definition of $\mathfrak{C}_{k,m,j}$.

(iii) The dependence on f in (11.122) is linear and continuous. Therefore, we can assume that $f = \Phi_{\lambda_l,0,k,m,j}$, $l \in \mathbb{Z}_+$, but then (11.122) becomes apparent by taking Proposition 11.10 into account.

(iv) Owing to (11.121) and (11.123),

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \lambda^2\right)\mathfrak{C}_{k,m,j}(\Theta_{\lambda,0,k,m,j})(t) = 0. \tag{11.126}$$

In addition,

$$\mathfrak{C}_{k,m,j}(\Theta_{\lambda,0,k,m,j})(0) = 1 \tag{11.127}$$

(see Lemma 11.4). Together, (11.126) and (11.127) yield (11.124) if $\mu = 0$. The general case is verified by differentiation with respect to λ. □

Chapter 12
The Case of Phase Space

This chapter completes our study of transmutation operators, focusing on the case of the phase space associated to the Heisenberg group.

The Heisenberg group H^n, $n \in \mathbb{N}$, is the Cartesian product $\mathbb{C}^n \times \mathbb{R}^1$ with the operation

$$(z,t)(w,s) = \left(z + w, t + s + \frac{1}{2}\operatorname{Im}\langle z, w\rangle_{\mathbb{C}}\right),$$

where $z, w \in \mathbb{C}^n$ and $t, s \in \mathbb{R}^1$. Under this multiplication, H^n becomes a nilpotent unimodular Lie group, the Haar measure being the Lebesgue measure $\mathrm{d}z\,\mathrm{d}t$ on $\mathbb{C}^n \times \mathbb{R}^1$.

Many problems about functions and operators on H^n respond very well to the technique of taking the Fourier transform in the t variable. In this way the study of convolution equations on H^n is closely related to the study of twisted convolution equations on the phase space $\mathbb{C}^n$.

Relevant facts about twisted convolution are collected in Sect. 12.1. In particular, we give here the relationship between the twisted convolution on $\mathbb{C}^n$ and the special Hermite operator $\mathfrak{L}$. This operator plays for the phase space a role analogous to that of the Euclidean Laplacian on $\mathbb{R}^n$.

The twisted convolution operator $f \to f \star T$ ($f \in \mathcal{D}'(\mathbb{C}^n)$, $T \in \mathcal{E}'_\natural(\mathbb{C}^n)$) commutes with the unitary group $\mathrm{U}(n)$. In this connection we shall need spherical harmonic expansions in $\mathbb{C}^n$ which are well adapted to the action of $\mathrm{U}(n)$. Such expansions are discussed in Sect. 12.2.

The study of transmutation operators on the phase space heavily depends on many properties of Laguerre functions. They arise as eigenfunctions of the operator $\mathfrak{L}$. In Sect. 12.3 we prove some important properties of Laguerre functions and their generalizations. Among these properties, we point out the analog of the Koornwinder integral representation for Jacobi functions (Proposition 12.8).

A holomorphic extension of the discrete Fourier–Laguerre transform leads to the transform $\mathcal{F}_l^{(p,q)}$ which is an analog of the spherical transform in the case under consideration. The basic properties of this transform are studied in Sect. 12.4. In

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particular, we prove a Paley–Wiener theorem for $\mathcal{F}_l^{(p,q)}$ using the above-mentioned integral representation for Laguerre functions.

In the last section we introduce transmutation operators on $\mathbb{C}^n$ associated with the Laguerre polynomials expansion. We then present their properties which will be needed for the study of twisted mean periodic functions in Part III.

12.1 The Twisted Convolution of Distributions on $\mathbb{C}^n$. Special Hermite Operator

Let $T_1, T_2 \in \mathcal{D}'(\mathbb{C}^n)$, $n \in \mathbb{N}$, and let $T_1 \otimes T_2$ be the tensor product of T_1, T_2 (see Sect. 6.2). Recall that $T_1 \otimes T_2 \in \mathcal{D}'(\mathbb{C}^{2n})$ and

$$\operatorname{supp}(T_1 \otimes T_2) = \operatorname{supp} T_1 \times \operatorname{supp} T_2, \tag{12.1}$$

where $\times$ on the right-hand side of (12.1) denotes the Cartesian product. If T_1 or T_2 belongs to $\mathcal{E}'(\mathbb{C}^n)$, put

$$\langle T_1 \star T_2, \psi\rangle = \big\langle T_1(z) \otimes T_2(w), \psi(z+w)e^{\frac{i}{2}\operatorname{Im}\langle z,w\rangle_{\mathbb{C}}}\big\rangle \tag{12.2}$$

for all $\psi \in \mathcal{D}(\mathbb{C}^n)$, $z, w \in \mathbb{C}^n$. It is not hard to make sure that $T_1 \star T_2$ is well defined by (12.2) as a distribution in $\mathcal{D}'(\mathbb{C}^n)$. This distribution is called the *twisted convolution* of T_1 with T_2.

Let us consider important particular cases of the definition (12.2).

Since the tensor product possesses the Fubini rule,

$$\begin{aligned}\langle T_1 \star T_2, \psi\rangle &= \big\langle T_1(z), \big\langle T_2(w), \psi(z+w)e^{\frac{i}{2}\operatorname{Im}\langle z,w\rangle_{\mathbb{C}}}\big\rangle\big\rangle \\ &= \big\langle T_2(w), \big\langle T_1(z), \psi(z+w)e^{\frac{i}{2}\operatorname{Im}\langle z,w\rangle_{\mathbb{C}}}\big\rangle\big\rangle.\end{aligned} \tag{12.3}$$

Relation (12.3) implies that $T_1 \star T_2 \in C^\infty(\mathbb{C}^n)$, provided that T_1 or T_2 is in $\mathcal{D}(\mathbb{C}^n)$. In this situation

$$(T_1 \star T_2)(z) = \big\langle T_2(w), T_1(z-w)e^{\frac{i}{2}\operatorname{Im}\langle z,w\rangle_{\mathbb{C}}}\big\rangle \quad \text{for } T_1 \in \mathcal{D}\big(\mathbb{C}^n\big) \tag{12.4}$$

and

$$(T_1 \star T_2)(w) = \big\langle T_1(z), T_2(w-z)e^{\frac{i}{2}\operatorname{Im}\langle z,w\rangle_{\mathbb{C}}}\big\rangle \quad \text{for } T_2 \in \mathcal{D}\big(\mathbb{C}^n\big). \tag{12.5}$$

In view of (12.3)–(12.5), equality (12.2) can be rewritten in the form

$$\langle T_1 \star T_2, \psi\rangle = \big\langle T_1, \check{T}_2 \star \psi\big\rangle = \big\langle T_2, \psi \star \check{T}_1\big\rangle, \quad \psi \in \mathcal{D}\big(\mathbb{C}^n\big), \tag{12.6}$$

where

$$\check{T}_i(z) = T_i(-z), \quad i = 1, 2.$$

By means of (12.6) we see that (12.4) (respectively (12.5)) remains valid when $T_1 \in C^\infty(\mathbb{C}^n)$, $T_2 \in \mathcal{E}'(\mathbb{C}^n)$ (respectively $T_1 \in \mathcal{E}'(\mathbb{C}^n)$, $T_2 \in C^\infty(\mathbb{C}^n)$). In addition,

$$(T_1 \star T_2)(z) = \int_{\mathbb{C}^n} T_1(z-w)T_2(w)e^{\frac{i}{2}\operatorname{Im}\langle z,w\rangle_{\mathbb{C}}}\, dm_n(w)$$

if $T_1 \in L^{1,\mathrm{loc}}(\mathbb{C}^n)$ and $T_2 \in (L^{1,\mathrm{loc}} \cap \mathcal{E}')(\mathbb{C}^n)$.

We now establish some elementary but basic properties of the twisted convolution.

For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, we put $\bar{z} = (\bar{z}_1, \dots, \bar{z}_n)$.

Proposition 12.1. *Let $T_i \in \mathcal{D}'(\mathbb{C}^n)$, $i = 1, 2, 3$, and suppose that at least two of the distributions T_i have compact supports. Then*

(i) $(\lambda T_1 + \mu T_2) \star T_3 = \lambda(T_1 \star T_3) + \mu(T_2 \star T_3)$, $\lambda, \mu \in \mathbb{C}$;

(ii) $(T_1 \star T_2)\check{} = \check{T}_1 \star \check{T}_2$;

(iii) $\operatorname{supp}(T_1 \star T_2) \subset \operatorname{supp} T_1 + \operatorname{supp} T_2$;

(iv) $\overline{T_1 \star T_2} = \overline{T_2} \star \overline{T_1}$;

(v) $(T_1 \star T_2)(\bar{z}) = (T_2 \star T_1)(z)$ *if* $T_i(z) = T_i(\bar{z})$, $i = 1, 2$. *In particular,*

$$T_1 \star T_2 = T_2 \star T_1$$

for radial distributions T_1, T_2;

(vi) $(T_1 \star T_2) \star T_3 = T_1 \star (T_2 \star T_3)$;

(vii) $(T_1(z+w)e^{\frac{i}{2}\operatorname{Im}\langle z,w\rangle_{\mathbb{C}}}) \star T_2(z) = (T_1 \star T_2)(z+w)e^{\frac{i}{2}\operatorname{Im}\langle z,w\rangle_{\mathbb{C}}}$;

(viii) $T_1 \star \delta_0 = \delta_0 \star T_1 = T_1$;

(ix) $\mathfrak{L}(T_1 \star T_2) = T_1 \star \mathfrak{L}T_2$, $\mathfrak{L}^*(T_1 \star T_2) = (\mathfrak{L}^* T_1) \star T_2$, $(\mathfrak{L}T_1) \star T_2 = T_1 \star \mathfrak{L}^* T_2$,

where

$$\begin{aligned} \mathfrak{L} &= \frac{1}{4}|z|^2 + \sum_{k=1}^n \left(z_k \frac{\partial}{\partial z_k} - \bar{z}_k \frac{\partial}{\partial \bar{z}_k} - 4\frac{\partial^2}{\partial z_k \partial \bar{z}_k} \right), \\ \mathfrak{L}^* &= \frac{1}{4}|z|^2 + \sum_{k=1}^n \left(\bar{z}_k \frac{\partial}{\partial \bar{z}_k} - z_k \frac{\partial}{\partial z_k} - 4\frac{\partial^2}{\partial z_k \partial \bar{z}_k} \right). \end{aligned} \tag{12.7}$$

Proof. Relations (i)–(viii) follow from (12.1)–(12.6) with the help of simple transformations. As for (ix), it is enough to use (12.4) and (12.6) and the formula

$$\mathfrak{L}\big(\psi(z-w)e^{\frac{i}{2}\operatorname{Im}\langle w,z\rangle_{\mathbb{C}}}\big) = (\mathfrak{L}\psi)(z-w)e^{\frac{i}{2}\operatorname{Im}\langle w,z\rangle_{\mathbb{C}}}, \tag{12.8}$$

which can be obtained by a direct calculation. □

Remark 12.1. It is easy to see that $\mathfrak{L}^*$ is the adjoint of the special Hermite operator $\mathfrak{L}$, i.e.,

$$\big\langle \mathfrak{L}^* T, \psi \big\rangle = \langle T, \mathfrak{L}\psi \rangle, \quad T \in \mathcal{D}'\big(\mathbb{C}^n\big), \psi \in \mathcal{D}\big(\mathbb{C}^n\big). \tag{12.9}$$

We have the equality

$$\mathfrak{L}^*\big(\overline{T}\big) = \overline{\mathfrak{L}T}.$$

Furthermore,

$$\mathfrak{L}^* T = \mathfrak{L} T \tag{12.10}$$

for any radial distribution $T \in \mathcal{D}'(\mathbb{C}^n)$.

Proposition 12.2. *Let $T, T_k \in \mathcal{D}'(\mathbb{C}^n)$, $k \in \mathbb{N}$. Assume that $T_k \to 0$ in $\mathcal{D}'(\mathbb{C}^n)$ and at least one of the following conditions hold:* 1) $T \in \mathcal{E}'(\mathbb{C}^n)$*;* 2) *there exists $R > 0$ such that* $\operatorname{supp} T_k \subset B_R$ *for all $k \in \mathbb{N}$, where $B_R = \{z \in \mathbb{C}^n : |z| < R\}$. Then $T_k \star T \to 0$ in $\mathcal{D}'(\mathbb{C}^n)$.*

Proof. For any $\psi \in \mathcal{D}(\mathbb{C}^n)$, we have

$$\langle T_k \star T, \psi \rangle = \big\langle T_k(z), \big\langle T(w), \psi(z+w) e^{\frac{i}{2} \operatorname{Im}\langle z,w \rangle_{\mathbb{C}}} \big\rangle\big\rangle. \tag{12.11}$$

Moreover, if $\operatorname{supp} T_k \subset B_R$, $k \in \mathbb{N}$, then for every $\eta \in \mathcal{D}(\mathbb{C}^n)$ such that $\eta = 1$ on B_R,

$$\langle T_k \star T, \psi \rangle = \big\langle T_k(z), \eta(z) \big\langle T(w), \psi(z+w) e^{\frac{i}{2} \operatorname{Im}\langle z,w \rangle_{\mathbb{C}}} \big\rangle\big\rangle. \tag{12.12}$$

Since $T_k \to 0$ in $\mathcal{D}'(\mathbb{C}^n)$, (12.11) and (12.12) give $\lim_{k\to\infty} \langle T_k \star T, \psi \rangle = 0$. This finishes the proof. □

Note, in conclusion, that if $\mathcal{O}_1, \mathcal{O}_2$ are nonempty open subsets of $\mathbb{C}^n$, $T_1 \in \mathcal{D}'(\mathcal{O}_1)$, $T_2 \in \mathcal{E}'(\mathbb{C}^n)$, and $\mathcal{O}_2 - \operatorname{supp} T_2 \subset \mathcal{O}_1$, then the convolution $T_1 \star T_2$ is well defined by (12.2) as a distribution in $\mathcal{D}'(\mathcal{O}_2)$. In this case the map $T_1 \to T_1 \star T_2$ is continuous from $\mathcal{D}'(\mathcal{O}_1)$ into $\mathcal{D}'(\mathcal{O}_2)$.

12.2 Expansions over Bigraded Spherical Harmonics

In this section we discuss expansions of functions and distributions which are well adapted to the action of the unitary group $\mathrm{U}(n)$. They play an important role for the further study in Sects. 12.3–12.5. For the rest of Chap. 12, it is assumed that $n \geqslant 2$. The case $n = 1$ requires minor changes.

Let $\mathcal{H}_2^{n,p,q}$ be the space of spherical harmonics of bidegree p, q on $\mathbb{S}^{2n-1}$ regarded as a subspace of $L^2(\mathbb{S}^{2n-1})$ (see Sect. 4.2). Denote by $d(n, p, q)$ the dimension of $\mathcal{H}_2^{n,p,q}$. As is well known (see Smith [201]),

$$d(n,p,q) = \frac{(p+n-2)!(q+n-2)!(p+q+n-1)}{p!q!(n-1)!(n-2)!}.$$

Let $\{S_l^{p,q}\}$, $l \in \{1, \dots, d(n,p,q)\}$, be a fixed orthonormal basis in $\mathcal{H}_2^{n,p,q}$, $S_1^{0,0} = 1/\sqrt{\omega_{2n-1}}$. Each function $f \in L^{1,\mathrm{loc}}(\mathcal{O})$, where $\mathcal{O}$ is a nonempty open $\mathrm{U}(n)$-invariant subset of $\mathbb{C}^n$, has a Fourier expansion of the form

$$f(z) \sim \sum_{p,q=0}^{\infty} \sum_{l=1}^{d(n,p,q)} f_{(p,q),l}(\varrho) S_l^{p,q}(\sigma), \quad z = \varrho\sigma, \ \sigma \in \mathbb{S}^{2n-1}, \tag{12.13}$$

where

$$f_{(p,q),l}(\varrho) = \int_{\mathbb{S}^{2n-1}} f(\varrho\sigma) \overline{S_l^{p,q}(\sigma)} \, d\omega(\sigma). \tag{12.14}$$

By the Fubini theorem the function $f_{(p,q),l}$ is locally summable on the set $\{r > 0 : S_r \subset \mathcal{O}\}$, where $S_r = \{z \in \mathbb{C}^n : |z| = r\}$. Similarly, the function $f^{(p,q),l}(z) = f_{(p,q),l}(\varrho) S_l^{p,q}(\sigma)$ belongs to the class $L^{1,\text{loc}}(\mathcal{O})$.

Let $\{t_{k,l}^{(p,q)}(\tau)\}$ $(k, l \in \{1, \dots, d(n, p, q)\}, \tau \in \mathrm{U}(n))$ be the matrix of the representation $T_2^{n,p,q}(\tau)$ (see Sect. 4.2) in the basis $\{S_l^{p,q}\}$, that is,

$$T_2^{n,p,q}(\tau) S_l^{p,q} = \sum_{k=1}^{d(n,p,q)} t_{k,l}^{(p,q)}(\tau) S_k^{p,q}.$$

In view of the orthonormality of $\{S_l^{p,q}\}$,

$$t_{k,l}^{(p,q)}(\tau) = \int_{\mathbb{S}^{2n-1}} S_l^{p,q}\left(\tau^{-1}\sigma\right) \overline{S_k^{p,q}(\sigma)} \, d\omega(\sigma).$$

We require the following statement.

Proposition 12.3. *The relation*

$$f_{(p,q),l}(\varrho) S_k^{p,q}(\sigma) = d(n, p, q) \int_{\mathrm{U}(n)} f\left(\tau^{-1} z\right) \overline{t_{k,l}^{(p,q)}(\tau)} \, d\tau, \tag{12.15}$$

where $d\tau$ *is the normalized Haar measure on the group* $\mathrm{U}(n)$*, holds for almost all* $\varrho \in \{r > 0 : S_r \subset \mathcal{O}\}$ *and all* $\sigma \in \mathbb{S}^{2n-1}$.

Proof. Using Theorem 4.3, we conclude that (12.15) follows just like equality (1.80). □

Formula (12.15) shows that if $f \in C^m(\mathcal{O})$ for some $m \in \mathbb{Z}_+ \cup \{\infty\}$, then the function

$$f^{(p,q),k,l}(z) = f_{(p,q),l}(\varrho) S_k^{p,q}(\sigma)$$

coincides almost everywhere with a function in the class $C^m(\mathcal{O})$. Furthermore, for each $\psi \in \mathcal{D}(\mathcal{O})$, one has

$$\begin{aligned} &\int_{\mathcal{O}} f^{(p,q),k,l}(z) \psi(z) \, dm_n(z) \\ &= d(n, p, q) \int_{\mathcal{O}} f(z) \int_{\mathrm{U}(n)} \psi\left(\tau^{-1} z\right) t_{l,k}^{(p,q)}(\tau) \, d\tau \, dm_n(z) \\ &= \int_{\mathcal{O}} f(z) \overline{(\overline{\psi})_{(p,q),k}(\varrho) S_l^{p,q}(\sigma)} \, dm_n(z). \end{aligned}$$

Now we can extend the map $f \to f^{(p,q),k,l}$ and expansion (12.13) to distributions $f \in \mathcal{D}'(\mathcal{O})$ as follows:

$$\begin{aligned}\left\langle f^{(p,q),k,l}, \psi\right\rangle &= \left\langle f, d(n,p,q)\int_{\mathrm{U}(n)} \psi\left(\tau^{-1}z\right)t_{l,k}^{(p,q)}(\tau)\,\mathrm{d}\tau\right\rangle \\ &= \left\langle f, \overline{(\overline{\psi})_{(p,q),k}(\varrho)S_l^{p,q}(\sigma)}\right\rangle, \quad \psi \in \mathcal{D}(\mathcal{O}),\end{aligned} \tag{12.16}$$

$$f \sim \sum_{p,q=0}^{\infty}\sum_{l=1}^{d(n,p,q)} f^{(p,q),l}, \tag{12.17}$$

where

$$f^{(p,q),l} = f^{(p,q),l,l}.$$

It can be proved that series (12.17) converges in $\mathcal{D}'(\mathcal{O})$ (respectively $C^\infty(\mathcal{O})$) for $f \in \mathcal{D}'(\mathcal{O})$ (respectively $f \in C^\infty(\mathcal{O})$). Owing to (12.16), the mapping $f \to f^{(p,q),k,l}$ is a continuous operator from $\mathcal{D}'(\mathcal{O})$ into $\mathcal{D}'(\mathcal{O})$.

Proposition 12.4. *Let $f \in \mathcal{D}'(\mathcal{O})$, and let T be a radial distribution in $\mathcal{E}'(\mathbb{C}^n)$. Assume that the set $\mathcal{U} = \{z \in \mathbb{C}^n : z - \operatorname{supp} T \subset \mathcal{O}\}$ is nonempty. Then*

$$(f \star T)^{(p,q),k,l} = f^{(p,q),k,l} \star T \quad \textit{in } \mathcal{U}. \tag{12.18}$$

In particular,

$$\left(P(\mathfrak{L})f\right)^{(p,q),k,l} = P(\mathfrak{L})f^{(p,q),k,l}$$

for every polynomial P.

Proof. The argument is quite parallel to the proof of (9.10) (see (12.3), (12.16), and Proposition 12.1(ix)). □

Let $\mathfrak{W}(\mathcal{O})$ be a given class of distributions on $\mathcal{O}$. Set

$$\mathfrak{W}_{(p,q),l}(\mathcal{O}) = \left\{f \in \mathfrak{W}(\mathcal{O}) : f = f^{(p,q),l}\right\}.$$

Clearly, $\mathfrak{W}_{(0,0),1}(\mathcal{O}) = \mathfrak{W}_\natural(\mathcal{O})$, where

$$\mathfrak{W}_\natural(\mathcal{O}) = \left\{f \in \mathfrak{W}(\mathcal{O}) : \langle f, \psi\rangle = \langle f, \psi \circ \tau\rangle \text{ for all } \psi \in \mathcal{D}(\mathcal{O}),\ \tau \in \mathrm{U}(n)\right\}.$$

It is easy to see that the support of a distribution $f \in \mathcal{D}'_{(p,q),l}(\mathcal{O})$ is $\mathrm{U}(n)$-invariant. For $f \in \mathcal{D}'_{(p,q),l}(\mathcal{O})$, we put $r(f) = \inf\{r \geqslant 0 : \operatorname{supp} f \subset \dot{B}_r\}$, where $\dot{B}_r = \{z \in \mathbb{C}^n : |z| \leqslant r\}$.

To close this section we present some formulas concerning the action of the operator $\mathfrak{L}$ on the space $C^2_{(p,q),l}(\mathcal{O})$. Let E be a nonempty open subset of $(0, +\infty)$. For any $s \in \mathbb{Z}$, we consider the differential operators $D_i(s)$, $i = 1, 2$, defined on $C^1(E)$ as follows:

$$\left(D_i(s)\varphi\right)(\varrho) = \varrho^s \mathrm{e}^{(-1)^{i+1}\varrho^2/4}\frac{\mathrm{d}}{\mathrm{d}\varrho}\left(\varrho^{-s}\mathrm{e}^{(-1)^i\varrho^2/4}\varphi(\varrho)\right), \quad \varphi \in C^1(E). \tag{12.19}$$

Proposition 12.5. *Suppose that* $f \in C^2_{(p,q),l}(\mathcal{O})$. *Then*

(i) $\big(\mathfrak{L} + (n+2q)\,\mathrm{Id}\big) f(z)$
$= -\big(D_2(1-2n-p-q)D_1(p+q) f_{(p,q),l}\big)(\varrho) S_l^{p,q}(\sigma);$

(ii) $\big(\mathfrak{L} - (n+2p)\,\mathrm{Id}\big) f(z)$
$= -\big(D_1(1-2n-p-q)D_2(p+q) f_{(p,q),l}\big)(\varrho) S_l^{p,q}(\sigma);$

(iii) $\big(\mathfrak{L} + (n+2q-2)\,\mathrm{Id}\big) f(z)$
$= -\big(D_1(p+q-1)D_2(2-2n-p-q) f_{(p,q),l}\big)(\varrho) S_l^{p,q}(\sigma);$

(iv) $\big(\mathfrak{L} - (n+2p-2)\,\mathrm{Id}\big) f(z)$
$= -\big(D_2(p+q-1)D_1(2-2n-p-q) f_{(p,q),l}\big)(\varrho) S_l^{p,q}(\sigma).$

Proof. By a straightforward computation using (12.7) we find that

$$(\mathfrak{L} f)(z) = (\mathfrak{L}_{p,q} f_{(p,q),l})(\varrho) S_l^{p,q}(\sigma), \tag{12.20}$$

where

$$\mathfrak{L}_{p,q} = -\frac{\mathrm{d}^2}{\mathrm{d}\varrho^2} - \frac{2n-1}{\varrho}\frac{\mathrm{d}}{\mathrm{d}\varrho} + \left(\frac{(p+q)(2n+p+q-2)}{\varrho^2} + \frac{1}{4}\varrho^2 + p - q\right)\mathrm{Id}\,. \tag{12.21}$$

Our result now follows from (12.19). □

Remark 12.2. Equalities (12.20) and (12.7) show that

$$\mathfrak{L} - \mathfrak{L}^* = (2p-2q)\,\mathrm{Id} \tag{12.22}$$

on the space $C^2_{(p,q),l}(\mathcal{O})$.

12.3 Derivatives of Generalized Laguerre Functions

Laguerre functions on the phase space $\mathbb{C}^n$ appear as radial eigenfunctions of the special Hermite operator $\mathfrak{L}$. In this section we investigate their generalizations which are analogues of functions (9.13) and (9.18) in the case under consideration.

Let $\eta, p, q \in \mathbb{Z}_+$, $\zeta \in \mathbb{C}$, and $\varrho \in (0, +\infty)$. Put

$$\begin{aligned} a &= p + \frac{n-\zeta^2}{2}, & b &= n+p+q, \\ x &= \frac{\varrho^2}{2}, & y &= \varrho^{p+q} \mathrm{e}^{-\varrho^2/4}. \end{aligned} \tag{12.23}$$

For $\lambda \in \mathbb{C}$, $l \in \{1, \ldots, d(n,p,q)\}$, and $z \in \mathbb{C}^n$, we define

$$\phi_{\lambda,\eta,p,q,l}(z) = \begin{cases} \sqrt{\omega_{2n-1}}\phi_{\lambda,\eta,p,q}(\varrho)S_l^{p,q}(\sigma) & \text{if } z=\varrho\sigma \in \mathbb{C}^n\backslash\{0\}, \\ \delta_{0,\eta}\delta_{0,p+q} & \text{if } z=0, \end{cases} \tag{12.24}$$

where

$$\phi_{\lambda,\eta,p,q}(\varrho) = \left(\frac{\partial}{\partial\zeta}\right)^{\varkappa}\big(y\,{}_1F_1(a;b;x)\big)\Big|_{\zeta=\lambda} \tag{12.25}$$

with

$$\varkappa = \begin{cases} \eta & \text{if } \lambda \neq 0, \\ 2\eta & \text{if } \lambda = 0. \end{cases}$$

It can easily be checked that $\phi_{\lambda,\eta,p,q,l} \in \mathrm{RA}(\mathbb{C}^n)$.

Next, we set

$$g(a,b,x) = \Gamma(a)\left((-1)^b\Psi(a,b;x) - \frac{\psi(a)}{\Gamma(b)\Gamma(a-b+1)}\,{}_1F_1(a;b;x)\right),$$

where $\psi(a) = \Gamma'(a)/\Gamma(a)$, and $\Psi(a,b;x)$ is the Tricomi confluent hypergeometric function. Since for $k \in \mathbb{Z}_+$,

$$\lim_{\lambda\to -k}\Gamma(\lambda)(\lambda+k) = \frac{(-1)^k}{\Gamma(k+1)}, \qquad \lim_{\lambda\to -k}\psi(\lambda)(\lambda+k) = -1,$$

and

$$\Psi(-k,b;x) = (-1)^k\frac{\Gamma(b+k)}{\Gamma(b)}\,{}_1F_1(-k;b;x)$$

(see Erdélyi (ed.) [73, 1.17 (11), 1.17 (12) and 6.9 (36)]), $g(a,b,x)$ is an even entire function with respect to the variable ζ. Now for $z \in \mathbb{C}^n \setminus \{0\}$, $\lambda \in \mathbb{C}$, we introduce $\psi_{\lambda,\eta,p,q,l}(z)$ by the formula

$$\psi_{\lambda,\eta,p,q,l}(z) = \psi_{\lambda,\eta,p,q}(\varrho)S_l^{p,q}(\sigma), \tag{12.26}$$

where

$$\psi_{\lambda,\eta,p,q}(\varrho) = \left(\frac{\partial}{\partial\zeta}\right)^{\varkappa}\big(yg(a,b,x)\big)\Big|_{\zeta=\lambda}. \tag{12.27}$$

Let us study basic properties of functions (12.24)–(12.27).

Proposition 12.6. *Let $D_i(s)$, $i = 1,2$, be the differential operators defined by* (12.19). *Then*

(i) $$D_1(b-n)\phi_{\lambda,0,p,q} = -\frac{n+2q+\lambda^2}{2b}\phi_{\lambda,0,p,q+1}, \quad p,q \in \mathbb{Z}_+; \tag{12.28}$$

(ii) $$D_2(b-n)\phi_{\lambda,0,p,q} = \frac{n+2p-\lambda^2}{2b}\phi_{\lambda,0,p+1,q}, \quad p,q \in \mathbb{Z}_+; \tag{12.29}$$

(iii) $$D_1(2-b-n)\phi_{\lambda,0,p,q} = 2(b-1)\phi_{\lambda,0,p-1,q}, \quad p \in \mathbb{N},\ q \in \mathbb{Z}_+; \tag{12.30}$$

(iv) $$D_2(2-b-n)\phi_{\lambda,0,p,q} = 2(b-1)\phi_{\lambda,0,p,q-1}, \quad p \in \mathbb{Z}_+,\ q \in \mathbb{N}. \tag{12.31}$$

Proof. Apply formula (7.72) with $k = 1$. Taking (7.70) and (7.71) into account, we arrive at (12.28)–(12.31). □

Theorem 12.1.

(i) *If* $k \in \mathbb{Z}_+$ *and* $\mu \in \mathbb{C}\backslash\{0\}$, *then in* $\mathbb{C}^n$ *the following relations are true:*

$$\mathfrak{L}^k \phi_{0,\eta,p,q,l} = \begin{cases} (-2\eta)_{2k}\phi_{0,\eta-k,p,q,l} & \text{if } k \leqslant \eta, \\ 0 & \text{if } k > \eta, \end{cases} \tag{12.32}$$

$$\mathfrak{L}^k \phi_{\mu,\eta,p,q,l} = \sum_{\nu=\max\{0,\eta-2k\}}^{\eta} \binom{\eta}{\nu} \frac{(2k)!\mu^{2k-\eta+\nu}}{(2k-\eta+\nu)!} \phi_{\mu,\nu,p,q,l}, \tag{12.33}$$

$$\begin{aligned}&\left(\mathfrak{L} - \mu^2\right)^k \phi_{\mu,\eta,p,q,l} \\ &= \sum_{\nu=\max\{0,\eta-2k\}}^{\eta} (-1)^{\nu-\eta} \frac{\eta! 2^{2k-\eta+\nu}(-k)_{2k-\eta+\nu}}{\nu!(2k-\eta+\nu)!} \mu^{2k-\eta+\nu} \phi_{\mu,\nu,p,q,l}.\end{aligned} \tag{12.34}$$

In particular, for $\lambda \in \mathbb{C}$,

$$\left(\mathfrak{L} - \lambda^2\right)^{\eta+1} \phi_{\lambda,\eta,p,q,l} = 0. \tag{12.35}$$

(ii) *In* (12.32)–(12.35) *the functions* $\phi_{\lambda,\eta,p,q,l}$ *may be replaced by the functions* $\psi_{\lambda,\eta,p,q,l}$. *In this case the equalities remain valid in* $\mathbb{C}^n\backslash\{0\}$. *Furthermore,* $\psi_{\lambda,0,0,0,1} \in L^{1,\mathrm{loc}}(\mathbb{C}^n)$ *and*

$$\left(\mathfrak{L} - \lambda^2\right)\psi_{\lambda,0,0,0,1} = \sqrt{\omega_{2n-1}}(-1)^n(n-1)!2^n\delta_0, \quad \lambda \in \mathbb{C}.$$

Proof. By virtue of (12.20), the equation

$$\left(\mathfrak{L} - \zeta^2\right)\left(f(\varrho)S_l^{p,q}(\sigma)\right) = 0$$

can be rewritten in the form

$$\begin{aligned}0 = f''(\varrho) &+ \frac{f'(\varrho)}{\varrho}(2n-1) \\ &- \frac{f(\varrho)}{\varrho^2}\left((p+q)(2n+p+q-2) + \left(p-q-\zeta^2\right)\varrho^2 + \frac{1}{4}\varrho^4\right).\end{aligned} \tag{12.36}$$

A substitution $f(\varrho) = yu(x)$ (see (12.23)) reduces (12.36) to confluent hypergeometric equation (7.67). This implies (12.35) for $\eta = 0$ and the same equality for $\psi_{\lambda,0,p,q,l}$ in $\mathbb{C}^n\backslash\{0\}$. Now we obtain the desired assertion in the same way as in the proof of Theorem 10.4 (see Proposition 12.6). □

Proposition 12.7. *For* $\varrho \in (0, +\infty)$ *and* $\zeta \in \mathbb{C}$, *we have*

$$\phi_{\zeta,0,p,q}(\varrho)\frac{\mathrm{d}}{\mathrm{d}\varrho}\psi_{\zeta,0,p,q}(\varrho) - \psi_{\zeta,0,p,q}(\varrho)\frac{\mathrm{d}}{\mathrm{d}\varrho}\phi_{\zeta,0,p,q}(\varrho) = \frac{(-1)^{b+1}\Gamma(b)2^b}{\varrho^{2n-1}}. \tag{12.37}$$

Proof. Denote by $f_\zeta(\varrho)$ the function on the left-hand side of (12.37). Because $f_\zeta(\varrho)$ is analytic in ζ, it suffices to show that equality (12.37) is true for $a \notin \mathbb{Z}\backslash\mathbb{N}$. In view of (12.35), the functions $\phi_{\zeta,0,p,q}(\varrho)$ and $\psi_{\zeta,0,p,q}(\varrho)$ satisfy equation (12.36). Therefore, by the Liouville–Ostrogradsky formula,

$$f_\zeta(\varrho) = \frac{c}{\varrho^{2n-1}},$$

where c is independent of ϱ. From (7.65) and (7.66) we find $c = (-1)^{b+1}\Gamma(b)2^b$, whence (12.37) follows. □

Remark 12.3. Let $\{\mu_1, \dots, \mu_r\}$ be a set of complex numbers such that the numbers $\{\mu_1^2, \dots, \mu_r^2\}$ are distinct, and let $\mathcal{O}$ be a nonempty open subset in $\mathbb{C}^n$. Then exactly as in Proposition 10.4 (but now working with Theorem 12.1 and Proposition 12.7) we see that the assertions of Proposition 10.4 remain true for the functions $\phi_{\mu_i,\nu,p,q,l}$ and $\psi_{\mu_i,\nu,p,q,l}$.

Proposition 12.8. *The integral representation*

$$\phi_{\lambda,0,p,q}(\varrho) = \frac{1}{\sqrt{\omega_{2n-1}}}\int_0^\varrho \cos(\lambda t)\mathfrak{K}_{n,p,q}(\varrho, t)\,\mathrm{d}t \tag{12.38}$$

holds, where $\mathfrak{K}_{n,p,q}(\varrho,\cdot) \in L^1[0,\varrho]$. *In addition,*

$$\mathfrak{K}_{n,p,p}(\varrho, t) = 4^{n+2p-1}\sqrt{\omega_{2n-1}}\frac{(\varrho^2 - t^2)^{n+2p-3/2}}{\varrho^{2n+2p-2}} \times \left(k_{n+2p}(u, v)\left(4u - v^2\right)^{3/2-n-2p}\right)\Big|_{u=\varrho^2/2,\ v=\sqrt{2}t},$$

where $k_{n+2p}(u, v)$ *is defined by* (7.81).

Proof. According to Proposition 7.11,

$$\phi_{\lambda,0,p,q}(\varrho) = \frac{1}{\varrho^{n+b-2}}\int_0^\varrho \cos\left(\sqrt{\lambda^2+q-p}\,t\right)\left(\varrho^2 - t^2\right)^{b-3/2}\mathcal{K}_{n,p,q}(\varrho, t)\,\mathrm{d}t \tag{12.39}$$

with

$$\mathcal{K}_{n,p,q}(\varrho, t) = 4^{b-1}k_b(u, v)\left(4u - v^2\right)^{3/2-b}\Big|_{u=\varrho^2/2, v=\sqrt{2}t}.$$

Using (12.39) and (7.87) and the Paley–Wiener theorem for the Fourier-cosine transform, we complete the proof. □

Remark 12.4. The proof of Proposition 12.8 shows that

$$\mathfrak{K}_{n,p,q}(\varrho,\cdot) \in L^2[0,\varrho], \quad \text{provided that } b > 1.$$

For $0 \leqslant r < R \leqslant +\infty$, we put $B_{r,R} = \{z \in \mathbb{C}^n : r < |z| < R\}$. Denote by $\dot{B}_{r,R}$ the closure of $B_{r,R}$.

Proposition 12.9. *Let* $0 < r < R < +\infty$, $\lambda \in \mathbb{C}$, $\eta, k \in \mathbb{Z}_+$, *and* $\varepsilon \in (0, 1)$. *If* $\eta < \varepsilon R|\lambda|$, *then the following estimates hold:*

$$\|\phi_{\lambda,\eta,p,q,l}\|_{C^k(\mathring{B}_R)} \leqslant \gamma_1\sqrt{1+\eta}(1+|\lambda|)^{k-p-q} R^\eta \mathrm{e}^{R|\mathrm{Im}\,\lambda|}, \tag{12.40}$$

$$\|\phi_{\lambda,\eta,p,q,l}\|_{C^k(\mathring{B}_{r,R})} \leqslant \gamma_2\sqrt{1+\eta}(1+|\lambda|)^{k-b+1/2} R^\eta \mathrm{e}^{R|\mathrm{Im}\,\lambda|}, \tag{12.41}$$

$$\|\psi_{\lambda,\eta,p,q,l}\|_{C^k(\mathring{B}_{r,R})} \leqslant \gamma_3\sqrt{1+\eta}(1+|\lambda|)^{k+b-3/2} \log(2+|\lambda|) R^\eta \mathrm{e}^{R|\mathrm{Im}\,\lambda|}, \tag{12.42}$$

where $\gamma_1, \gamma_2, \gamma_3 > 0$ *are independent of* λ, η.

Proof. Let $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_+^n$, and let $l(\alpha, \beta) = \alpha_1 + \cdots + \alpha_n + \beta_1 + \cdots + \beta_n$. Put

$$f(t) = \mathrm{e}^{-t/4} {}_1F_1\left(p + \frac{n-\lambda^2}{2}; b; \frac{t}{2}\right).$$

By induction it is easy to make sure that

$$\frac{\partial^{l(\alpha,\beta)}}{\partial z^\alpha \partial \overline{z}^\beta}\phi_{\lambda,0,p,q,l}(z) = \sum_{k \geqslant \max\{0,m/2\}}^{l(\alpha,\beta)} f^{(k)}\big(|z|^2\big) Q_{2k-m}(z), \tag{12.43}$$

where $m = l(\alpha, \beta) - p - q$, and Q_{2k-m} is a homogeneous polynomial in $\mathbb{C}^n$ of degree $2k - m$. In view of (7.72), equality (12.43) can be written in the form

$$\begin{aligned}\frac{\partial^{l(\alpha,\beta)}}{\partial z^\alpha \partial \overline{z}^\beta}\phi_{\lambda,0,p,q,l}(z) = &\sum_{k \geqslant \max\{0,m/2\}}^{l(\alpha,\beta)} \sum_{j=0}^{k} (-1)^{k-j} \binom{j}{k} \frac{2^j}{4^k} \frac{(p+(n-\lambda^2)/2)_j}{(b)_j} \\ &\times {}_1F_1\left(p+j+\frac{n-\lambda^2}{2}; b+j; \frac{|z|^2}{2}\right) Q_{2k-m}(z)\mathrm{e}^{-|z|^2/4}.\end{aligned} \tag{12.44}$$

Suppose that $|\lambda|^{-1} \leqslant |z| \leqslant R$. Then applying (7.90), we conclude from (12.44) that

$$\frac{\partial^{l(\alpha,\beta)}}{\partial z^\alpha \partial \overline{z}^\beta}\phi_{\lambda,0,p,q,l}(z) = O\big(|\lambda|^{l(\alpha,\beta)-p-q} \mathrm{e}^{R|\mathrm{Im}\,\lambda|}\big). \tag{12.45}$$

Similarly, for $|z| \leqslant |\lambda|^{-1}$, relations (12.44) and (7.91) give

$$\frac{\partial^{l(\alpha,\beta)}}{\partial z^\alpha \partial \overline{z}^\beta}\phi_{\lambda,0,p,q,l}(z) = O\big(|\lambda|^{l(\alpha,\beta)-p-q}\big). \tag{12.46}$$

Combining (12.45) and (12.46), we arrive at (12.40) if $\eta = 0$ and, hence, by Proposition 6.11, in general. Estimates (12.41) and (12.42) follow from Propositions 6.11 and 7.12, (7.72), (7.73), and the Stirling asymptotic formula (see the proof of Proposition 11.6). □

Proposition 12.10. *For fixed $p, q \in \mathbb{Z}_+$ and $r \in (0, +\infty)$, the following statements are valid.*

(i) *The function $\phi_{\lambda,0,p,q}(r)$ has infinitely many zeroes. All the zeroes of $\phi_{\lambda,0,p,q}(r)$ are real and simple and are located symmetrically relative to the point $\lambda = 0$.*
(ii) *Let $z_l = z_l(p, q, r, n)$, $l \in \mathbb{N}$, be the sequence of all positive zeroes of $\phi_{\lambda,0,p,q}(r)$ numbered in the ascending order and suppose that $0 < r_1 \leqslant r \leqslant r_2 < +\infty$. Then*

$$rz_l = \pi\left(\frac{2b+1}{4} + l + j(p, q, r, n)\right) + \left(\frac{r^3}{12} + (p-q)r - \frac{(2b-3)(2b-1)}{4r}\right)\frac{1}{2z_l} + O\left(\frac{1}{z_l^3}\right),$$

where $j(p, q, r, n)$ belongs to $\mathbb{Z}$ and does not depend on l, and the constant in O depends only on p, q, n, r_1, r_2.

Proof. From (7.74) we have

$$\left(\lambda^2 - \mu^2\right)\int_0^r \varrho^{2n-1}\phi_{\lambda,0,p,q}(\varrho)\phi_{\mu,0,p,q}(\varrho)\mathrm{d}\varrho = r^{2n-1}\left(\phi_{\lambda,0,p,q}(r)\phi'_{\mu,0,p,q}(r) - \phi_{\mu,0,p,q}(r)\phi'_{\lambda,0,p,q}(r)\right). \tag{12.47}$$

The rest of the proof now duplicates Proposition 7.6 (see (12.39) and (7.87)). □

Denote

$$N_{p,q}(r) = \left\{\lambda > 0 : \phi_{\lambda,0,p,q}(r) = 0\right\}.$$

For $\lambda, \mu \in N_{p,q}(r)$, define

$$\eta(\lambda, \mu) = \int_0^r \varrho^{2n-1}\phi_{\lambda,0,p,q}(\varrho)\phi_{\mu,0,p,q}(\varrho)\,\mathrm{d}\varrho.$$

Using (7.90), (12.47), (12.39), Proposition 12.10, and the proofs from Lemmas 7.1 and 7.2, we obtain the following statements.

Lemma 12.1. *Let $\lambda, \mu \in N_{p,q}(r)$. Then $\eta(\lambda, \mu) = 0$ if $\lambda \neq \mu$, and $\eta(\lambda, \lambda)\lambda^{2b} > c$, where $c > 0$ is independent of λ.*

Lemma 12.2. *Assume that $v \in L^1[0, r]$ and*

$$\int_0^r \varrho^{2n-1}v(\varrho)\phi_{\lambda,0,p,q}(\varrho)\,\mathrm{d}\varrho = 0$$

for all $\lambda \in N_{p,q}(r)$. Then $f = 0$.

Next, if $u \in L^1[0, r]$ and $\lambda \in N_{p,q}(r)$, put

$$c_\lambda(u) = \left(\eta(\lambda, \lambda)\right)^{-1}\int_0^r \varrho^{2n-1}u(\varrho)\phi_{\lambda,0,p,q}(\varrho)\,\mathrm{d}\varrho.$$

Also let $\mathfrak{L}_{p,q}$ be the differential operator given by (12.21).

Proposition 12.11. *Suppose that for some $\zeta > n+(p+q+1)/2$, $\zeta \in \mathbb{N}$, a function u enjoys the following properties:*

(1) $\mathfrak{L}^s_{p,q}u \in C^2[0,r]$ *if* $s = 0, 1, \dots, \zeta - 1$, *and* $\mathfrak{L}^\zeta_{p,q}u \in C[0,r]$.
(2) $(\mathfrak{L}^s_{p,q}u)(r) = 0$, $s = 0, 1, \dots, \zeta - 1$.

Then $c_\lambda(u) = O(\lambda^{-2\zeta+b+n})$ *as* $\lambda \to +\infty$, *and the expansion*

$$u(\varrho) = \sum_{\lambda \in N_{p,q}(r)} c_\lambda(u)\phi_{\lambda,0,p,q}(\varrho), \quad \varrho \in [0,r],$$

holds in which the series converges in $C^s[0,r]$ *with* $s < \zeta - n - (p+q+1)/2$.

The proof of this statement is an immediate extension of that of Theorem 7.1 (see (7.90), (7.91), (7.72), Proposition 12.6, and Lemmas 12.1 and 12.2).

For $j \in \mathbb{Z}_+$, we set

$$\lambda_j = \sqrt{2p+n+2j}, \tag{12.48}$$

$$\mu_j = \frac{2^{1-b}}{\omega_{2n-1}(b-1)!}\binom{b+j-1}{b-1}. \tag{12.49}$$

In terms of the Laguerre polynomials L^α_j (see Erdélyi (ed.) [73, 6.9 (36)]), we can write

$$\binom{b+j-1}{b-1}\phi_{\lambda_j,0,p,q}(\varrho) = \varrho^{p+q}\mathrm{e}^{-\varrho^2/4}L^{b-1}_j\big(\varrho^2/2\big).$$

Proposition 12.12. *The system of functions* $\{\phi_{\lambda_j,0,p,q,l}\}_{j=0}^\infty$ *forms an orthogonal basis in* $L^2_{(p,q),l}(\mathbb{C}^n)$. *In addition,*

$$\int_{\mathbb{C}^n} |\phi_{\lambda_j,0,p,q,l}(z)|^2 \,\mathrm{d}m_n(z) = \frac{1}{\mu_j}.$$

Proof. Let $g \in L^2((0,+\infty), t^{b-1}\mathrm{e}^{-t}\,\mathrm{d}t)$ and $G(z) = \varrho^{p+q}\mathrm{e}^{-\varrho^2/4}g(\varrho^2/2)S^{p,q}_l(\sigma)$. Then

$$\int_{\mathbb{C}^n} |G(z)|^2 \,\mathrm{d}m_n(z) = \int_0^\infty \int_{\mathbb{S}^{2n-1}} \varrho^{2b-1}\mathrm{e}^{-\varrho^2/2}\big|g\big(\varrho^2/2\big)\big|^2\big|S^{p,q}_l(\sigma)\big|^2 \,\mathrm{d}\varrho\,\mathrm{d}\omega(\sigma)$$
$$= 2^{b-1}\int_0^\infty t^{b-1}\mathrm{e}^{-t}|g(t)|^2\,\mathrm{d}t.$$

This relation shows that the mapping $g \to G$ is an isomorphism between the space $L^2((0,+\infty), (2t)^{b-1}\mathrm{e}^{-t}\mathrm{d}t)$ and $L^2_{(p,q),l}(\mathbb{C}^n)$. As the polynomials L^α_j, $j \in \mathbb{Z}_+$, form an orthogonal basis in $L^2((0,+\infty), t^\alpha\mathrm{e}^{-t}\mathrm{d}t)$ and

$$\int_0^\infty t^\alpha\mathrm{e}^{-t}\big(L^\alpha_j(t)\big)^2\,\mathrm{d}t = \frac{\Gamma(\alpha+j+1)}{j!}$$

(see [73, 10.12 (2)]), we complete the proof. □

12.4 Analogues of the Spherical Transform

Let $p, q \in \mathbb{Z}_+$ and $l \in \{1, \dots, d(n, p, q)\}$. For each $f \in \mathcal{E}'_{(p,q),l}(\mathbb{C}^n)$, we set

$$\mathcal{F}_l^{(p,q)}(f)(\lambda) = \left\langle f, \overline{\phi_{\bar{\lambda},0,p,q,l}}\right\rangle = \sqrt{\omega_{2n-1}}\left\langle f, \phi_{\lambda,0,p,q}(\varrho)\overline{S_l^{p,q}(\sigma)}\right\rangle, \quad \lambda \in \mathbb{C}, \tag{12.50}$$

where the distribution f acts with respect to the variable $z = \varrho\sigma$. According to (12.25), $\mathcal{F}_l^{(p,q)}(f)$ is an even entire function of λ. If $f \in \mathcal{E}'_{\natural}(\mathbb{C}^n)$, we shall write $\widetilde{f}(\lambda)$ for $\mathcal{F}_1^{(0,0)}(f)(\lambda)$, i.e.,

$$\widetilde{f}(\lambda) = \langle f, \phi_{\lambda,0,0,0,1}\rangle. \tag{12.51}$$

We are going to study basic properties of the transform $\mathcal{F}_l^{(p,q)}$.

Proposition 12.13. *Let $T \in \mathcal{E}'_{\natural}(\mathbb{C}^n)$. Assume that $R \in (r(T), +\infty]$, $f \in \mathcal{D}'(B_R)$ and*

$$\mathfrak{L} f = \lambda^2 f \quad (\textit{respectively}, \ \mathfrak{L}^* f = \lambda^2 f) \tag{12.52}$$

for some $\lambda \in \mathbb{C}$. Then

$$f \star T = \widetilde{T}(\lambda) f \quad (\textit{respectively}, \ T \star f = \widetilde{T}(\lambda) f) \tag{12.53}$$

in the ball $B_{R-r(T)}$.

Proof. We treat the case where $\mathfrak{L} f = \lambda^2 f$. (If $\mathfrak{L}^* f = \lambda^2 f$, the statement is proved analogously.) As $\mathfrak{L}$ is an elliptic operator, the distribution f belongs to $\mathrm{RA}(B_R)$ (see Hörmander [126], Chap. 8.6). Fix $w \in B_{R-r(T)}$ and introduce the function

$$f_w(z) = \int_{\mathrm{U}(n)} f(\tau z + w) \mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}\langle \tau z, w\rangle_{\mathbb{C}}}\, \mathrm{d}\tau, \quad z \in B_{R-|w|}.$$

From the definition of f_w we see that

$$f_w \in \mathrm{RA}_{\natural}(B_{R-|w|}) \quad \text{and} \quad f_w(0) = f(w). \tag{12.54}$$

Furthermore, because of (12.52) and (12.8),

$$(\mathfrak{L} f_w)(z) = \lambda^2 f_w(z), \quad z \in B_{R-|w|}. \tag{12.55}$$

By means of (12.54), (12.55), and (12.20) we have

$$f_w(z) = f(w)\phi_{\lambda,0,0,0,1}(z).$$

This, together with the assumption on T, gives

$$\begin{aligned}\widetilde{T}(\lambda) f(w) &= \left\langle T(z), f_w(z)\right\rangle \\ &= \left\langle T(z), f(z+w)\mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}\langle z, w\rangle_{\mathbb{C}}}\right\rangle\end{aligned}$$

$$= \left\langle T(z), f(w-z)\mathrm{e}^{\frac{\mathrm{i}}{2}\operatorname{Im}\langle w,z\rangle_{\mathbb{C}}}\right\rangle$$
$$= (f \star T)(w),$$

as required. □

Proposition 12.14. *If $f \in \mathcal{E}'_{(p,q),l}(\mathbb{C}^n)$ and $T \in \mathcal{E}'_{\natural}(\mathbb{C}^n)$, then*

$$\mathcal{F}_l^{(p,q)}(f \star T)(\lambda) = \mathcal{F}_l^{(p,q)}(f)(\lambda)\widetilde{T}(\lambda) \tag{12.56}$$

and

$$\mathcal{F}_l^{(p,q)}(T \star f)(\lambda) = \mathcal{F}_l^{(p,q)}(f)(\lambda)\widetilde{T}\left(\sqrt{\lambda^2 + 2q - 2p}\right). \tag{12.57}$$

In particular, for an arbitrary polynomial P,

$$\mathcal{F}_l^{(p,q)}\big(P(\mathfrak{L})f\big)(\lambda) = P\big(\lambda^2\big)\mathcal{F}_l^{(p,q)}(f)(\lambda) \tag{12.58}$$

and

$$\mathcal{F}_l^{(p,q)}\big(P\big(\mathfrak{L}^*\big)f\big)(\lambda) = P\big(\lambda^2 + 2q - 2p\big)\mathcal{F}_l^{(p,q)}(f)(\lambda). \tag{12.59}$$

Proof. In view of (12.6), Proposition 12.1(iv), (12.35), and (12.53),

$$\left\langle f \star T, \overline{\Phi_{\overline{\lambda},0,p,q,l}}\right\rangle = \left\langle f, \overline{\Phi_{\overline{\lambda},0,p,q,l} \star \overline{T}}\right\rangle = \left\langle f, \overline{\widetilde{\overline{T}}\big(\overline{\lambda}\big)}\overline{\Phi_{\overline{\lambda},0,p,q,l}}\right\rangle = \widetilde{T}(\lambda)\left\langle f, \overline{\Phi_{\overline{\lambda},0,p,q,l}}\right\rangle,$$

which proves (12.56). Similarly, one deduces (12.57) (see (12.22)). Put $T = P(\mathfrak{L})\delta_0$ in (12.56) (respectively, $T = P(\mathfrak{L}^*)\delta_0$ in (12.57)). Taking (12.9), (12.10), (12.35), and Proposition 12.1(viii), (ix) into account, we obtain (12.58) and (12.59). □

Proposition 12.15. *The transform $\mathcal{F}_l^{(p,q)}$ is injective on $\mathcal{E}'_{(p,q),l}(\mathbb{C}^n)$.*

Proof. Let $f \in \mathcal{E}'_{(p,q),l}(\mathbb{C}^n)$ and $\mathcal{F}_l^{(p,q)}(f) = 0$. By (12.56),

$$\mathcal{F}_l^{(p,q)}(f \star \varphi)(\lambda) = 0, \quad \lambda \in \mathbb{C}, \tag{12.60}$$

for every function $\varphi \in \mathcal{D}_{\natural}(\mathbb{C}^n)$. Relations (12.60), (12.18), and (12.39) imply that

$$\int_0^a \cos\left(\sqrt{\lambda^2 + q - p}x\right)\int_x^a \varrho^{1-p-q}(f \star \varphi)_{(p,q),l}(\varrho)\big(\varrho^2 - x^2\big)^{n+p+q-3/2}$$
$$\times \mathcal{K}_{n,p,q}(\varrho, x)\,\mathrm{d}\varrho\,\mathrm{d}x = 0,$$

where $a = r(f \star \varphi)$. Hence,

$$\int_x^a \varrho^{1-p-q}(f \star \varphi)_{(p,q),l}(\varrho)\big(\varrho^2 - x^2\big)^{n+p+q-3/2}\mathcal{K}_{n,p,q}(\varrho, x)\,\mathrm{d}\varrho = 0. \tag{12.61}$$

From (12.61) we have

$$\int_u^{a^2} \left(\sqrt{s}\right)^{-p-q} (f \star \varphi)_{(p,q),l}\left(\sqrt{s}\right)(s-u)^{n+p+q-3/2}\mathcal{K}_{n,p,q}\left(\sqrt{s},\sqrt{u}\right)\,\mathrm{d}s = 0 \tag{12.62}$$

for $0 < u < a^2$. Take $t \in (0, a^2)$. Multiply (12.62) by $(u-t)^{n+p+q-3/2}$ and integrate with respect to u from t to a^2. Changing the order of integration, we get

$$\int_t^{a^2} \left(\sqrt{s}\right)^{-p-q} (f \star \varphi)_{(p,q),l}\left(\sqrt{s}\right) \int_t^s \left((s-u)(u-t)\right)^{n+p+q-3/2} \times \mathcal{K}_{n,p,q}\left(\sqrt{s},\sqrt{u}\right)\,\mathrm{d}u\,\mathrm{d}s = 0.$$

The substitution $(s-t)x = s+t-2u$ in the inner integral yields

$$\int_t^{a^2} \left(\sqrt{s}\right)^{-p-q} (f \star \varphi)_{(p,q),l}\left(\sqrt{s}\right)(s-t)^{2n+2p+2q-2} g(s,t)\,\mathrm{d}s = 0, \tag{12.63}$$

where

$$g(s,t) = \int_{-1}^1 \left(1-x^2\right)^{n+p+q-3/2}\mathcal{K}_{n,p,q}\left(\sqrt{s}, \sqrt{\frac{s+t-(s-t)x}{2}}\right)\mathrm{d}x \in C^\infty\left(\mathbb{R}^2\right).$$

Differentiating $2n+2p+2q-1$ times with respect to t in (12.63), we find

$$\frac{(f \star \varphi)_{(p,q),l}(\sqrt{t})}{(\sqrt{t})^{p+q}} - \int_t^{a^2} \frac{(f \star \varphi)_{(p,q),l}(\sqrt{s})}{(\sqrt{s})^{p+q}} k(s,t)\,\mathrm{d}s = 0,$$

where

$$k(s,t) = \frac{2^{n+p+q-2}}{(2n+2p+2q-2)!}\left(\frac{\partial}{\partial t}\right)^{2n+2p+2q-1}\left((s-t)^{2n+2p+2q-2} g(s,t)\right).$$

Thus, $(\sqrt{t})^{-p-q}(f \star \varphi)_{(p,q),l}(\sqrt{t})$ is a solution of the homogeneous integral Volterra equation of the second kind with kernel $k \in C^\infty(\mathbb{R}^2)$. Therefore, $(f \star \varphi)_{(p,q),l} = 0$ and $f \star \varphi = 0$. Since φ can be chosen arbitrary, it follows that $f = 0$. □

Proposition 12.16.

(i) *Suppose that* $f \in (C^s \cap \mathcal{E}'_{(p,q),l})(\mathbb{C}^n)$ *for some* $s \in \mathbb{Z}_+$. *Then*

$$\left|\mathcal{F}_l^{(p,q)}(f)(\lambda)\right| \leqslant c\frac{\mathrm{e}^{r(f)|\mathrm{Im}\,\lambda|}}{(1+|\lambda|)^{s+p+q}}, \quad \lambda \in \mathbb{C}, \tag{12.64}$$

where the constant c *is independent of* λ.

(ii) *Let* $f \in \mathcal{E}'_{(p,q),l}(\mathbb{C}^n)$ *and* $s \in \mathbb{Z}_+$. *Assume that*

$$\mathcal{F}_l^{(p,q)}(f)(\lambda_j) = O\left(j^{-n-\frac{p+q+s}{2}-1}\right) \quad \text{as } j \to +\infty,$$

where λ_j *is defined by* (12.48). *Then* $f \in C^s(\mathbb{C}^n)$.

Proof. (i) By (12.50),

$$\mathcal{F}_l^{(p,q)}(f)(\lambda) = \sqrt{\omega_{2n-1}} \int_0^\infty \varrho^{2n-1} f_{(p,q),l}(\varrho)\phi_{\lambda,0,p,q}(\varrho)\,\mathrm{d}\varrho. \tag{12.65}$$

For the function $\phi_{\lambda,0,p,q}$, we have differentiation formulas (12.28)–(12.31). Therefore, repeated integration by parts in (12.65) gives

$$\mathcal{F}_l^{(p,q)}(f)(\lambda) = \kappa \int_0^{r(f)} \varrho^{2n-1}\big(d_2^{s-2[s/2]}(d_1 d_2)^{[s/2]} f_{(p,q),l}\big)(\varrho) \times \phi_{\lambda,0,p+s-2[s/2],q}(\varrho)\,\mathrm{d}\varrho,$$

where $d_1 = D_1(1-2n-p-q)$, $d_2 = D_2(p+q)$,

$$\kappa = \frac{\sqrt{\omega_{2n-1}}}{(2p+n-\lambda^2)^{[s/2]}}\left(\frac{-1}{2(n+p+q)}\right)^{s-2[s/2]}.$$

Now (i) follows from (12.40).

(ii) This can be proved directly with the aid of Proposition 12.12, (12.50), and (12.40). □

We now prove an analogue of the Paley–Wiener theorem for the transform $\mathcal{F}_l^{(p,q)}$.

Theorem 12.2.

(i) *Let* $f \in \mathcal{E}'_{(p,q),l}(\mathbb{C}^n)$ *and* supp $f \subset \mathring{B}_r$. *Then*

$$\left|\mathcal{F}_l^{(p,q)}(f)(\lambda)\right| \leqslant c_1(1+|\lambda|)^{c_2}\mathrm{e}^{r|\mathrm{Im}\,\lambda|}, \quad \lambda \in \mathbb{C}, \tag{12.66}$$

where $c_1, c_2 > 0$ *are independent of* λ. *Conversely, for each even entire function* $w(\lambda)$ *satisfying the estimate of the form* (12.66)*, there exists a distribution* $f \in \mathcal{E}'_{(p,q),l}(\mathbb{C}^n)$ *such that*

$$\operatorname{supp} f \subset \mathring{B}_r \quad \textit{and} \quad \mathcal{F}_l^{(p,q)}(f) = w. \tag{12.67}$$

(ii) *If* $f \in \mathcal{D}_{(p,q),l}(\mathbb{C}^n)$ *and* supp $f \subset \mathring{B}_r$*, then for every* $N \in \mathbb{Z}_+$*, there exists a constant* $c_N > 0$ *such that*

$$\left|\mathcal{F}_l^{(p,q)}(f)(\lambda)\right| \leqslant c_N(1+|\lambda|)^{-N}\mathrm{e}^{r|\mathrm{Im}\,\lambda|}, \quad \lambda \in \mathbb{C}. \tag{12.68}$$

Conversely, for each even entire function $w(\lambda)$ *satisfying the estimate of the form* (12.68) *for all* $N \in \mathbb{Z}_+$*, there exists a function* $f \in \mathcal{D}_{(p,q),l}(\mathbb{C}^n)$ *such that conditions* (12.67) *hold.*

Proof. (i) Using (12.40) and the definition of ord f, we obtain that for each $\varepsilon > 0$,

$$\left|\mathcal{F}_l^{(p,q)}(f)(\lambda)\right| \leqslant \varkappa_\varepsilon \mathrm{e}^{(r+\varepsilon)|\mathrm{Im}\,\lambda|}(1+|\lambda|)^{\mathrm{ord}\, f - p - q}, \quad \lambda \in \mathbb{C},$$

where $\varkappa_\varepsilon > 0$ does not depend on λ. Applying the Phragmén–Lindelöf principle, we arrive at (12.66).

Let us prove the converse assertion. First assume that the number of zeros of the function w is finite. Then, by Proposition 6.1(iv), w is an even polynomial. Consider the differential operator $S_l^{p,q}(\partial)$ associated with $S_l^{p,q}(z) = \varrho^{p+q} S_l^{p,q}(\sigma)$ (i.e., we replace each monomial $z_1^{\alpha_1} \cdots z_n^{\alpha_n} \overline{z}_1^{\beta_1} \cdots \overline{z}_n^{\beta_n}$ by

$$2^{p+q}\left(\frac{\partial}{\partial \overline{z}_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial \overline{z}_n}\right)^{\alpha_n} \left(\frac{\partial}{\partial z_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial z_n}\right)^{\beta_n}$$

in $S_l^{p,q}(z)$). By virtue of (5.10) and (5.19), $S_l^{p,q}(\partial)^*\delta_0 \in \mathcal{E}'_{(p,q),l}(\mathbb{C}^n)$ and

$$\mathcal{F}_l^{(p,q)}\big(S_l^{p,q}(\partial)^*\delta_0\big)(\lambda) = \frac{2^{p+q}(n)_{p+q}}{\sqrt{\omega_{2n-1}}}, \tag{12.69}$$

where $S_l^{p,q}(\partial)^*$ is the adjoint to the operator $S_l^{p,q}(\partial)$. Combining (12.69) with Proposition 12.14, we infer that conditions (12.67) hold for the distribution $f = P_1(\mathfrak{L}) S_l^{p,q}(\partial)^*\delta_0$, where

$$P_1(\lambda) = \frac{\sqrt{\omega_{2n-1}}}{2^{p+q}(n)_{p+q}} w\big(\sqrt{\lambda}\big).$$

Next, let the function w have infinitely many zeroes. Pick an even polynomial P_2 for which the function w/P_2 is entire and

$$\sup_{\lambda \in \mathbb{C}} \frac{(1+|\lambda|)^{2n+3}}{\mathrm{e}^{r|\mathrm{Im}\,\lambda|}}\left|\frac{w(\lambda)}{P_2(\lambda)}\right| < \infty.$$

By the Paley–Wiener theorem for the Fourier-cosine transform, there exists an even function $\varphi \in C^{2n}(\mathbb{R}^1)$ such that $\operatorname{supp}\varphi \subset [-r, r]$ and

$$\frac{w(\lambda)}{P_2(\lambda)} = \int_0^r \cos(\lambda t)\varphi(t)\,\mathrm{d}t, \quad \lambda \in \mathbb{C}. \tag{12.70}$$

Let $\psi \in C(0, r]$ be a function satisfying

$$\varphi(t) = \omega_{2n-1}\int_t^r \varrho\psi(\varrho)\big(\varrho^2 - t^2\big)^{n-3/2}\mathcal{K}_{n,0,0}(\varrho, t)\,\mathrm{d}\varrho, \quad 0 < t < r \tag{12.71}$$

(as above, this equation reduces to a Volterra integral equation of the second kind). Put

$$h(z) = \begin{cases} \psi(|z|), & z \in B_r, \\ 0, & z \in \mathbb{C}^n \backslash B_r. \end{cases}$$

In the same manner as in the proof of Theorem 11.2(i), we see that $h \in L_\natural^{1,\mathrm{loc}}(\mathbb{C}^n)$. In addition, owing to (12.51), (12.39), (12.70), and (12.71), $\widetilde{h}(\lambda) = w(\lambda)/P_2(\lambda)$.

Now, as in the first case, we conclude that conditions (12.67) hold for the distribution $f = P_3(\mathfrak{L})S_l^{p,q}(\partial)^*\delta_0 \star h$, where

$$P_3(\lambda) = \frac{\sqrt{\omega_{2n-1}}}{2^{p+q}(n)_{p+q}} P_2\big(\sqrt{\lambda}\big).$$

So part (i) is proved. Part (ii) is immediate from (i) and Proposition 12.16. $\square$

Corollary 12.1. *Let* $r > 0$, $u \in L^1[0, r]$, *and suppose that*

$$\int_0^{\varrho} u(t)\mathfrak{K}_{n,p,q}(\varrho, t)\,\mathrm{d}t = 0$$

for almost all $\varrho \in (0, r)$, *where* $\mathfrak{K}_{n,p,q}(\varrho, t)$ *is given by* (12.38)*. Then* $u = 0$.

Proof. Let $\psi \in \mathcal{D}_\natural(-r, r)$. By Theorem 12.2 there exists $h \in \mathcal{D}_{(p,q),1}(\mathbb{C}^n)$ such that $\operatorname{supp} h \subset B_r$ and $\widehat{\psi} = \mathcal{F}_1^{(p,q)}(h)$. Using (12.50) and (12.38), we find

$$\psi(t) = \frac{1}{2}\int_t^r \varrho^{2n-1} h_{(p,q),1}(\varrho)\mathfrak{K}_{n,p,q}(\varrho, t)\,\mathrm{d}\varrho, \quad t \in (0, r).$$

Then by hypothesis

$$\int_0^r u(t)\psi(t)\,\mathrm{d}t = 0.$$

Since ψ can be chosen arbitrarily, it follows that $u = 0$. $\square$

Remark 12.5. From Theorem 12.2(i) and Propositions 12.16 and 12.14 we can conclude:

(i) If $f \in \mathcal{E}'_{(p,q),l}(\mathbb{C}^n)$, then

$$\mathcal{F}_l^{(p,q)}(f)(\lambda) = O\big((1 + |\lambda|)^{\operatorname{ord} f - p - q} \mathrm{e}^{r(f)|\operatorname{Im}\lambda|}\big), \quad \lambda \in \mathbb{C}.$$

(ii) For the order of a distribution $f \in \mathcal{E}'_{(p,q),l}(\mathbb{C}^n)$ satisfying (12.66), one has $\operatorname{ord} f \leqslant \max\{0, c_2 + 2n + p + q + 4\}$.

Because of Theorems 12.2 and 6.3, the mapping $\Lambda^{(p,q),l} : \operatorname{conj}(\mathcal{E}'_{(p,q),l}(\mathbb{C}^n)) \to \mathcal{E}'_\natural(\mathbb{R}^1)$ given by

$$\widehat{\Lambda^{(p,q),l}(T)}(\lambda) = \overline{\mathcal{F}_l^{(p,q)}(\overline{T})(\overline{\lambda})} = \langle T, \phi_{\lambda,0,p,q,l}\rangle, \quad \lambda \in \mathbb{C},$$

is a bijection and $r(\Lambda^{(p,q),l}(T)) = r(T)$. Furthermore (see Remark 12.5 and Theorem 6.3),

$$\operatorname{ord} \Lambda^{(p,q),l}(T) \leqslant \max\{0, \operatorname{ord} T - p - q + 1\}.$$

In the special case $p = q = 0$, $l = 1$, we write Λ instead of $\Lambda^{(0,0),1}$. If $\mathfrak{W}(\mathbb{R}^1)$ denotes one of the classes $\mathfrak{M}(\mathbb{R}^1)$, $\mathfrak{N}(\mathbb{R}^1)$, $\mathfrak{E}(\mathbb{R}^1)$, $\mathrm{Inv}_+(\mathbb{R}^1)$, or $\mathrm{Inv}(\mathbb{R}^1)$, we set

$$\mathfrak{W}(\mathbb{C}^n) = \{T \in \mathcal{E}'_\natural(\mathbb{C}^n) : \Lambda(T) \in \mathfrak{W}(\mathbb{R}^1)\}.$$

We conclude this section with the inversion formula for the transform $\mathcal{F}_l^{(p,q)}$.

Proposition 12.17. *Let $f \in (\mathcal{E}'_{(p,q),l} \cap C^s)(\mathbb{C}^n)$ for some $s \geqslant 2n+2$. Then*

$$\mu_j \mathcal{F}_l^{(p,q)}(f)(\lambda_j) = O(j^{\frac{1}{2}(2n+p+q-2-s)}) \quad \text{as } j \to +\infty \tag{12.72}$$

and

$$f(z) = \sum_{j=0}^{\infty} \mu_j \mathcal{F}_l^{(p,q)}(f)(\lambda_j)\phi_{\lambda_j,0,p,q,l}(z) \tag{12.73}$$

for $z \in \mathbb{C}^n$, where λ_j and μ_j are defined by (12.48) *and* (12.49).

Proof. Estimate (12.72) follows from (12.64). Relation (12.73) is essentially a special case of an expansion in Laguerre polynomials (see Proposition 12.12 and estimate (12.40)). □

12.5 Transmutation Mappings Generated by the Laguerre Polynomials Expansion

Let $p, q \in \mathbb{Z}_+$ and $l \in \{1, \dots, d(n, p, q)\}$. In this section we introduce and study the operator $\mathfrak{A}_{(p,q),l}$, which is an analogue of the operator $\mathfrak{A}_{k,m,j}$ from Sect. 11.4 for the twisted convolution on $\mathbb{C}^n$. It is closely related to the Laguerre polynomial expansion obtained in Proposition 12.17.

Let $R \in (0, +\infty]$ and $f \in \mathcal{D}'_{(p,q),l}(B_R)$. For $\psi \in \mathcal{D}(-R, R)$, we select $\eta \in \mathcal{D}_\natural(B_R)$ so that $\eta = 1$ in $B_{r_0(\psi)+\varepsilon}$ with some $\varepsilon \in (0, R - r_0(\psi))$. Put

$$\langle \mathfrak{A}_{(p,q),l}(f), \psi \rangle = \sum_{j=0}^{\infty} \mu_j \mathcal{F}_l^{(p,q)}(f\eta)(\lambda_j) \int_{-R}^{R} \psi(t) \cos(\lambda_j t)\, \mathrm{d}t \tag{12.74}$$

(for notation, see Sects. 12.3 and 12.4). Taking Propositions 12.8, 12.14, and 12.17 into account, we see from Corollary 12.1 and the proof of Lemma 9.2 that $\mathfrak{A}_{(p,q),l}(f)$ is well defined by (12.74) as a distribution in $\mathcal{D}'_\natural(-R, R)$, and

$$\mathfrak{A}_{(p,q),l}(f|_{B_r}) = \mathfrak{A}_{(p,q),l}(f)|_{(-r,r)}$$

for every $r \in (0, R]$. In addition, using (12.40), (12.35), Remark 12.1, and Proposition 12.13 and repeating the arguments in the proof of Theorem 9.3, we obtain the following:

Theorem 12.3. *For $R \in (0, +\infty]$, $N \in \mathbb{Z}_+$, and $\nu = 2 + 2n + p + q + N$, the following statements hold.*

(i) *Let $f \in \mathcal{D}'_{(p,q),l}(B_R)$ and $r \in (0, R]$. Then $f = 0$ in B_r if and only if $\mathfrak{A}_{(p,q),l}(f) = 0$ on $(-r, r)$.*
(ii) *If $f \in C^{\nu}_{(p,q),l}(B_R)$, then $\mathfrak{A}_{(p,q),l}(f) \in C^N_{\natural}(-R, R)$. Furthermore,*

$$f_{(p,q),l}(\varrho) = \int_0^{\varrho} \mathfrak{A}_{(p,q),l}(f)(t)\mathfrak{K}_{n,p,q}(\varrho, t)\,\mathrm{d}t, \quad 0 < \varrho < R,$$

where $\mathfrak{K}_{n,p,q}(\varrho, t)$ is given by (12.38), and

$$\mathfrak{A}_{(p,q),l}(f)(0) = \frac{1}{\sqrt{\omega_{2n-1}}} \lim_{\varrho \to 0} f_{(p,q),l}(\varrho)\varrho^{-p-q}.$$

(iii) *The mapping $\mathfrak{A}_{(p,q),l}$ is continuous from $\mathcal{D}'_{(p,q),l}(B_R)$ into $\mathcal{D}'_{\natural}(-R, R)$ and from $C^{\nu}_{(p,q),l}(B_R)$ into $C^N_{\natural}(-R, R)$.*
(iv) *Let $f \in \mathcal{D}'_{(p,q),l}(B_R)$ and* $\operatorname{ord} f = N$. *Then* $\operatorname{ord} \mathfrak{A}_{(p,q),l}(f) \leqslant \nu$.
(v) *Assume that $f \in C^{\nu}_{(p,q),l}(B_R)$ has all derivatives of order $\leqslant \nu$ vanishing at* 0. *Then*

$$\mathfrak{A}_{(p,q),l}(f)^{(s)}(0) = 0, \quad s = 0, \dots, N.$$

(vi) *For $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{Z}_+$, we have*

$$\mathfrak{A}_{(p,q),l}(\phi_{\lambda,\mu,p,q,l}) = u_{\lambda,\mu},$$

where $u_{\lambda,\mu}$ is defined in (9.60).
(vii) *Let $T \in \operatorname{conj}(\mathcal{E}'_{(p,q),l}(\mathbb{C}^n))$, $r(T) < R$, and $F \in C^s_{(p,q),l}(B_R)$ with* $s = \max\{2n + p + q + 2, \operatorname{ord} T + 2n + 3\}$. *Then*

$$\langle T, f\rangle = \big\langle \Lambda^{(p,q),l}(T), \mathfrak{A}_{(p,q),l}(f)\big\rangle.$$

(viii) *If $f \in \mathcal{D}'_{(p,q),l}(B_R)$, $T \in \mathcal{E}'_{\natural}(\mathbb{C}^n)$, and $r(T) < R$, then*

$$\mathfrak{A}_{(p,q),l}(f \star T) = \mathfrak{A}_{(p,q),l}(f) * \Lambda(T)$$

on $(r(T) - R, R - r(T))$. In particular,

$$\mathfrak{A}_{(p,q),l}\big(P(\mathfrak{L})f\big) = P\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right)\mathfrak{A}_{(p,q),l}(f)$$

for each polynomial P.

Remark 12.6. If $f \in C^{\nu}_{(p,q),l}(\mathring{B}_r)$, $r \in (0, +\infty)$, put $\mathfrak{A}_{(p,q),l}(f) = \mathfrak{A}_{(p,q),l}(f_1)|_{[-r,r]}$, where f_1 is a continuation of f on $\mathbb{C}^n$ belonging to $C^{\nu}_{(p,q),l}(\mathbb{C}^n)$. In view of Theorem 12.3(i), (ii), $\mathfrak{A}_{(p,q),l}(f)$ does not depend on the choice of f_1, and $\mathfrak{A}_{(p,q),l}(f) \in C^N_{\natural}[-r, r]$.

Theorem 12.4. *Let $r \in (0, +\infty)$. Then there is a constant $c > 0$ such that*

$$\int_{-r}^{r} \left|\mathfrak{A}_{(p,q),l}(f)^{(M)}(t)\right| \mathrm{d}t \leqslant c \sum_{i=0}^{n+2+[(p+q)/2]} \int_{B_r} \left|\mathfrak{L}^{[(M+1)/2]+i} f(z)\right| \mathrm{d}m_n(z)$$

for all $M \in \mathbb{Z}_+$ and $f \in C^s_{(p,q),l}(\mathring{B}_r)$, where $s = 2([(M+1)/2] + n + 2 + [(p+q)/2])$.

Owing to Theorem 12.3, Proposition 12.16, Remark 12.5, and Lemma 12.1, the proof of Theorem 12.4 does not essentially differ from the proof of Theorem 9.4, so we omit it.

Next, let $F \in \mathcal{D}'_\natural(-R, R)$ and $w \in \mathcal{D}(B_R)$. Consider $\eta \in \mathcal{D}_\natural(-R, R)$ such that $\eta = 1$ on $(-r_0(w) - \varepsilon, r_0(w) + \varepsilon)$ for some $\varepsilon \in (0, R - r_0(w))$, where $r_0(w) = \inf\{r > 0 : \operatorname{supp} w \subset B_r\}$. Then, as above, we see from the proof of Lemma 9.3 and Theorem 9.5 that the relation

$$\begin{aligned}\left\langle \mathfrak{B}_{(p,q),l}(F), w \right\rangle &= \frac{1}{\pi} \int_0^\infty \widehat{F\eta}(\lambda) \mathcal{F}_l^{(p,q)}\left(\overline{(\overline{w})_{(p,q),l}(\varrho)} S_l^{(p,q)}(\sigma)\right)(\lambda)\, \mathrm{d}\lambda \\ &= \frac{1}{\pi} \int_0^\infty \widehat{F\eta}(\lambda) \langle w, \phi_{\lambda,0,p,q,l}\rangle\, \mathrm{d}\lambda\end{aligned}$$

defines $\mathfrak{B}_{(p,q),l}(F)$ as a distribution in $\mathcal{D}'_{(p,q),l}(B_R)$, and we have the following result.

Theorem 12.5.

(i) *Let $F \in \mathcal{D}'_\natural(-R, R)$, $r \in (0, R]$. Then $F = 0$ on $(-r, r)$ if and only if $\mathfrak{B}_{(p,q),l}(F) = 0$ in B_r.*

(ii) *If $F \in C^s_\natural(-R, R)$, $s \geqslant 2$, then $\mathfrak{B}_{(p,q),l}(F) \in C^{s+p+q-2}_{(p,q),l}(B_R)$. In addition,*

$$\mathfrak{B}_{(p,q),l}(F)(z) = \int_0^\varrho F(t) \mathfrak{K}_{n,p,q}(\varrho, t)\, \mathrm{d}t\, S_l^{(p,q)}(\sigma)$$

for $z \in B_R \backslash \{0\}$, and

$$\lim_{z \to 0} \frac{\mathfrak{B}_{(p,q),l}(F)(z)(S_l^{(p,q)}(z/|z|))^{-1}}{|z|^{p+q}} = \sqrt{\omega_{2n-1}} F(0).$$

(iii) *The mapping $\mathfrak{B}_{(p,q),l}$ is continuous from $\mathcal{D}'_\natural(-R, R)$ into $\mathcal{D}'_{(p,q),l}(B_R)$ and from $C^s_\natural(-R, R)$, $s \geqslant 2$, into $C^{s+p+q-2}_{(p,q),l}(B_R)$.*

(iv) *If $F \in \mathcal{D}'_\natural(-R, R)$, then* $\operatorname{ord} \mathfrak{B}_{(p,q),l}(F) \leqslant \max\{0, \operatorname{ord} F - p - q + 3\}$.

(v) *Suppose that $s \in \{2, 3, \ldots\}$, $F \in C^s_\natural(-R, R)$, and $F^{(\nu)}(0) = 0$ with $\nu = 0, \ldots, s$. Then all derivatives of $\mathfrak{B}_{(p,q),l}(F)$ of order $\leqslant s + p + q - 2$ vanish at* 0.

(vi) *For $F \in \mathcal{D}'_\natural(-R, R)$, one has $\mathfrak{A}_{(p,q),l}(\mathfrak{B}_{(p,q),l}(F)) = F$.*

(vii) *If* $T \in \operatorname{conj}(\mathcal{E}'_{(p,q),l}(\mathbb{C}^n))$, $r(T) < R$, $m = \max\{2, \operatorname{ord} T - p - q + 2\}$, *and* $F \in C^m_\natural(-R, R)$, *then*

$$\langle T, \mathfrak{B}_{(p,q),l}(F)\rangle = \langle \Lambda^{(p,q),l}(T), F\rangle.$$

(viii) *If* $F \in \mathcal{D}'_\natural(-R, R)$, $T \in \mathcal{E}'_\natural(\mathbb{C}^n)$, *and* $r(T) < R$, *then*

$$\mathfrak{B}_{(p,q),l}(F) \star T = \mathfrak{B}_{(p,q),l}\big(F * \Lambda(T)\big)$$

in $B_{R-r(T)}$. *In particular,*

$$P(\mathfrak{L})\mathfrak{B}_{(p,q),l}(F) = \mathfrak{B}_{(p,q),l}\left(P\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right)F\right)$$

for every polynomial P.

Remark 12.7. Theorems 12.3 and 12.5 show that the mapping $\mathfrak{A}_{(p,q),l}$ is a homeomorphism of $\mathcal{D}'_{(p,q),l}(B_R)$ onto $\mathcal{D}'_\natural(-R, R)$ and $C^\infty_{(p,q),l}(B_R)$ onto $C^\infty_\natural(-R, R)$. In addition, $\mathfrak{A}^{-1}_{(p,q),l} = \mathfrak{B}_{(p,q),l}$.

In complete analogy with Remark 9.3 we can now define $\mathfrak{B}_{(p,q),l}$ on $C^s_\natural[-r, r]$, $s \geqslant 2$, $r \in (0, +\infty)$. Then by Theorem 12.5(vii), (viii) and the method developed in the proof of Theorem 9.6 we have the following:

Theorem 12.6. *There exists a constant* $c > 0$ *such that for all* $N \in \mathbb{Z}_+$ *and* $F \in C^{2N+2}_\natural[-r, r]$,

$$\int_{B_r} \left|\mathfrak{L}^N \mathfrak{B}_{(p,q),l}(F)(z)\right| \mathrm{d}m_n(z) \leqslant c \int_{-r}^{r} \left(\left|F^{(2N)}(t)\right| + \left|F^{(2N+2)}(t)\right|\right) \mathrm{d}t.$$

To go further, we set

$$\mathcal{A}^{(p,q)}_l = \mathfrak{A}^{-1}_{(0,0),1}\mathfrak{A}_{(p,q),l}.$$

The operator $\mathcal{A}^{(p,q)}_l$ possesses the following properties:

(a)

$$\mathcal{A}^{(p,q)}_l(\phi_{\lambda,\mu,p,q,l}) = \phi_{\lambda,\mu,0,0,1}; \tag{12.75}$$

(b)

$$\mathcal{A}^{(p,q)}_l(f \star T) = \mathcal{A}^{(p,q)}_l(f) \star T \quad \text{in } B_{R-r(T)} \tag{12.76}$$

for $f \in \mathcal{D}'_{(p,q),l}(B_R)$, $T \in \mathcal{E}'_\natural(\mathbb{C}^n)$, and $r(T) < R$.

Relations (12.75), (12.30), and (12.31) lead to the following:

Lemma 12.3. *Let* $f \in C^{p+q+2n+4}_{(p,q),l}(B_R)$, *and let*

$$c_s = \frac{1}{\sqrt{\omega_{2n-1}}} 2^{-s} \frac{(n-1)!}{(n+s-1)!}, \quad s \in \mathbb{Z}_+.$$

Then

$$\mathcal{A}_l^{(p,q)}(f) = f \tag{12.77}$$

if $p = q = 0$,

$$\mathcal{A}_l^{(p,q)}(f) = c_p D_1(1-2n)\cdots D_1(2-p-2n)(f_{(p,q),l}) \tag{12.78}$$

if $p \geqslant 1$, $q = 0$,

$$\mathcal{A}_l^{(p,q)}(f) = c_q D_2(1-2n)\cdots D_2(2-q-2n)(f_{(p,q),l}) \tag{12.79}$$

if $p = 0$, $q \geqslant 1$, *and*

$$\mathcal{A}_l^{(p,q)}(f) = c_{p+q} D_1(1-2n)\cdots D_1(2-p-2n) D_2(1-p-2n)\cdots \times D_2(2-p-q-2n)\big(f_{(p,q),l}\big) \tag{12.80}$$

if $p \geqslant 1$, $q \geqslant 1$.

The proof of this lemma is similar to that of Lemma 9.4, the only change being that instead of using (9.47) we now use (12.73).

To close we note that equalities (12.77)–(12.80) allow one to extend the operator $\mathcal{A}_l^{(p,q)}$ on $\mathcal{D}'_{(p,q),l}(\mathcal{O})$, where $\mathcal{O}$ is a nonempty open $\mathrm{U}(n)$-invariant subset of $\mathbb{C}^n$ (see the end of Sect. 9.4).

Part III
Mean Periodicity

Mean periodic functions are a far-reaching generalization of ordinary periodic functions. The notion of a mean periodic function was introduced by Delsarte [54] and was afterwards developed by Schwartz [188]. According to Schwartz, an infinitely differentiable function f on $\mathbb{R}^n$ is said to be *mean periodic* if the closed subspace $V(f)$ generated by f and all its translates is proper in $\mathcal{E}(\mathbb{R}^n)$. Equivalently, by the Hahn–Banach theorem, $f \in C^\infty(\mathbb{R}^n)\backslash\{0\}$ is mean periodic if and only if there exists a compactly supported distribution $T \neq 0$ such that $f * T = 0$. These equations generalize linear partial differential equations with constant coefficients. If $f * T = 0$, we say that f is mean periodic with respect to T. Analogously, by a mean periodic function on a symmetric space or on the Heisenberg group we mean a solution of convolution equation of compact support.

In the course of the study of mean periodic functions a complicated theory arises, which can be developed rather far. A remarkable feature of this theory is that its results are closely related to a wide variety of problems in contemporary mathematics.

In each case there is a striking difference between the behavior of mean periodic functions. We now illustrate this by some examples.

Example I. Let f be a mean periodic function on $\mathbb{R}^1$. Then $V(f)$ contains an exponential $e^{i\lambda x}$ for some $\lambda \in \mathbb{C}$. This result is due to Schwartz [188]. An exact analogue of the Schwartz theorem fails to be true in the case $\mathbb{R}^n$, $n \geqslant 2$ (see Gurevich [102]).

Example II. Let M be a compact Riemannian manifold and L the Laplace–Beltrami operator on M. If f is a twice differentiable function on M such that $Lf \geqslant 0$ everywhere, then f is a constant function in view of the classical Hopf lemma (see Kobayashi and Nomizu [137], Vol. II, Note 14). This statement shows that a compact symmetric space has no mean periodic functions with respect to $T = L\,\delta_0$ except for constant functions. On the other hand, such functions exist on symmetric spaces of noncompact type (see Sect. 10.3).

Example III. Let μ_r stand for the normalized surface measure on the sphere $S_r = \{z \in \mathbb{C}^n : |z| = r\}$, and let $\varphi_k(z) = e^{-|z|^2/4}\ {}_1F_1(-k, n; |z|^2/2)$, $k \in \mathbb{Z}_+$. If $\varphi_k|_{S_r} = 0$, then it follows from (12.35) and (12.53) that φ_k is mean periodic with respect to μ_r. Note that φ_k is a Schwartz class function. This is in sharp contrast with the case of ordinary mean periodic functions on $\mathbb{R}^n$. As is well known, no mean periodic function on $\mathbb{R}^n$ can be integrable.

In this part we wish to arrive at a better understanding of mean periodic functions on the spaces under consideration. Results related to this theme are numerous and diverse. In our treatment of mean periodic functions we discriminate between the following aspects: (i) group and infinitesimal properties of mean periodic functions; (ii) support theorems; (iii) characterization of uniqueness sets; (iv) multidimensional analogues of the distribution ζ_T; (v) description of various classes of mean periodic functions; (vi) a periodic in the mean extension; (vii) analogues of Liouville's property; (viii) approximation theorems.

Chapters 13 and 14 treat the case of Euclidean spaces. A detailed study of mean periodic functions on symmetric spaces is contained in Chaps. 15 and 16. Chapter 17 is devoted to the case of the phase space and the Heisenberg group.

Chapter 13
Mean Periodic Functions on Subsets of the Real Line

We have already made sporadic use of mean periodic functions on $\mathbb{R}^1$ (see Chap. 8). Our intention here is to provide a much more systematic treatment than in Chap. 8.

We continue the study of the distributions $T_{\lambda,\eta}$ begun in Part II and indicate how these techniques can be applied to the subject matter. In particular, in Sect. 13.2 we investigate the basic properties of mean periodic functions which are equal to zero on some open subsets of $\mathbb{R}^1$. We describe here the uniqueness sets of distributions f satisfying the equation

$$f * T = 0 \tag{13.1}$$

with $T \in (\mathrm{Inv}_+ \cap \mathrm{Inv}_-)(\mathbb{R}^1)$ (Theorem 13.5). In addition, we prove a number of sharp uniqueness results for solutions of (13.1) in the case where the zero set of f contains $\operatorname{supp} T$.

Sections 13.3 and 13.4 are devoted to the study of the structure of solutions of (13.1). We prove local analogues of Schwartz's fundamental principle and give other characterizations of mean periodic functions. For example, if $T \in (\mathrm{Inv}_+ \cap \mathrm{Inv}_-)(\mathbb{R}^1)$, we derive the representation $f = \zeta_T * u$, where $u \in \mathcal{E}'(\mathbb{R}^1)$. For various classes of distributions $T \in \mathcal{E}'(\mathbb{R}^1)$, we obtain the series development

$$f(t) = \sum_{\lambda \in \mathcal{Z}(\widehat{T})} \sum_{\eta=0}^{m(\lambda,T)} \gamma_{\lambda,\eta} (\mathrm{i}t)^{\eta} \mathrm{e}^{\mathrm{i}\lambda t} \tag{13.2}$$

with detailed information on the coefficients that appear in it. Another central aspect of Chap. 13 is the problem of mean periodic continuation (see Sect. 13.5). Here, interesting effects arise: the possibility of the extension depends to a large extent on the behavior of $\mathrm{Im}\,\lambda$ as $\lambda \to \infty$, $\lambda \in \mathcal{Z}(\widehat{T})$.

In Sect. 13.6 we specify the form of series (13.2) for the case where f satisfies some growth conditions at infinity. In particular, we indicate the exact dependence between the growth of f and the set on nonzero coefficients in (13.2).

V.V. Volchkov, V.V. Volchkov, *Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group,* Springer Monographs in Mathematics,

13.1 Main Classes of Mean Periodic Functions

Let $T \in \mathcal{E}'(\mathbb{R}^1)$, $T \neq 0$, and suppose that

$$-\infty \leqslant a < b \leqslant +\infty, \quad b - a > 2r(T). \tag{13.3}$$

We denote

$$(a, b)_T = \{t \in \mathbb{R}^1 : t - \operatorname{supp} T \subset (a, b)\}.$$

If $f \in \mathcal{D}'(a, b)$, the convolution $f * T$ is well defined in $(a, b)_T$. Throughout this chapter we shall study the convolution equation

$$(f * T)(t) = 0, \quad t \in (a, b)_T, \tag{13.4}$$

with unknown $f \in \mathcal{D}'(a, b)$.

Let $\mathcal{D}'_T(a, b)$ denote the set of all distributions $f \in \mathcal{D}'(a, b)$ satisfying (13.4). Also let

$$C^k_T(a, b) = (\mathcal{D}'_T \cap C^k)(a, b) \quad \text{for } k \in \mathbb{Z}^+ \text{ or } k = \infty,$$
$$C_T(a, b) = C^0_T(a, b), \qquad \mathrm{QA}_T(a, b) = (\mathcal{D}'_T \cap \mathrm{QA})(a, b),$$
$$\mathrm{RA}_T(a, b) = (\mathcal{D}'_T \cap \mathrm{RA})(a, b), \qquad G^\alpha_T(a, b) = (\mathcal{D}'_T \cap G^\alpha)(a, b), \quad \alpha > 0.$$

If the interval (a, b) is symmetric with respect to origin, we set

$$\mathcal{D}'_{T,\natural}(a, b) = (\mathcal{D}'_T \cap \mathcal{D}'_\natural)(a, b).$$

Let us now consider some simplest properties of the class $\mathcal{D}'_T(a, b)$ needed later.

Proposition 13.1.

(i) *If $f \in \mathcal{D}'_T(a, b)$, $\Phi \in \mathcal{E}'(\mathbb{R}^1)$, $\Phi \neq 0$, and $b - a > 2(r(T) + r(\Phi))$, then $f * \Phi \in \mathcal{D}'_T((a, b)_\Phi)$. In particular, if $f \in \mathcal{D}'_T(a, b)$, then*

$$p\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) f \in \mathcal{D}'_T(a, b)$$

for each polynomial p.

(ii) *Let $f = \mathrm{e}^{\lambda,\eta}$, where $\lambda \in \mathbb{C}$ and $\eta \in \mathbb{Z}_+$ (see formula (8.40)). Then*

$$f * T = \sum_{\nu=0}^{\eta} \binom{\eta}{\nu} \widehat{T}^{(\nu)}(\lambda) \mathrm{e}^{\lambda,\eta-\nu}. \tag{13.5}$$

In particular, $f \in \mathcal{D}'_T(a, b)$ if and only if $\lambda \in \mathcal{Z}(\widehat{T})$ and $\eta \leqslant m(\lambda, T)$.

(iii) *Let $\operatorname{supp} T \subset [-r(T), r(T)]$, $f \in \mathcal{D}'_T(a, b)$, $\lambda \in \mathcal{Z}(\widehat{T})$, $n_\lambda = n_\lambda(\widehat{T})$. Then*

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{i}\lambda\right) f * T_{\lambda, n_\lambda - 1} = 0 \quad \text{in } (a, b)_T.$$

In addition, if $n_\lambda \geqslant 2$*, then*

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{i}\lambda\right) f * T_{\lambda,\eta} = \mathrm{i}(\eta+1) f * T_{\lambda,\eta+1} \quad \text{in } (a,b)_T$$

for all $\eta \in \{0, \dots, n_\lambda - 2\}$.

Proof. Part (i) is clear from the definition of $\mathcal{D}'_T(a,b)$. Turning to (ii), let $u_\lambda(t) = \mathrm{e}^{\mathrm{i}\lambda t}$, $\lambda \in \mathbb{C}$. Since

$$(u_\lambda * T)(t) = \widehat{T}(\lambda)\mathrm{e}^{\mathrm{i}\lambda t},$$

one has

$$(f * T)(t) = \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^{\eta} (u_\lambda * T)(t) = \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^{\eta} \big(\widehat{T}(\lambda)\mathrm{e}^{\mathrm{i}\lambda t}\big).$$

This yields (ii). Assertion (iii) follows immediately from (8.20), (8.25), and (8.27). □

The following result describes the class $\mathcal{D}'_T(a,b)$ for the case where $r(T) = 0$.

Proposition 13.2. *Assume that* $r(T) = 0$ *and* $f \in \mathcal{D}'(a,b)$*. Then* $f \in \mathcal{D}'_T(a,b)$ *if and only if*

$$f = \sum_{\lambda \in \mathcal{Z}(\widehat{T})} \sum_{\eta=0}^{m(\lambda,T)} \gamma_{\lambda,\eta} \mathrm{e}^{\lambda,\eta},$$

where $\gamma_{\lambda,\eta} \in \mathbb{C}$.

Proof. According to Corollary 6.2, (13.4) can be reduced to the differential equation

$$\prod_{\lambda \in \mathcal{Z}(\widehat{T})} \left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{i}\lambda\right)^{m(\lambda,T)+1} f = 0.$$

The desired statement is now obvious. □

Some far-reaching generalizations of this result will be given later (see Sect. 13.4). We conclude this section with the following version of the smoothing procedure for solutions of convolution equations.

Proposition 13.3. *Let* $f \in \mathcal{D}'_T(a,b)$ *and assume that* p *is a polynomial such that* $\mathcal{Z}(p) \cap \mathcal{Z}(\widehat{T}) = \varnothing$*. Then there exists* $g \in \mathcal{D}'_T(a,b)$ *such that* $p(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t})g = f$.

Proof. If $p = 0$, then $\mathcal{Z}(\widehat{T}) = \varnothing$, and Proposition 13.2 yields $f = 0$. In this case the desired statement is evident. Next, it is easy to dispense with the case where p is a nonzero identically constant. Assume now that $p(z) = \alpha(z - \lambda)$, where $\alpha, \lambda \in \mathbb{C}$ and $\alpha \neq 0$. Then there exists $F \in \mathcal{D}'(a,b)$ such that

$$p\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right) F = f.$$

Setting $u = F * T$, we obtain

$$p\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right)u = 0 \quad \text{in } (a,b)_T.$$

Hence, $u(t) = c_\lambda e^{\mathrm{i}\lambda t}$ for some $c_\lambda \in \mathbb{C}$, and the distribution

$$g(t) = F(t) - \frac{c_\lambda e^{\mathrm{i}\lambda t}}{\widehat{T}(\lambda)}, \quad t \in (a,b),$$

satisfies the requirement. Thus, the desired result is valid if the polynomial p is linear. Now one may proceed iteratively to complete the proof in the general case. □

The example

$$\widehat{T}(z) = p(z) = z$$

shows that Proposition 13.3 is no longer valid if $\mathcal{Z}(p) \cap \mathcal{Z}(\widehat{T}) \neq \varnothing$.

Corollary 13.1. *Let E be an infinite subset of $\mathbb{C}$, let $E \cap \mathcal{Z}(\widehat{T}) = \varnothing$, $f \in \mathcal{D}'_T(a,b)$, and suppose that*

$$a < a' < b' < b, \qquad b' - a' > 2r(T).$$

Then for each $m \in \mathbb{Z}_+$, there exists $g \in C^m_T(a',b')$ such that $p(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t})g = f$ in (a',b') for some polynomial p. Moreover, all the zeros of the polynomial p are simple and $\mathcal{Z}(p) \subset E$.

Proof. By assumption the distribution f is of finite order in (a',b'). Now the desired conclusion follows by using Proposition 13.3 on a polynomial p such that $\mathcal{Z}(p) \cap \mathcal{Z}(p') = \varnothing$, $\mathcal{Z}(p) \subset E$, and the degree of p is large enough. □

13.2 Structure of Zero Sets

The object of this section is to establish main properties of mean periodic functions which are equal to zero on some open subsets of $\mathbb{R}^1$.

Throughout the section we assume that

$$T \in \mathcal{E}'(\mathbb{R}^1), \quad r(T) > 0, \quad \text{and} \quad \operatorname{supp} T \subset [-r(T), r(T)]. \tag{13.6}$$

Next, for $m \in \mathbb{Z}$, $m < 0$, $p \in [1,+\infty]$, and $(\alpha,\beta) \subset \mathbb{R}^1$, let us define

$$L^{p,\mathrm{loc}}_m(\alpha,\beta) = \left\{f \in \mathcal{D}'(\alpha,\beta) : f = g^{(-m)} \text{ for some } g \in L^{p,\mathrm{loc}}(\alpha,\beta)\right\}$$

and

$$L^p_m(\alpha,\beta) = \left\{f \in \mathcal{D}'(\alpha,\beta) : f = g^{(-m)} \text{ for some } g \in L^p(\alpha,\beta)\right\}.$$

Recall that the classes $L_m^{p,\mathrm{loc}}$ and L_m^p for $m \in \mathbb{Z}_+$ are defined in Sect. 1.2.
The first result we would like to establish is as follows.

Theorem 13.1.

(i) *Suppose that*

$$-\infty \leqslant a < -r(T), \quad r(T) < b \leqslant +\infty, \tag{13.7}$$

let $f \in \mathcal{D}'_T(a,b)$ *and* $f = 0$ *in* $(-r(T), r(T))$. *Assume that there exist* $\varepsilon > 0$, $p \in [1, +\infty]$, *and* $m \in \mathbb{Z}$ *such that*

$$\big(-r(T) - \varepsilon, r(T) + \varepsilon\big) \subset (a,b), \quad f \in L_m^{p,\mathrm{loc}}\big(r(T) - \varepsilon, r(T) + \varepsilon\big)$$

and

$$T \in L_{-m-1}^{q,\mathrm{loc}}\big(-r(T) - \varepsilon, -r(T) + \varepsilon\big),$$

where $p^{-1} + q^{-1} = 1$. *Then* $f = 0$ *on* (a,b).

(ii) *Let* $m \in \mathbb{Z}$, $p, q \in [1, +\infty)$, *and* $p^{-1} + q^{-1} > 1$. *Then there exist nonzero distributions*

$$T \in \big(\mathcal{E}'_\natural \cap L_{-m-1}^q\big)\big(\mathbb{R}^1\big) \quad \textit{and} \quad f \in \big(\mathcal{D}'_T \cap L_m^{p,\mathrm{loc}}\big)\big(\mathbb{R}^1\big)$$

such that $r(T) > 0$ *and* $f = 0$ *in* $(-r(T), r(T))$.

Proof. To prove (i), first observe that $\mathcal{Z}(\widehat{T}) \neq \varnothing$ (see (13.6) and the beginning of Sect. 8.1). Let $\lambda \in \mathcal{Z}(\widehat{T})$ and $\eta \in \{0, \dots, m(\lambda, T)\}$. In view of (8.22), (8.26), and (8.28), we deduce that

$$r\big(T^{\lambda,\eta}\big) = r(T), \quad T^{\lambda,\eta} \in L_{-m+\eta}^{q,\mathrm{loc}}\big(-r(T) - \varepsilon, -r(T) + \varepsilon\big),$$

and that the convolution $F = f * T^{\lambda,\eta}$ satisfies

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{i}\lambda\right)^{1+\eta} F = 0 \quad \text{in } \big(a + r(T), b - r(T)\big). \tag{13.8}$$

In particular, $F \in C^\infty(a + r(T), b - r(T))$. Let $s \in \{0, \dots, \eta\}$. We claim that

$$F^{(s)}(0) = 0.$$

First consider the case $m \leqslant 0$. By the hypothesis, there exists $g \in \mathcal{D}'(a,b)$ such that

$$g = 0 \quad \text{on } \big(-r(T), r(T)\big), \quad g \in L^{p,\mathrm{loc}}\big(r(T) - \varepsilon, r(T) + \varepsilon\big),$$

and $g^{(-m)} = f$. Then $F^{(s)} = g * (T^{\lambda,\eta})^{(s-m)}$, and the Hölder inequality yields

$$\big|F^{(s)}(t)\big| \leqslant \|g\|_{L^p[r(T), r(T)+t]} \big\|\big(T^{\lambda,\eta}\big)^{(s-m)}\big\|_{L^q[-r(T)-t, -r(T)]}$$

for each $t \in (0, \varepsilon)$. Letting t tend to zero, this gives $F^{(s)}(0) = 0$.

Now assume that $m > 0$. Then there exists $Q \in \mathcal{D}'(\mathbb{R}^1)$ such that

$$Q = 0 \quad \text{on } \big(r(T), +\infty\big), \qquad Q \in L^{q,\mathrm{loc}}_{\eta}\big(-r(T) - \varepsilon, -r(T) + \varepsilon\big),$$

and $Q^{(m)} = T^{\lambda,\eta}$. Define $f_1 \in \mathcal{D}'(-\infty, b)$ by letting $f_1 = f$ on $(0, b)$ and $f_1 = 0$ on $(-\infty, r(T))$. Then

$$F^{(s)}(t) = \big(f_1^{(m)} * Q^{(s)}\big)(t)$$

for $t \in (0, \varepsilon)$, and the previous argument shows that $F^{(s)}(0) = 0$. Relation (13.8) now implies that $F = 0$. To complete the proof of (i), we have only to apply Corollary 8.5.

Turning to (ii), let $\gamma, r > 0$ and

$$H_{\gamma,r}(t) = \begin{cases} \big(r^2 - t^2\big)^{-\gamma-1} & \text{if } t \in (-r, r), \\ 0 & \text{if } t \in \mathbb{R}^1 \backslash (-r, r). \end{cases}$$

Select $\alpha \in (1 - p^{-1}, q^{-1})$ and define $T \in \mathcal{E}'_{\natural}(\mathbb{R}^1)$ by letting $T = H_{\alpha+m,r}$ if $m < 0$ and $T = H^{(m+1)}_{\alpha-1,r}$ if $m \geqslant 0$. Then

$$T \in \big(\mathcal{E}'_{\natural} \cap L^{q}_{-m-1}\big)\big(\mathbb{R}^1\big) \quad \text{and} \quad r(T) = r.$$

Putting $f = \zeta_T$, we see from Proposition 8.20 and Theorem 8.6 that f is nonzero, $f \in (\mathcal{D}'_T \cap L^{p,\mathrm{loc}}_m)(\mathbb{R}^1)$, and $f = 0$ in $(-r(T), r(T))$. This gives (ii), proving the theorem. □

Corollary 13.2. *Assume that (13.7) is satisfied, let $f \in \mathcal{D}'_T(a, b)$ and $f = 0$ on $(-r(T), r(T))$. Suppose that for some $\varepsilon \in (0, \min\{|a|, |b|\} - r(T))$, at least one of the following assumptions holds:*

(1) *$f \in C^{\infty}(r(T) - \varepsilon, r(T) + \varepsilon)$;*
(2) *$T \in C^{\infty}(-r(T) - \varepsilon, -r(T) + \varepsilon)$.*

Then $f = 0$ on (a, b).

Proof. Since $T \in \mathcal{E}'(\mathbb{R}^1)$, we derive that T is a distribution of finite order on $\mathbb{R}^1$. It is easy to check that f has the same property on $(r(T) - \varepsilon, r(T) + \varepsilon)$. The required conclusion is now evident from Theorem 13.1. □

We now focus on the case $(a, b) = \mathbb{R}^1$.

Theorem 13.2. *Let $f \in \mathcal{D}'_T(\mathbb{R}^1)$, assume that $f = 0$ in $(-r(T), r(T))$, and let*

$$T = T_1 * T_2,$$

where $T_1, T_2 \in \mathcal{E}'(\mathbb{R}^1)$, $r(T_2) > 0$, and $\operatorname{supp} T_2 \subset [-r(T_2), r(T_2)]$.

(i) *If* $T_2 \in \mathrm{Inv}(\mathbb{R}^1)$ *and*

$$\frac{\operatorname{Im}\lambda}{\log(2+|\lambda|)} \to \infty \quad \textit{as}\ \lambda \to \infty,\ \lambda \in \mathcal{Z}(\widehat{T_2}), \tag{13.9}$$

then $f = 0$.

(ii) *If* $T_2 \in \mathfrak{E}(\mathbb{R}^1)$ *and* f *is of finite order, then* $f = 0$.

Since $T = \delta_0 * T$, assertions (i) and (ii) remain valid, provided that T_2 is replaced by T. We point out that the assumptions in Theorem 13.2 cannot be considerably relaxed in general (see Theorem 13.3(ii) and Corollary 13.7 below).

Proof of Theorem 13.2. Setting $F = f * T_1$, we infer that

$$F \in \mathcal{D}'_{T_2}\big(\mathbb{R}^1\big) \quad \text{and} \quad F = 0 \quad \text{on } \big(-r(T_2), r(T_2)\big). \tag{13.10}$$

Suppose that $T_2 \in \mathrm{Inv}(\mathbb{R}^1)$. Then (13.9) implies that $F \in C^\infty(\mathbb{R}^1)$ (see Hörmander [126], Theorem 16.6.5). Owing to Corollary 13.2, $F = 0$ in $\mathbb{R}^1$, whence $f \in \mathcal{D}'_{T_1}(\mathbb{R}^1)$. Since $r(T_2) > 0$, one sees that $r(T_1) < r(T)$ (see Theorem 6.2). This, together with Corollary 13.2 and Proposition 13.2, yields (i).

Assume now that $T_2 \in \mathfrak{E}(\mathbb{R}^1)$ and f is of finite order in $\mathbb{R}^1$. Then F is of finite order in $\mathbb{R}^1$, too. It follows by (13.10) that $F \in C^\infty(\mathbb{R}^1)$ (see Corollary 13.5 in Sect. 13.4). The rest of the proof now duplicates (i). □

Theorem 13.3.

(i) *Let* $T \notin (\mathrm{Inv}_+ \cap \mathrm{Inv}_-)(\mathbb{R}^1)$, $f \in \mathcal{D}'_T(\mathbb{R}^1)$, *and assume that* $f = 0$ *in* $(-r(T), r(T))$. *Then* $f = 0$ *in* $\mathbb{R}^1$.

(ii) *If* $T \in (\mathrm{Inv}_+ \cap \mathrm{Inv}_-)(\mathbb{R}^1)$, *then there exists nonzero* $f \in \mathcal{D}'_T(\mathbb{R}^1)$ *such that* $f = 0$ *in* $(-r(T), r(T))$.

(iii) *If* $T \in (\mathrm{Inv}_+ \cap \mathrm{Inv}_-)(\mathbb{R}^1)$, *then for each* $\varepsilon \in (0, r(T))$, *there exists nonzero* $f \in C^\infty_T(\mathbb{R}^1)$ *such that* $f = 0$ *in* $(-r(T)+\varepsilon, r(T)-\varepsilon)$.

(iv) *If* $r > 0$ *and* $\varepsilon \in (0, r)$, *then there exists nonzero* $T \in \mathcal{E}'(\mathbb{R}^1)$ *such that* $\operatorname{supp} T \subset [-r, r]$, $T \notin \mathrm{Inv}(\mathbb{R}^1)$, *and*

$$\big\{f \in C^\infty_T\big(\mathbb{R}^1\big) : f = 0 \textit{ on } (-r+\varepsilon, r-\varepsilon)\big\} \neq \{0\}. \tag{13.11}$$

Proof. In (i), we can write $f = f^+ + f^-$, where $f^+, f^- \in \mathcal{D}'(\mathbb{R}^1)$, $\operatorname{supp} f^+ \subset [r(T), +\infty)$, $\operatorname{supp} f^- \subset (-\infty, -r(T)]$. Then

$$f^+ * T = f^- * T = 0 \quad \text{on } \mathbb{R}^1 \setminus \{0\}. \tag{13.12}$$

If either $r(T) \notin \operatorname{supp} f^+$ or $-r(T) \notin \operatorname{supp} f^-$, then the desired statement follows by (13.12) and Corollary 13.2. Suppose now that $r(T) \in \operatorname{supp} f^+$ and $-r(T) \in \operatorname{supp} f^-$. Because of (13.12) and Corollary 6.2,

$$f^+ * T = p_+\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\delta_0 \quad \text{and} \quad f^- * T = p_-\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\delta_0 \tag{13.13}$$

for some nonzero polynomials p_+ and p_-. It is not difficult to verify that there exist $g^+, g^- \in \mathcal{D}'(\mathbb{R}^1)$ such that

$$p_+\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)g^+ = f^+, \qquad p_-\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)g^- = -f^-,$$

and

$$\operatorname{supp} g^+ \subset [r(T), +\infty), \qquad \operatorname{supp} g^- \subset (-\infty, r(T)]. \tag{13.14}$$

Relations (13.13) and (13.14) yield

$$g^+ * T = -g^- * T = \delta_0.$$

Thus, $T \in (\mathrm{Inv}_+ \cap \mathrm{Inv}_-)(\mathbb{R}^1)$, which contradicts the hypothesis and so proves (i).

As for (ii), it is enough to put $f = \zeta_T$ (see Proposition 8.20). Part (iii) follows from (ii) by regularization. Turning to (iv), let $T = T_1 * w_1$, where

$$T_1 \in (\mathrm{Inv}_+ \cap \mathrm{Inv}_-)\big(\mathbb{R}^1\big), \quad \operatorname{supp} T_1 = [-r + \varepsilon/2, r - \varepsilon/2], \quad w_1 \in \mathcal{D}\big(\mathbb{R}^1\big),$$

and $\operatorname{supp} w_1 = [-\varepsilon/2, \varepsilon/2]$. Then $T \in \mathcal{D}(\mathbb{R}^1)$, $T \neq 0$, $\operatorname{supp} T \subset [-r, r]$, and estimate (6.34) yields $T \notin \mathrm{Inv}(\mathbb{R}^1)$. In addition, for some $w_2 \in \mathcal{D}(\mathbb{R}^1)$ with $\operatorname{supp} w_2 \subset [-\varepsilon/2, \varepsilon/2]$, the example $f = \zeta_{T_1} * w_2$ shows that (13.11) holds (see Proposition 8.20). Hence the theorem. □

The next result enables us to supplement Theorem 13.3(ii), (iii).

Theorem 13.4. *Let $T \in (\mathrm{Inv}_+ \cap \mathrm{Inv}_-)(\mathbb{R}^1)$ and assume that (13.7) is fulfilled. Then the following statements are valid.*

(i) *If $f \in \mathcal{D}'_T(a, b)$ and $r \in (0, r(T))$, then in order that $f = 0$ on $(-r, r)$, it is necessary and sufficient that*

$$f = \zeta_T * u \quad \text{in } (a, b) \tag{13.15}$$

for some $u \in \mathcal{E}'(\mathbb{R}^1)$ with $\operatorname{supp} u \subset [r - r(T), r(T) - r]$.

(ii) *Let T be a distribution of order l, suppose that $m \in \mathbb{Z}_+$, and let $f \in C_T^{m+l}(a, b)$. Then $f^{(s)}(0) = 0$ for each $s \in \{0, \dots, m\}$ if and only if relation (13.15) holds for some $u \in C^l(\mathbb{R}^1)$ with $\operatorname{supp} u \subset [-r(T), r(T)]$.*

Proof. To prove (i), first suppose that $f \in \mathcal{D}'_T(a, b)$ and $f = 0$ on $(-r, r)$ for some $r \in (0, r(T))$. We define $f^+ \in \mathcal{D}'(-\infty, b)$ and $f^- \in \mathcal{D}'(a, +\infty)$ so that

$$\operatorname{supp} f^+ \subset [r, b), \quad \operatorname{supp} f^- \subset (a, -r], \quad \text{and} \quad f^+ + f^- = f \quad \text{in } (a, b).$$

Let $u_1 = f^+ * T$ and $u_2 = f^- * T$. Then $\operatorname{supp} u_1 \subset [r - r(T), \rho_1]$ and $\operatorname{supp} u_2 \subset [\rho_2, r(T) - r]$, where

$$\rho_1 = \min\{r(T) - r, b - r(T)\}, \qquad \rho_2 = \max\{a + r(T), r - r(T)\}.$$

In addition, $u_1 = -u_2$ on $(a + r(T), b - r(T))$. Hence, there exists $u \in \mathcal{E}'(\mathbb{R}^1)$ such that $u = u_1$ on $(-\infty, b - r(T))$ and $u = -u_2$ on $(a + r(T), +\infty)$. Using relations (8.89) and (8.90), one has

$$f^+ = f^+ * \delta_0 = u * \zeta_T^+$$

and

$$f^- = f^- * \delta_0 = u * \zeta_T^-.$$

Now (13.15) follows from (8.92) and the definitions of f^+, f^-, u.

Conversely, if $u \in \mathcal{E}'(\mathbb{R}^1)$ and $\operatorname{supp} u \subset [r - r(T), r(T) - r]$, then $\zeta_T * u \in \mathcal{D}'_T(\mathbb{R}^1)$ and $\zeta_T * u = 0$ on $(-r, r)$. This completes the proof of (i).

Part (ii) is proved in the same way as (i). This concludes the proof. □

Remark 13.1. Let $T \in (\mathrm{Inv}_+ \cap \mathrm{Inv}_- \cap \mathcal{E}'_\natural)(\mathbb{R}^1)$, $u \in \mathcal{E}'(\mathbb{R}^1)$, $\operatorname{supp} u \subset [-r(u), r(u)]$, and suppose that

$$\zeta_T * u \in \mathcal{D}'_\natural(-a, a)$$

for some $a > 0$. According to Proposition 8.20 (ii), we can write

$$\zeta_T * u = \zeta'_T * v \quad \text{in } (-a, a),$$

where $v \in \mathcal{E}'_\natural(\mathbb{R}^1)$ is defined by

$$v'(t) = \frac{1}{2}\big(u(t) - u(-t)\big).$$

To continue, let E be a nonempty open subset of $\mathbb{R}^1$. Then E can be represented as

$$E = \bigcup_{k=1}^{\infty} (\alpha_k, \beta_k),$$

where $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots$ is a collection of intervals in $\mathbb{R}^1$ (it is possible that some of these intervals have common points or coincide). We set

$$t_k = \frac{1}{2}(\alpha_k + \beta_k), \qquad r_k = \frac{1}{2}(\beta_k - \alpha_k).$$

Suppose that

$$-\infty \leqslant a < t_k - r(T) \quad \text{and} \quad t_k + r(T) < b \leqslant +\infty$$

for all k. Now define

$$\mathcal{N}_T\big((a, b), E\big) = \big\{f \in \mathcal{D}'_T(a, b) \colon f|_E = 0\big\}.$$

Corollary 13.2 shows that

$$\mathcal{N}_T\big((a, b), E\big) = \{0\}$$

if $r_k > r(T)$ for some $k \in \mathbb{N}$.

Theorem 13.5. *Let $T \in (\mathrm{Inv}_+ \cap \mathrm{Inv}_-)(\mathbb{R}^1)$ and assume that $r_k \in (0, r(T)]$ for all $k \in \mathbb{N}$. Then $\mathcal{N}_T((a,b), E) \neq \{0\}$ if and only if for each $k \in \mathbb{N}$, there exists nonzero $u_k \in \mathcal{E}'(\mathbb{R}^1)$ with the following properties:*

(1) $\operatorname{supp} u_k \subset [r_k - r(T), r(T) - r_k]$;
(2) *For all $k, l \in \mathbb{N}$, there exists $\psi_{k,l} \in \mathcal{E}'(\mathbb{R}^1)$ such that*

$$u_k(t - t_k) - u_l(t - t_l) = (T * \psi_{k,l})(t) \quad \text{in } \mathbb{R}^1.$$

Moreover, if $\mathcal{N}_T((a,b), E) \neq \{0\}$ and $f \in \mathcal{D}'(a,b)$ is nonzero, then in order that $f \in \mathcal{N}_T((a,b), E)$ it is necessary and sufficient that

$$f = \zeta_T * v_k \quad \text{in } (a,b)$$

for each $k \in \mathbb{N}$, where

$$v_k(t) = u_k(t - t_k).$$

Proof. First assume that $f \in \mathcal{N}_T((a,b), E)$ and let

$$f_k(t) = f(t + t_k).$$

Then $f_k \in \mathcal{D}'_T(a - t_k, b - t_k)$ and $f_k = 0$ in $(-r_k, r_k)$. By Theorem 13.4(i) there exists $u_k \in \mathcal{E}'(\mathbb{R}^1)$ with $\operatorname{supp} u_k \subset [r_k - r(T), r(T) - r_k]$ such that

$$f_k = \zeta_T * u_k \quad \text{in } (a - t_k, b - t_k).$$

Theorem 13.4(i) and Theorem 6.2 imply that u_k is determined uniquely by f_k. Thus, the required statements follow from the definitions of f_k and Theorem 13.4(i). □

We are now able to prove the following result.

Theorem 13.6. *Let $T \in (\mathrm{Inv}_+ \cap \mathrm{Inv}_-)(\mathbb{R}^1)$, and let*

$$E = \bigcup_{k=1}^{3} \big(t_k - r(T), t_k + r(T)\big),$$

where $t_1 < t_2 < t_3$ and $(t_1 - t_2)/(t_2 - t_3) \notin \mathbf{Q}$. Assume that $-\infty \leqslant a < t_1 - r(T)$ and $t_3 + r(T) < b \leqslant +\infty$. Then

$$\mathcal{N}_T\big((a,b), E\big) = \{0\}.$$

Proof. Let us assume the opposite. Then Theorem 13.5 ensures us that there exist nonzero $u_1, u_2, u_3 \in \mathcal{E}'(\mathbb{R}^1)$ such that $\operatorname{supp} u_k = \{0\}$ and

$$\widehat{u}_k(z)\mathrm{e}^{-\mathrm{i}zt_k} - \widehat{u}_l(z)\mathrm{e}^{-\mathrm{i}zt_l} = \widehat{T}(z)\widehat{\psi}_{k,l}(z) \tag{13.16}$$

for all $k, l \in \{1,2,3\}$, where $\psi_{k,l} \in \mathcal{E}'(\mathbb{R}^1)$. By Corollary 6.2 we conclude that $\widehat{u}_1, \widehat{u}_2, \widehat{u}_3$ are polynomials. Since $(t_1 - t_2)/(t_2 - t_3) \notin \mathbf{Q}$, relation (13.16) leads to the conclusion that there exists a nonzero function w of the form

$$w(z) = p_1(z) + p_2(z)e^{i\alpha z}$$

such that p_1 and p_2 are polynomials, $\alpha \in (-r(T), r(T))$, and the function $w(z)/\widehat{T}(z)$ is entire. This, however, contradicts Corollary 6.3, and the theorem is proved. □

The condition $(t_1 - t_2)/(t_2 - t_3) \notin \mathbf{Q}$ in the previous theorem is necessary as, for example, the following statement shows.

Theorem 13.7. *Let $r > 0$ and $a \geqslant 2r$. Then there exists even $T \in \mathfrak{N}(\mathbb{R}^1)$ with the following properties:*

(1) $r(T) = r$;
(2) *All the zeros of $\widehat{T}$ are real and simple;*
(3) $\mathcal{N}_T(\mathbb{R}^1, E) \neq \{0\}$, *where* $E = \bigcup_{m\in\mathbb{Z}}(am - r, am + r)$.

Proof. This is immediate from Theorem 8.7 and Proposition 8.20(i). □

To conclude we complement Theorem 13.6 by the following result.

Theorem 13.8. *Let $T \in (\mathrm{Inv}_+ \cap \mathrm{Inv}_-)(\mathbb{R}^1)$ and assume that the set $\mathcal{Z}(\widehat{T}) \cap \mathbb{R}^1$ is infinite. Let*

$$E = \big(t_1 - r(T), t_1 + r(T)\big) \cup \big(t_2 - r(T), t_2 + r(T)\big)$$

and suppose $\mathcal{N}_T((a, b), E) \neq \{0\}$ for some $a, b \in \mathbb{R}^1$ such that $E \subset (a, b)$. Then there exist $\gamma \in \mathbb{R}^1$ and $R > 0$ such that the function

$$\big(e^{i(\gamma+(t_1-t_2)z)} - 1\big)\big/\widehat{T}(z)$$

is holomorphic in $\{z \in \mathbb{C} : |z| > R\}$.

Proof. By the hypothesis and Theorem 13.7 one sees that for some polynomials p_1 and p_2, the function

$$w(z) = \big(p_1(z)e^{-it_1 z} - p_2(z)e^{-it_2 z}\big)\big/\widehat{T}(z)$$

is entire and nonzero (see the beginning of the proof of Theorem 13.6). If $\alpha = t_1 - t_2$ and $\lambda \in \mathcal{Z}(\widehat{T}) \cap \mathbb{R}^1$, we obtain

$$p_1(\lambda) = p_2(\lambda)e^{i\alpha\lambda}.$$

Hence, $|p_1(\lambda)|^2 = |p_2(\lambda)|^2$, and by assumption on $\mathcal{Z}(\widehat{T}) \cap \mathbb{R}^1$ one has

$$|p_1(t)|^2 = |p_2(t)|^2 \quad \text{for all } t \in \mathbb{R}^1.$$

This implies that $\mathcal{Z}(p_1) = \mathcal{Z}(p_2)$ and $n_\mu(p_1) = n_\mu(p_2)$ for each $\mu \in \mathcal{Z}(p_1)$. Thus, there exists $\gamma \in \mathbb{R}^1$ such that $p_2(z) = p_1(z)\mathrm{e}^{\mathrm{i}\gamma}$ for all $z \in \mathbb{C}$. Since w is entire, this completes the proof. □

13.3 Nonharmonic Fourier Series

Throughout this section we assume that

$$T \in \mathcal{E}'(\mathbb{R}^1), \quad T \neq 0, \quad \operatorname{supp} T \subset [-r(T), r(T)],$$

and that (13.3) holds. Let $\lambda \in \mathcal{Z}(\widehat{T})$, $\eta \in \{0, \dots, m(\lambda, T)\}$ and $f \in \mathcal{D}'_T(a, b)$. By Proposition 13.1(iii),

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{i}\lambda\right)^{m(\lambda,T)+1} \big(f * T_{\lambda,0}\big) = 0 \quad \text{in } \big(a + r(T), b - r(T)\big).$$

Hence, there exist constants $c_{\lambda,\eta}(T, f) \in \mathbb{C}$ such that

$$f * T_{\lambda,0} = \sum_{\eta=0}^{m(\lambda,T)} c_{\lambda,\eta}(T, f)\mathrm{e}^{\lambda,\eta} \quad \text{in } \big(a + r(T), b - r(T)\big). \tag{13.17}$$

In this section we assemble basic properties of the coefficients $c_{\lambda,\eta}(T, f)$.

Proposition 13.4.

(i) *If* $(c, d) \subset (a, b)$ *and* $d - c > 2r(T)$, *then*

$$c_{\lambda,\eta}(T, f|_{(c,d)}) = c_{\lambda,\eta}(T, f) \tag{13.18}$$

for each $f \in \mathcal{D}'_T(a, b)$.

(ii) *If* $f_1, \dots, f_k \in \mathcal{D}'_T(a, b)$ *and* $\gamma_1, \dots, \gamma_k \in \mathbb{C}$, *then*

$$c_{\lambda,\eta}\left(T, \sum_{\nu=1}^{k} \gamma_\nu f_\nu\right) = \sum_{\nu=1}^{k} \gamma_\nu c_{\lambda,\eta}(T, f_\nu). \tag{13.19}$$

(iii) *If* $f \in \mathcal{D}'_T(a, b)$ *and* $\gamma \in \mathbb{R}^1$, *then*

$$c_{\lambda,\eta}(T, f) = \sum_{\nu=\eta}^{m(\lambda,T)} c_{\lambda,\eta}\big(T, f(\cdot - \gamma)\big)\binom{\nu}{\eta}(\mathrm{i}\gamma)^{\nu-\eta}\mathrm{e}^{\mathrm{i}\lambda\gamma}. \tag{13.20}$$

The proof follows immediately from the definition of $c_{\lambda,\eta}(T, f)$.

Proposition 13.5.

(i) *If* $\nu \in \{0, \dots, m(\lambda, T)\}$ *and* $f \in \mathcal{D}'_T(a, b)$, *then*

$$f * T_{\lambda,\nu} = \sum_{\mu=0}^{m(\lambda,T)-\nu} \binom{\nu+\mu}{\nu} c_{\lambda,\nu+\mu}(T, f) e^{\lambda,\mu} \quad \textit{in } \big(a + r(T), b - r(T)\big). \tag{13.21}$$

(ii) *Let* $f_k \in \mathcal{D}'_T(a, b)$, $k = 1, 2, \dots$, *and suppose that* $f_k \to f$ *in* $\mathcal{D}'(a, b)$ *as* $k \to \infty$. *Then*

$$c_{\lambda,\eta}(T, f_k) \to c_{\lambda,\eta}(T, f) \quad \textit{as } k \to \infty.$$

(iii) *Let* $f \in \mathcal{D}'_T(a, b)$, $u \in \mathcal{E}'(\mathbb{R}^1)$, *and let* $2(r(u) + r(T)) < b - a$. *Then*

$$c_{\lambda,\eta}(T, f * u) = \sum_{\nu=\eta}^{m(\lambda,T)} c_{\lambda,\nu}(T, f) \binom{\nu}{\eta} \widehat{u}^{(\nu-\eta)}(\lambda).$$

In particular, for each polynomial p,

$$c_{\lambda,\eta}\left(T, p\left(\frac{d}{dt}\right) f\right) = \sum_{\nu=\eta}^{m(\lambda,T)} c_{\lambda,\nu}(T, f) \binom{\nu}{\eta} i^{\nu-\eta} p^{(\nu-\eta)}(i\lambda). \tag{13.22}$$

Proof. Part (i) follows by induction on ν (see Proposition 13.1(iii) and (13.17)). Using now (i) repeatedly for $\nu = m(\lambda, T), \dots, 0$, we arrive at (ii). Part (iii) is a consequence of (13.17) and Proposition 13.1(ii). □

Proposition 13.6.

(i) *If* $f = e^{\mu,\nu}$, *where* $\mu \in \mathcal{Z}(\widehat{T})$ *and* $\nu \in \{0, \dots, m(\mu, T)\}$, *then* $f \in \mathcal{D}'_T(\mathbb{R}^1)$ *and*

$$c_{\lambda,\eta}(T, f) = \delta_{\lambda,\mu}\delta_{\eta,\nu}. \tag{13.23}$$

(ii) *If* $f \in \mathcal{D}'_T(a, b)$ *and* $c_{\lambda,\eta}(T, f) = 0$ *for all* $\lambda \in \mathcal{Z}(\widehat{T})$ *and* $\eta \in \{0, \dots, m(\lambda, T)\}$, *then* $f = 0$.

(iii) *If* $[-r(T), r(T)] \subset (a, b)$, $f \in C^k_T(a, b)$, $k = \max\{0, \operatorname{ord} T - 1\}$, *then*

$$c_{\lambda,\eta}(T, f) = \big\langle T_{\lambda,\eta}, f(-\cdot)\big\rangle. \tag{13.24}$$

Proof. Relation (13.23) follows from (13.17), (13.5), and (8.24). Part (ii) is obvious from Proposition 13.5(i) and Remark 8.1. Next, using Proposition 8.5(iii), Corollary 8.1, and equality (13.21), we obtain (iii). □

Proposition 13.7. *Let* $f \in \mathcal{D}'_T(a, b)$ *and assume that*

$$T = \left(\frac{d}{dt} - i\lambda_1\right)^{s_1} \dots \left(\frac{d}{dt} - i\lambda_l\right)^{s_l} Q, \tag{13.25}$$

where $\{\lambda_1, \dots, \lambda_l\}$ *is a set of distinct complex numbers,* $s_1, \dots, s_l \in \mathbb{N}$, *and* $Q \in \mathcal{E}'(\mathbb{R}^1)$. *Then*

$$\operatorname{supp} Q \subset [-r(T), r(T)], \qquad s_j \leqslant n_{\lambda_j}(\widehat{T}),$$

for all $j \in \{1, \dots, l\}$, *and the distribution*

$$g = f - \sum_{j=1}^{l} \sum_{\eta=m(\lambda_j,T)+1-s_j}^{m(\lambda_j,T)} c_{\lambda_j,\eta}(T, f)\mathrm{e}^{\lambda_j,\eta}$$

is in the class $\mathcal{D}'_Q(a, b)$. *In addition, if* $\lambda \in \mathcal{Z}(\widehat{Q})$, *then* $n_\lambda(\widehat{Q}) \leqslant n_\lambda(\widehat{T})$ *and*

$$c_{\lambda,\eta}(Q, g) = c_{\lambda,\eta}(T, f) \tag{13.26}$$

for all $\eta \in \{0, \dots, m(\lambda, Q)\}$.

Proof. We can assume, without loss of generality, that $l = s_1 = 1$. The general case reduces to this one by iteration. Formula (13.25) now yields

$$\widehat{T}(z) = \mathrm{i}(z - \lambda_1)\widehat{Q}(z).$$

Hence, $s_1 \leqslant n_{\lambda_1}(\widehat{T})$ and $n_\lambda(\widehat{Q}) \leqslant n_\lambda(\widehat{T})$ for all $\lambda \in \mathbb{C}$. In addition, (13.25) implies that $\operatorname{supp} Q \subset [-r(T), r(T)]$. The definition of $T_{\lambda_1,m(\lambda_1,T)}$ shows that $Q = \gamma T_{\lambda_1,m(\lambda_1,T)}$, where $\gamma = -\mathrm{i}(a^{\lambda_1,m(\lambda_1,T)}_{m(\lambda_1,T)})^{-1}$. By (13.21),

$$(g * Q)(t) = \gamma\big(c_{\lambda_1,m(\lambda_1,T)}(T, f)\mathrm{e}^{\mathrm{i}\lambda t} - c_{\lambda_1,m(\lambda_1,T)}(T, f)(u * T_{\lambda_1,m(\lambda_1,T)})(t)\big),$$

where

$$u(t) = (\mathrm{i}t)^{m(\lambda_1,T)}\mathrm{e}^{\mathrm{i}\lambda t}.$$

Owing to (13.5) and (8.24), we conclude that $g \in \mathcal{D}'_Q(a, b)$. Equality (13.26) now follows from Proposition 8.8 and (13.17). □

To every $f \in \mathcal{D}'_T(a, b)$ we now assign the generalized nonharmonic Fourier series

$$f \sim \sum_{\lambda \in \mathcal{Z}(\widehat{T})} \sum_{\eta=0}^{m(\lambda,T)} c_{\lambda,\eta}(T, f)\mathrm{e}^{\lambda,\eta}. \tag{13.27}$$

Theorem 13.9.

(i) *If* $f \in \mathcal{D}'_T(a, b)$ *and the series in* (13.27) *converges in* $\mathcal{D}'(a, b)$, *then its sum coincides with* f.

(ii) *Let* $f \in \mathcal{D}'(a, b)$ *and assume that*

$$f = \sum_{\lambda \in \mathcal{Z}(\widehat{T})} \sum_{\eta=0}^{m(\lambda,T)} \gamma_{\lambda,\eta}\mathrm{e}^{\lambda,\eta},$$

where $\gamma_{\lambda,\eta} \in \mathbb{C}$, *and the series converges in* $\mathcal{D}'(a, b)$. *Then* $f \in \mathcal{D}'_T(a, b)$ *and* $\gamma_{\lambda,\eta} = c_{\lambda,\eta}(T, f)$.

Proof. To prove (i), suppose that the series in (13.27) converges to $g \in \mathcal{D}'(a,b)$. In view of (13.5), $g \in \mathcal{D}'_T(a,b)$. Then it follows by (13.19), Proposition 13.6(i), and Proposition 13.5(ii) that

$$c_{\lambda,\eta}(T, f-g) = 0$$

for all λ, η. This, together with Proposition 13.6(ii), gives (i). The proof of (ii) is quite similar to that of (i). □

We shall now obtain some upper estimates for the constants $c_{\lambda,\eta}(T, f)$.

Theorem 13.10. *Let $r(T) > 0$. Then the following assertions hold.*

(i) *Let $f \in \mathcal{D}'_T(a,b)$, and let p be a nonzero polynomial. Then there exist $\gamma_1, \gamma_2 > 0$ independent of f such that for all $\lambda \in \mathcal{Z}(\widehat{T})$, $|\lambda| > \gamma_1$, the following estimate holds:*

$$\max_{0\leqslant\eta\leqslant m(\lambda,T)} |c_{\lambda,\eta}(T,f)| \leqslant \frac{\gamma_2}{|p(\mathrm{i}\lambda)|} \max_{0\leqslant\eta\leqslant m(\lambda,T)} \left| c_{\lambda,\eta}\left(T, p\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) f\right)\right|.$$

(ii) *Let $k \in \mathbb{Z}_+$, $f \in C^k_T(a,b)$, and*

$$[a',b'] \subset (a,b), \qquad b'-a' = 2r(T). \tag{13.28}$$

Then there exist $\gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7 > 0$ independent of f, k such that for all $\lambda \in \mathcal{Z}(\widehat{T})$, $|\lambda| > \gamma_3$,

$$\begin{aligned}
&\max_{0\leqslant\eta\leqslant m(\lambda,T)} |c_{\lambda,\eta}(T,f)| \\
&\quad\leqslant \gamma_4^{k+1}\gamma_5^{m(\lambda,T)}|\lambda|^{\gamma_6-k}\sigma_\lambda\big(\widehat{T}\big)\left(\int_{a'}^{b'} \left|f^{(k)}(t)\right| \mathrm{d}t + \gamma_7^k\gamma_8\right) \\
&\qquad\times \exp\left(\frac{1}{2}(a'+b')\operatorname{Im}\lambda\right),
\end{aligned}$$

where $\gamma_8 > 0$ is independent of k, λ. In addition, if $a' = -r(T)$ and $b' = r(T)$, then the same is true with $\gamma_5 = 1$.

(iii) *Let $f \in \mathcal{D}'_T(\mathbb{R}^1)$ and assume that f is of order $q < +\infty$ in $\mathbb{R}^1$. Then for each $\alpha \geqslant 1$,*

$$|c_{\lambda,\eta}(T,f)| \leqslant \gamma_9\sigma_\lambda\big(\widehat{T}\big)(1+|\lambda|)^{d_T+q+1}(\alpha\mathrm{e})^{m(\lambda,T)}m(\lambda,T)!\mathrm{e}^{-\alpha|\operatorname{Im}\lambda|},$$

where $\gamma_9 > 0$ is independent of λ, η.

Proof. To prove (i) it is enough to consider the case where p is a polynomial of degree one. In this case relation (13.22) can be written as

$$c_{\lambda,\eta}(T,f) = \frac{1}{p(\mathrm{i}\lambda)}\left(c_{\lambda,\eta}\left(T, p\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)f\right) - (\eta+1)p'(\mathrm{i}\lambda)c_{\lambda,\eta+1}(T,f)\right), \tag{13.29}$$

provided that $m(\lambda, T) > 0$ and $\eta < m(\lambda, T)$. Let $\varepsilon \in (0, 1)$. If $|\lambda|$ is large enough, relation (13.29) yields

$$|c_{\lambda,\eta}(T, f)| \leqslant \frac{1}{|p(\mathrm{i}\lambda)|}\left|c_{\lambda,\eta}\left(T, p\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)f\right)\right| + \varepsilon|c_{\lambda,\eta+1}(T, f)| \tag{13.30}$$

(see (8.4)). In addition, (13.22) implies that

$$c_{\lambda,m(\lambda,T)}(T, f) = \frac{1}{p(\mathrm{i}\lambda)}c_{\lambda,m(\lambda,T)}\left(T, p\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)f\right).$$

This, together with (13.30), proves (i).

Turning to (ii), select a polynomial p such that the function $\widehat{T}/p$ is entire, $\mathcal{Z}(\widehat{T}/p) \cap \mathcal{Z}(p) = \varnothing$, and

$$T = p\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right)Q \tag{13.31}$$

for some $Q \in (\mathcal{E}' \cap C)(\mathbb{R}^1)$. In view of Proposition 13.7, for all $\lambda \in \mathcal{Z}(\widehat{T}) \setminus \mathcal{Z}(p)$ and $\eta \in \{0, \dots, m(\lambda, T)\}$,

$$c_{\lambda,\eta}(T, f) = c_{\lambda,\eta}(Q, f - h), \tag{13.32}$$

where

$$h = \sum_{\lambda\in\mathcal{Z}(p)} \sum_{\eta=0}^{m(\lambda,T)} c_{\lambda,\eta}(T, f)\mathrm{e}^{\lambda,\eta}. \tag{13.33}$$

Let $g(t) = f(t + c') - h(t + c')$, where $c' = (a' + b')/2$. Using (13.24), (8.44), and Proposition 6.6(iv), one has

$$|c_{\lambda,\eta}(Q, g)| \leqslant \gamma_{10}(2 + |\lambda|)^{\gamma_{11}}\sigma_\lambda\big(\widehat{T}\big) \int_{-r(T)}^{r(T)} |g(t)|\,\mathrm{d}t,$$

where $\lambda \in \mathcal{Z}(\widehat{T}) \setminus \mathcal{Z}(p)$ and $\eta \in \{0, \dots, m(\lambda, T)\}$, and $\gamma_{10}, \gamma_{11} > 0$ are independent of λ, η, f. We now see that (ii) must be true as a consequence of (13.32), (13.33), (13.20), and (i).

Let us pass to (iii). As in the proof of (ii), we define $Q \in (\mathcal{E}' \cap C^q)(\mathbb{R}^1)$ by (13.31), where p is a polynomial of degree $d_T + q + 1$ such that the function $\widehat{T}/p$ is entire and $\mathcal{Z}(\widehat{T}/p) \cap \mathcal{Z}(p) = \varnothing$. Let $\lambda \in \mathcal{Z}(\widehat{T})\setminus\mathcal{Z}(p)$. Then $\lambda \in \mathcal{Z}(\widehat{Q})$, $m(\lambda, T) = m(\lambda, Q)$, and for all $\eta \in \{0, \dots, m(\lambda, T)\}$, relations (13.32) and (13.33) hold.

By assumption on f and (8.44),

$$\big|\big((f - h) * Q_{\lambda,\eta}\big)(t)\big| \leqslant \gamma_{12}\sigma_\lambda\big(\widehat{Q}\big), \quad t \in \mathbb{R}^1,$$

where $\gamma_{12} > 0$ is independent of λ, η. Combining this with (13.21), we obtain

$$|c_{\lambda,m(\lambda,Q)}(Q, f - h)| \leqslant \gamma_{12}\sigma_\lambda\big(\widehat{Q}\big)\mathrm{e}^{-|t\,\mathrm{Im}\,\lambda|}. \tag{13.34}$$

In addition, if $m(\lambda, Q) > 1$ and $\nu \in \{0, \dots, m(\lambda, Q) - 1\}$, then

$$|c_{\lambda,\nu}(Q, f-h)| \leqslant \gamma_{12}\sigma_\lambda\big(\widehat{Q}\big)e^{-|t\,\mathrm{Im}\,\lambda|} + \sum_{\mu=1}^{m(\lambda,Q)-\nu} \binom{\mu+\nu}{\nu} |t|^\mu |c_{\lambda,\nu+\mu}(Q, f-h)|. \tag{13.35}$$

By induction on ν we find from (13.34) and (13.35) that

$$|c_{\lambda,\nu}(Q, f-h)| \leqslant \frac{\gamma_{11}\sigma_\lambda(\widehat{Q})m(\lambda, Q)!(te)^{m(\lambda,Q)-\nu+1}}{\nu! e^{t|\mathrm{Im}\,\lambda|}}$$

for all $t \geqslant 1$ and $\nu \in \{0, \dots, n(\lambda, T)\}$. Now in order to complete the proof, one needs only (13.32) and Proposition 6.6(iv). □

Corollary 13.3. *Assume that (13.28) holds. Then the following statements are valid.*

(i) *Let $f \in \mathcal{D}'_T(a, b)$. Then*

$$|c_{\lambda,\eta}(T, f)| \leqslant (2 + |\lambda|)^{\gamma_1}\gamma_2^{m(\lambda,T)}\sigma_\lambda\big(\widehat{T}\big)\exp\bigg(\frac{1}{2}(a' + b')\,\mathrm{Im}\,\lambda\bigg),$$

where $\gamma_1, \gamma_2 > 0$ are independent of λ, η. If $[-r(T), r(T)] \subset (a, b)$, then the same is true with $\gamma_2 = 1$. In particular, if $T \in \mathfrak{M}(\mathbb{R}^1)$, then

$$|c_{\lambda,\eta}(T, f)| \leqslant (2 + |\lambda|)^{\gamma_3},$$

where $\gamma_3 > 0$ is independent of λ, η.

(ii) *If $T \in \mathfrak{M}(\mathbb{R}^1)$ and $f \in C_T^\infty(a, b)$, then for each $\alpha > 0$,*

$$|c_{\lambda,\eta}(T, f)| \leqslant \gamma_4(2 + |\lambda|)^{-\alpha},$$

where $\gamma_4 > 0$ is independent of λ, η.

(iii) *If $\alpha > 0$, $T \in \mathfrak{G}_\alpha(\mathbb{R}^1)$, $r(T) > 0$, and $f \in C_T^\infty(a, b) \cap G^\alpha[a', b']$, then*

$$|c_{\lambda,\eta}(T, f)| \leqslant \gamma_5 \exp\big(-\gamma_6|\lambda|^{1/\alpha}\big),$$

where $\gamma_5, \gamma_6 > 0$ is independent of λ, η.

(iv) *If $T \in \mathfrak{M}(\mathbb{R}^1)$, $r(T) > 0$ and $f \in C_T^\infty(a, b) \cap \mathrm{QA}[a', b']$, then*

$$\max_{0\leqslant\eta\leqslant m(\lambda,T)} |c_{\lambda,\eta}(T, f)| \leqslant M_q(1 + |\lambda|)^{-q} \quad \text{for all } \lambda \in \mathcal{Z}\big(\widehat{T}\big),\ q \in \mathbb{N},$$

where the constants $M_q > 0$ are independent of λ, and

$$\sum_{\nu=1}^{\infty} \frac{1}{\inf_{q\geqslant\nu} M_q^{1/q}} = +\infty. \tag{13.36}$$

(v) *Let* $T \in \mathfrak{E}(\mathbb{R}^1)$, $f \in \mathcal{D}'_T(\mathbb{R}^1)$, *and suppose that* f *is of finite order in* $\mathbb{R}^1$. *Then for each* $\alpha > 0$,

$$|c_{\lambda,\eta}(T, f)| \leqslant \gamma_7 e^{-\alpha|\operatorname{Im}\lambda|},$$

where $\gamma_7 > 0$ *is independent of* λ, η.

Proof. We suppose that $r(T) > 0$, otherwise, there is nothing to prove (see Proposition 13.2). To show (i), let $(c, d) \subset (a, b)$ and $d - c > 2r(T)$. By Corollary 13.1 there exists a polynomial q such that

$$f = q\left(\frac{d}{dt}\right)F$$

for some $F \in C_T(c, d)$. Taking (13.18) into account, we see from (13.22) that

$$c_{\lambda,\eta}(T, f) = \sum_{\nu=\eta}^{m(\lambda,T)} c_{\lambda,\nu}(T, F)\binom{\nu}{\eta} i^{\nu-\eta} q^{(\nu-\eta)}(i\lambda).$$

Using now Theorem 13.10(ii) and (8.4), we arrive at (i).

Part (ii) follows immediately from Theorem 13.10(ii). For (iii), it is enough to apply Theorem 13.10(ii) with

$$k = \left[\gamma|\lambda|^{1/\alpha}\right], \quad \gamma \in (0, 1), \ \lambda \in \mathcal{Z}\big(\widehat{T}\big),$$

where $|\lambda|$ is sufficiently large. Assertion (iv) can easily be deduced from Theorem 13.10(ii) and Lemma 8.1(i). Finally, part (v) is clear from Theorem 13.10(iii). □

As another application of Theorem 13.10, we now give the following uniqueness result.

Theorem 13.11. *Let* $r(T) > 0$, $f \in C_T^\infty(a, b)$, *and suppose that* (13.28) *holds. Assume that there exist* $\alpha > 0$, $\beta \geqslant 0$ *such that*

$$\liminf_{q\to+\infty} \alpha^{-q} q^{-\beta} \int_{a'}^{b'} \left|f^{(q)}(t)\right| dt = 0. \tag{13.37}$$

Then $c_{\lambda,\eta}(T, f) = 0$, *provided that* $|\lambda| > \alpha$, *and the same is true if* $|\lambda| = \alpha$, $\eta \geqslant \beta$. *In particular, if*

$$\alpha \leqslant \min_{\lambda\in\mathcal{Z}(\widehat{T})} |\lambda| \quad \textit{and} \quad \beta = 0,$$

then $f = 0$.

The example $f = e^{\lambda,\eta}$ shows that this result cannot be considerably sharpened (see also (8.71)).

Proof of Theorem 13.11. It follows by Theorem 13.10(ii) and (13.37) that

$$c_{\lambda,\eta}(T, f) = 0$$

if $|\lambda|$ is large enough. This, together with Theorem 13.9(i), (8.71), and (13.37), leads to the desired conclusion. □

Corollary 13.4. *Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function, let $f\big|_{\mathbb{R}^1} \in C_T^\infty(\mathbb{R}^1)$, and assume that*

$$|f(z)| \leqslant \gamma_1(1+|z|)^{\gamma_2} e^{\gamma_3|z|}, \quad z \in \mathbb{C}, \tag{13.38}$$

where the constants $\gamma_1, \gamma_2, \gamma_3 > 0$ are independent of z. Then

$$f = \sum_{\substack{\lambda \in \mathcal{Z}(\widehat{T}) \\ |\lambda|<\gamma_3}} \sum_{\eta=0}^{m(\lambda,T)} c_{\lambda,\eta}(T,f) e^{\lambda,\eta} + \sum_{\substack{\lambda \in \mathcal{Z}(\widehat{T}) \\ |\lambda|=\gamma_3}} \sum_{\eta=0}^{\min\{m(\lambda,T),\gamma_2\}} c_{\lambda,\eta}(T,f) e^{\lambda,\eta}. \tag{13.39}$$

Proof. Bearing Proposition 6.10 in mind, we infer from (13.38) and Theorem 13.11 that $c_{\lambda,\eta}(T,f) = 0$ when $|\lambda|$ is sufficiently large. Applying now Theorem 13.9(i) and (13.38), we arrive at (13.39). □

We end this section with a result that will be needed in Part IV.

Proposition 13.8. *Let U, V be nonzero distributions in the class $\mathcal{E}'(\mathbb{R}^1)$, and let* $\operatorname{supp} U \subset [-r(U), r(U)]$, $\operatorname{supp} V \subset [-r(V), r(V)]$, $\lambda \in \mathcal{Z}(\widehat{U})$. *Assume that $n_\lambda(\widehat{U}) > n_\lambda(\widehat{V})$ and $\eta \in \{n_\lambda(\widehat{V}), \dots, n_\lambda(\widehat{U}) - 1\}$, where the number $n_\lambda(\widehat{V})$ is set to be equal to zero for $\lambda \notin \mathcal{Z}(\widehat{V})$. Then the following assertions hold.*

(i) *If $R > r(U) + r(V)$ and $f \in (\mathcal{D}'_U \cap \mathcal{D}'_V)(-R,R)$, then $c_{\lambda,\eta}(U,f) = 0$.*
(ii) *If $R = r(U) + r(V)$ and $f \in C^\infty[-R,R] \cap (\mathcal{D}'_U \cap \mathcal{D}'_V)(-R,R)$, then $c_{\lambda,\eta}(U,f) = 0$.*
(iii) *If $R = r(U) + r(V)$ and $f \in (\mathcal{D}'_U \cap \mathcal{D}'_V)(-R,R) \cap \mathcal{D}'(\mathbb{R}^1)$ and at least one of the distributions U, V belongs to $\mathcal{D}(\mathbb{R}^1)$, then $c_{\lambda,\eta}(U,f) = 0$.*

Proof. Using Proposition 13.5(iii) with $u = V$, we obtain

$$\sum_{\mu=\nu}^{m(\lambda,U)} c_{\lambda,\mu}(U,f) \binom{\mu}{\nu} \widehat{V}^{(\mu-\nu)}(\lambda) = 0$$

for all $\nu \in \{m(\lambda,V)+1, \dots, m(\lambda,U)\}$. This implies (i).

Turning to (ii), first note that for each $k \in \mathbb{N}$ there exists $F \in C^k(\mathbb{R}^1)$ such that $F = f$ in $[-R,R]$. If k is large enough, we have

$$F * U_{\lambda,\nu} * V \in C\big(\mathbb{R}^1\big) \quad \text{and} \quad (F * U_{\lambda,\nu} * V)(0) = 0 \tag{13.40}$$

for all $\nu \in \{0, \dots, m(\lambda,U)\}$. On the other hand, (13.21) yields

$$(F * U_{\lambda,\nu} * V)(0) = \sum_{\mu=0}^{m(\lambda,U)-\nu} c_{\lambda,\nu+\mu}(U,f) \binom{\nu+\mu}{\nu} \widehat{V}^{(\mu)}(\lambda). \tag{13.41}$$

Comparing this with (13.40), we arrive at (ii).

As for (iii), observe that (13.40) and (13.41) hold with $F = f$. Thus, the desired conclusion follows. □

Remark 13.2. Examining the above proof, we see that (ii) remains to be true with

$$f \in C^k[-R, R] \cap \big(\mathcal{D}'_T \cap \mathcal{D}'_{T_1}\big)(-R, R),$$

where $k \in \mathbb{N}$ is large enough. In addition, part (iii) remains valid, provided that at least one of the distributions U, V is in $(C^k \cap \mathcal{E}')(\mathbb{R}^1)$ for sufficiently large $k \in \mathbb{N}$.

13.4 Local Analogues of the Schwartz Fundamental Principle

The aim of this section is to describe some classes of solutions of (13.4) and thereby to create a tool for the study of the extendability of mean periodic functions.

Throughout the section we suppose that

$$T \in \mathcal{E}'\big(\mathbb{R}^1\big), \quad T \neq 0 \quad \text{and} \quad \operatorname{supp} T \subset [-r(T), r(T)].$$

Owing to Theorem 6.3 and Proposition 6.1(ii), one can select $\gamma \in (0, \pi/2)$ so that $\{\lambda \in \mathcal{Z}(\widehat{T})\backslash\{0\} : |\arg \lambda| = \gamma\} = \varnothing$ and

$$\lim_{r\to+\infty} \frac{\log|\widehat{T}(re^{i\gamma})|}{r} = \lim_{r\to+\infty} \frac{\log|\widehat{T}(re^{-i\gamma})|}{r} = r(T)|\sin\gamma|.$$

Recall from Leont'ev [144, Chap. 1, Sect. 4.1] that there exist a number $c > 0$ and an increasing sequence $\{r_k\}_{k=1}^{\infty}$ of positive numbers with the following properties:

(a) $r_{k+1}/r_k \to 1$ as $k \to \infty$;
(b) for each $\varepsilon > 0$, there is $k_\varepsilon > 0$ such that

$$\log\big|\widehat{T}(z)\big| > r(T)|\operatorname{Im} z| - \varepsilon|z|, \quad \text{provided that } |z| \in (r_k - c, r_k + c),\ k > k_\varepsilon.$$

Given $f \in \mathcal{D}'_T(a, b)$, we introduce the functions

$$f^+_{k,\gamma} = \sum_{\lambda\in A^+_{k,\gamma}} \sum_{\eta=0}^{m(\lambda,T)} c_{\lambda,\eta}(T, f)e^{\lambda,\eta}$$

and

$$f^-_{k,\gamma} = \sum_{\lambda\in A^-_{k,\gamma}} \sum_{\eta=0}^{m(\lambda,T)} c_{\lambda,\eta}(T, f)e^{\lambda,\eta},$$

where

$$A^+_{k,\gamma} = \big\{\lambda \in \mathcal{Z}\big(\widehat{T}\big)\backslash\{0\} : |\arg\lambda| < \gamma,\ |\lambda| \leqslant r_k\big\},$$

$$A^-_{k,\gamma} = \left\{\lambda \in \mathcal{Z}\big(\widehat{T}\big)\backslash A^+_{k,\gamma} : |\lambda| \leqslant r_k\right\}.$$

Next, for $\delta \in (0, (b-a)/2)$, we set

$$U^+_\delta = \{z \in \mathbb{C} : \operatorname{Re} z \in (a+\delta, b-\delta),\ \operatorname{Im} z > 0\}$$

and

$$U^-_\delta = \left\{z \in \mathbb{C} : \operatorname{Re} z \in (a+\delta, b-\delta),\ -\delta(4\cos\gamma)^{-1} < \operatorname{Im} z < 0\right\}.$$

Then a local version of the Schwartz fundamental principle can be formulated as follows.

Theorem 13.12. *For each $f \in \mathcal{D}'_T(a,b)$, the limit $\lim_{k\to\infty} f^+_{k,\gamma}(z) = f^+_\gamma(z)$ (respectively $\lim_{k\to\infty} f^-_{k,\gamma}(z) = f^-_\gamma(z)$) exists for all $z \in U^+_\delta$ (respectively $z \in U^-_\delta$). Moreover, the following assertions hold.*

(i) *$f^+_{k,\gamma} \to f^+_\gamma$ (respectively $f^-_{k,\gamma} \to f^-_\gamma$) uniformly in every compact subset of U^+_δ (respectively U^-_δ).*
(ii) *$f^+_\gamma(\cdot + i\varepsilon) + f^-_\gamma(\cdot - i\varepsilon) \to f$ in $\mathcal{D}'(a+\delta, b-\delta)$ as $\varepsilon \to +0$.*
(iii) *If $f \in C^\infty_T(a,b)$, then $f^+_\gamma(\cdot + i\varepsilon) + f^-_\gamma(\cdot - i\varepsilon) \to f$ in $\mathcal{E}(a+\delta, b-\delta)$ as $\varepsilon \to +0$.*

Proof. If $r(T) = 0$, the desired result is a consequence of Proposition 13.2. Let $r(T) > 0$. First, consider the case $T \in (\mathcal{E}' \cap C)(\mathbb{R}^n)$. By Corollary 13.1 there exists $g \in C_T(a+\delta/2, b-\delta/2)$ such that $p(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t})g = f$ for some polynomial f. Relation (13.17) yields

$$p\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right)g^+_{k,\gamma} = f^+_{k,\gamma} \quad \text{and} \quad p\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right)g^-_{k,\gamma} = f^-_{k,\gamma}. \tag{13.42}$$

The proof of Theorem 13.9 and Leont'ev [144, Chap. 5, Sect. 1.1] show that the limits

$$\lim_{k\to\infty} g^+_{k,\gamma}(z) = g^+_\gamma(z) \quad \text{and} \quad \lim_{k\to\infty} g^-_{k,\gamma}(z) = g^-_\gamma(z)$$

exist for all $z \in U^+_\delta$ and $z \in U^-_\delta$ respectively. In addition, assertions (i) and (ii) are true with f replaced by g. Next, $f^{(m)} \in \mathcal{D}'_T(a,b)$ for each $m \in \mathbb{N}$. Repeating the above argument with $f^{(m)}$ instead of f, we see that (iii) holds if f is replaced by g. This, together with (13.42), proves the theorem for $T \in (\mathcal{E}' \cap C)(\mathbb{R}^n)$. Let us now consider the general case. Since $r(T) > 0$, there exists $Q \in (\mathcal{E}' \cap C)(\mathbb{R}^n)$ such that equality (13.25) is valid with $\lambda_j \in \mathcal{Z}(\widehat{T}),\ s_j = m(\lambda_j, T) + 1,\ j \in \{1, \dots, l\}$. Because of Proposition 13.7, the argument in the earlier case is now applicable if f is replaced by

$$f - \sum_{j=1}^{l} \sum_{\eta=0}^{m(\lambda_j,T)} c_{\lambda_j,\eta}(T, f)\mathrm{e}^{\lambda_j,\eta},$$

and T by Q. Hence the theorem. □

We shall now give a more precise characterization of some classes of solutions of (13.4).

Theorem 13.13. *Let* $T \in (\mathrm{Inv}_+ \cap \mathrm{Inv}_-)(\mathbb{R}^1)$ *and assume that* (13.7) *is fulfilled and that* $f \in \mathcal{D}'(a,b)$. *Then* $f \in \mathcal{D}'_T(a,b)$ *if and only if*

$$f = \zeta_T * u \quad \text{on } (a,b) \tag{13.43}$$

for some $u \in \mathcal{E}'(\mathbb{R}^1)$ *with* $\operatorname{supp} u \subset [-r(T), r(T)]$.

Proof. First, suppose that $f \in \mathcal{D}'_T(a,b)$ and select $\alpha, \beta \in \mathbb{R}^1$ so that $\alpha < -r(T)$, $\beta > r(T)$, and $[\alpha,\beta] \subset (a,b)$. Owing to Corollary 13.1, there exists $F \in C_T(\alpha,\beta)$ such that

$$p\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)F = f$$

on (α,β) for some polynomial p. Define $F^+ \in L^{1,\mathrm{loc}}(-\infty,b)$ and $F^- \in L^{1,\mathrm{loc}}(a,+\infty)$ by letting

$$F^+(t) = F(t) \quad \text{if } t \in [0,b), \qquad F^+(t) = 0 \quad \text{if } t < 0,$$
$$F^-(t) = F(t) \quad \text{if } t \in (a,0], \qquad F^-(t) = 0 \quad \text{if } t > 0.$$

Repeating the arguments in the proof of Theorem 13.4(i) with F^+, F^- instead of f^+, f^-, we conclude that $F = \zeta_T * U$ on (α,β) for some $U \in \mathcal{E}'(\mathbb{R}^1)$ such that $\operatorname{supp} U \subset [-r(T), r(T)]$. Therefore, equality in (13.43) is satisfied in (α,β) for

$$u = p\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)U.$$

Since $f - \zeta_T * u \in \mathcal{D}'_T(a,b)$, we derive from Corollary 13.2 the "only if" part of the theorem. The "if" part immediately follows from (13.43) and Proposition 8.20(i). □

Theorem 13.14. *Let* $T \in \mathfrak{M}(\mathbb{R}^1)$, *assume that* (13.3) *holds, and let* $f \in \mathcal{D}'(a,b)$. *Then the following results are true.*

(i) *In order that* $f \in \mathcal{D}'_T(a,b)$, *it is necessary and sufficient that*

$$f = \sum_{\lambda \in \mathcal{Z}(\widehat{T})} \sum_{\eta=0}^{m(\lambda,T)} \gamma_{\lambda,\eta} e^{\lambda,\eta}, \tag{13.44}$$

where $\gamma_{\lambda,\eta} \in \mathbb{C}$, *and the series converges in* $\mathcal{D}'(a,b)$.

(ii) $f \in C_T^\infty(a,b)$ *if and only if equality* (13.44) *is satisfied, where the series converges in* $\mathcal{E}(a,b)$.

(iii) $f \in \mathrm{QA}_T(a,b)$ *if and only if relation* (13.44) *is satisfied, where the series converges in* $\mathcal{E}(a,b)$,

$$\max_{0\leqslant\eta\leqslant m(\lambda,T)} |\gamma_{\lambda,\eta}| \leqslant M_q(1+|\lambda|)^{-q} \quad \textit{for all } \lambda\in\mathcal{Z}\big(\widehat{T}\big),\ q\in\mathbb{N},$$

and the constants $M_q > 0$ *are independent of* λ *and satisfy* (13.36).

Proof. The necessity is clear from Proposition 13.2, Corollary 13.3, and Theorem 13.9(i). The sufficiency follows from Propositions 8.17 and 8.18. □

It can be shown that the requirement for T in Theorem 13.14 cannot be considerably weakened (see V.V. Volchkov [225], Part III, Theorem 1.5).

Theorem 13.15.

(i) *Let* $\alpha > 0$, $T \in \mathfrak{G}_\alpha(\mathbb{R}^1)$, *suppose that*

$$-\infty \leqslant a < a' < b' < b \leqslant +\infty, \quad b'-a' = 2r(T),$$

and let $f \in C^\infty(a,b)\cap G^\alpha[a',b']$. *Then in order that* $f \in C_T^\infty(a,b)$, *it is necessary and sufficient that* f *is of the form* (13.44), *where the series converges in* $\mathcal{E}(a,b)$.

(ii) *Let* $T \in \mathfrak{E}(\mathbb{R}^1)$, $f \in \mathcal{D}'(\mathbb{R}^1)$, *and assume that* f *is of finite order in* $\mathbb{R}^1$. *Then* $f \in \mathcal{D}'_T(\mathbb{R}^1)$ *if and only if relation* (13.44) *holds throughout* $\mathbb{R}^1$ *and the series in* (13.44) *converges in* $\mathcal{E}(\mathbb{R}^1)$.

Proof. Once Corollary 13.3(iii), (v) and Proposition 8.19 have been established, the proof of our theorem is identical to that of Theorem 13.14. □

Corollary 13.5. *Let* $T \in \mathfrak{E}(\mathbb{R}^1)$, $f \in \mathcal{D}'_T(\mathbb{R}^1)$, *and suppose that* f *is of finite order in* $\mathbb{R}^1$. *Then* $f \in C_T^\infty(\mathbb{R}^1)$.

The proof follows at once from Theorem 13.15(ii).

13.5 The Problem of Mean Periodic Continuation

Throughout the section we suppose that

$$T \in \mathcal{E}'\big(\mathbb{R}^1\big), \quad T \neq 0, \quad \operatorname{supp} T \subset [-r(T), r(T)],$$

and that (13.3) holds. We are interested in the following problem. Given $f \in \mathcal{D}'_T(a,b)$, when does there exist $F \in \mathcal{D}'_T(\mathbb{R}^1)$ such that $F|_{(a,b)} = f$ and how does F depend on f?

The first result we would like to state is as follows.

Theorem 13.16. *Let* $T \in \mathcal{E}'(\mathbb{R}^1)$ *and suppose that* (13.3) *holds. Then the following statements are valid.*

(i) *If* $T \in (\mathrm{Inv}_+ \cap \mathrm{Inv}_-)(\mathbb{R}^1)$, *then for each* $f \in \mathcal{D}'_T(a,b)$, *there exists a unique* $F \in \mathcal{D}'_T(\mathbb{R}^1)$ *such that* $F|_{(a,b)} = f$.

(ii) *If* $T \in \mathfrak{M}(\mathbb{R}^1)$, *then for each* $f \in C_T^\infty(a,b)$, *there exists a unique* $F \in C_T^\infty(\mathbb{R}^1)$ *such that* $F|_{(a,b)} = f$.

(iii) *If* $T \in \mathfrak{M}(\mathbb{R}^1)$, *then for each* $f \in \mathrm{QA}_T(a,b)$, *there exists a unique* $F \in \mathrm{QA}_T(\mathbb{R}^1)$ *such that* $F|_{(a,b)} = f$.

(iv) *If* $\alpha > 0$, $T \in \mathfrak{G}_\alpha(\mathbb{R}^1)$, *then for each* $f \in G_T^\alpha(a,b)$, *there exists a unique* $F \in G_T^\alpha(\mathbb{R}^1)$ *such that* $F|_{(a,b)} = f$.

Proof. To prove (i) we choose $c \in (a,b)$ such that $[c - r(T), c + r(T)] \subset (a,b)$. Then

$$f(t+c) = (\zeta_T * u)(t)$$

on $(a-c, b-c)$ for some $u \in \mathcal{E}'(\mathbb{R}^1)$ (see Theorem 13.13). Now it is enough to put

$$F = \zeta_T * v,$$

where $v(t) = u(t-c)$. Corollary 13.2 implies that F is uniquely determined by f.

Parts (ii)–(iv) follow from the fact that F is determined uniquely by the series in the right-hand side of (13.44) (see Propositions 8.17–8.19, Theorems 13.14 and 13.15, and Corollary 13.2). □

We note that if $r(T) > 0$, $[-r(T), r(T)] \subset (a,b)$, and $f \in C_T^\infty(a,b)$, then

$$\left\langle T, f^{(\nu)}(-\cdot)\right\rangle = 0 \quad \text{for all } \nu \in \mathbb{Z}_+. \tag{13.45}$$

In this case Theorem 13.16(ii)–(iv) can be refined as follows.

Theorem 13.17. *Let* $r(T) > 0$, $f \in C^\infty[-r(T), r(T)]$, *and assume that* (13.45) *holds. Then the following results are true.*

(i) *If* $T \in \mathfrak{M}(\mathbb{R}^1)$, *then there exists a unique* $F \in C_T^\infty(\mathbb{R}^1)$ *such that* $F|_{[-r(T),r(T)]} = f$. *Furthermore, if* $f \in \mathrm{QA}[-r(T), r(T)]$, *then* $F \in \mathrm{QA}_T(\mathbb{R}^1)$.

(ii) *If* $\alpha > 0$, $T \in \mathfrak{G}_\alpha(\mathbb{R}^1)$, *and* $f \in G^\alpha[-r(T), r(T)]$, *then there exists a unique* $F \in G_T^\alpha(\mathbb{R}^1)$ *such that* $F|_{[-r(T),r(T)]} = f$.

This theorem is proved similarly to Theorem 13.16(ii)–(iv) by using Corollaries 8.8, 8.9, and 8.10.

Our next object is to show that the assumptions on T in Theorem 13.16 cannot be considerably relaxed.

Theorem 13.18. *Let* $T \in \mathcal{E}'(\mathbb{R}^1)$, $a \in \mathbb{R}^1$, *and*

$$\sup_{\lambda \in \mathcal{Z}(\widehat{T})} \frac{\operatorname{Im} \lambda}{\log(2 + |\lambda|)} = +\infty. \tag{13.46}$$

Then there exists $f \in C_T^\infty(a, +\infty)$ *such that if* $\varepsilon > 0$ *and* $F \in \mathcal{D}'(a - \varepsilon, a + \varepsilon)$, *then* $F|_{(a,a+\varepsilon)} \neq f|_{(a,a+\varepsilon)}$.

Proof. By hypothesis there exists a sequence $\{\zeta_k\}_{k=1}^\infty$ of zeros of $\widehat{T}$ such that for all $k \in \mathbb{N}$, the following estimates hold:

(a) $\operatorname{Im} \zeta_k > 1$ and $|\zeta_{k+1}|/|\zeta_k| \geqslant 2$;
(b) $\operatorname{Im} \zeta_{k+1} > (2k + \operatorname{Im} \zeta_k) \log(3 + |\zeta_{k+1}|)$.

Setting

$$f(t) = \sum_{k=1}^{\infty} |\zeta_k|^k \mathrm{e}^{\mathrm{i}\zeta_k (t-a)}, \quad t > a, \tag{13.47}$$

one can easily see from (a) and (b) that the series in (13.47) converges in $C^\infty(a, +\infty)$. Thus, $f \in C_T^\infty(a, +\infty)$. Assume now that for some $\varepsilon > 0$, there exists $F \in \mathcal{D}'(a - \varepsilon, a + \varepsilon)$ such that $F = f$ on $(a, a + \varepsilon)$. It is not difficult to see that

$$F = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\nu} u$$

on $(a - \varepsilon/2, a + \varepsilon/2)$ for some $u \in C(a - \varepsilon/2, a + \varepsilon/2)$, $\nu \in \mathbb{N}$. Using (13.47), we find that

$$u(t) = p(t) + \sum_{k=1}^{\infty} |\zeta_k|^k (\mathrm{i}\zeta_k)^{-\nu} \mathrm{e}^{\mathrm{i}\zeta_k (t-a)}, \quad t > a,$$

for some polynomial p. Let $s \in \mathbb{N}$, $s > \nu$. Then

$$\sum_{k=1}^{s-1} |\zeta_k|^{k-\nu} \leqslant c|\zeta_{s-1}|^{s-1-\nu},$$

where $c > 0$ is independent of s. In addition, (a) and (b) imply that

$$\sum_{k=s+1}^{\infty} |\zeta_k|^{k-\nu} \exp\left(-\frac{\operatorname{Im} \zeta_k}{\operatorname{Im} \zeta_s}\right) = O(1) \quad \text{as } s \to +\infty.$$

Therefore, if s is large enough, then

$$u(a + 1/\operatorname{Im} \zeta_s) > |\zeta_s|^{s-\nu}/2.$$

This is a contradiction because $u \in C(a - \varepsilon/2, a + \varepsilon/2)$. Theorem 13.18 is thereby established. □

Corollary 13.6. *Let $T \in \mathcal{E}'(\mathbb{R}^1)$, $a \in \mathbb{R}^1$, and*

$$\inf_{\lambda \in \mathcal{Z}(\widehat{T})} \frac{\operatorname{Im} \lambda}{\log(2 + |\lambda|)} = -\infty.$$

Then there exists $f \in C^\infty(-\infty, a)$ such that if $\varepsilon > 0$ and $F \in \mathcal{D}'(a - \varepsilon, a + \varepsilon)$, then $F|_{(a-\varepsilon,a)} \neq f|_{(a-\varepsilon,a)}$.

Proof. This corollary results from using Theorem 13.18 on the distribution $T(-\cdot)$. □

Remark 13.3. Let f be defined by (13.47) with $a < 0$ and assume that $\{\zeta_k\}_{k=1}^{\infty}$ is a sequence of complex numbers such that $i\zeta_k \in (-\infty, 0)$ for all k and that estimates (a) and (b) in the proof of Theorem 13.18 hold. Then f admits holomorphic extension to the half-plane $\{z \in \mathbb{C}\colon\ \mathrm{Re}\, z > a\}$. In addition, for each $r \in (0, |a|)$, there exists $T \in \mathcal{E}'(\mathbb{R}^1)$ such that

$$\mathrm{supp}\, T \subset [-r, r] \quad \text{and} \quad \{\zeta_k\}_{k=1}^{\infty} \subset \mathcal{Z}\big(\widehat{T}\big)$$

(see Leont'ev [144], Theorem 1.4.3). Hence, $f \in \mathrm{RA}_T(a, +\infty)$ and the proof of Theorem 13.18 shows that if $\varepsilon > 0$ and $F \in \mathcal{D}'(a - \varepsilon, a + \varepsilon)$, then $F|_{(a,a+\varepsilon)} \neq f|_{(a,a+\varepsilon)}$. Therefore, assumption on T in Theorem 13.16(iii) cannot be dropped.

Remark 13.4. Let $\alpha > 0$, $T \in \mathcal{E}'(\mathbb{R}^1)$, $T \neq 0$, and

$$\sup_{\lambda \in \mathcal{Z}(\widehat{T})} \frac{\mathrm{Im}\, \lambda}{(1 + |\lambda|)^{1/\alpha}} = +\infty.$$

Then there exists a sequence $\{\zeta_k\}_{k=1}^{\infty}$ of zeros of $\widehat{T}$ such that $\mathrm{Im}\, \zeta_k > 1$, $|\zeta_{k+1}|/|\zeta_k| \geqslant 2$, and

$$\mathrm{Im}\, \zeta_{k+1} > \big((\alpha + 1)k + 3 + \mathrm{Im}\, \zeta_k\big)(3 + |\zeta_{k+1}|)^{\alpha}$$

for all $k \in \mathbb{N}$. Now define f by (13.47) with $a < 0$. It is easy to verify that $f \in G_T^{\alpha}(a, +\infty)$ (see Proposition 8.19). In addition, the proof of Theorem 13.18 leads to the conclusion that $f|_{(a,a+\varepsilon)} \neq F|_{(a,a+\varepsilon)}$ for all $\varepsilon > 0$, $F \in \mathcal{D}'(a - \varepsilon, a + \varepsilon)$. Thus, the assumption on T in Theorem 13.15(i) cannot be omitted.

Theorem 13.19. *Let $T \in \mathcal{E}'(\mathbb{R}^1)$, $R > r(T)$, and*

$$\sup_{\lambda \in \mathcal{Z}(\widehat{T})} \frac{m(\lambda, T)}{\max\{\mathrm{Im}\, \lambda, \log(2 + |\lambda|)\}} = +\infty. \tag{13.48}$$

Then there exists $f \in C_T^{\infty}(-R, R)$ such that if $\varepsilon \in (0, R)$ and $F \in \mathcal{D}'(R-\varepsilon, R+\varepsilon)$, then $F|_{(R-\varepsilon,R)} \neq f|_{(R-\varepsilon,R)}$.

The proof depends upon the following technical lemma.

Lemma 13.1. *Let $\nu, l \in \mathbb{N}$, $\lambda \in \mathbb{C}$, $|\lambda| \geqslant 1$, and let*

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\nu} \left(\sum_{\eta=0}^{l} \alpha_\eta e^{\lambda,\eta}(t)\right) = e^{\lambda,l}(t), \quad t \in \mathbb{R}^1,$$

for some $\alpha_\eta \in \mathbb{C}$. Then $\alpha_l = (i\lambda)^{-\nu}$ and

$$\sum_{\eta=0}^{l-1} |\alpha_\eta| t^{\eta} \leqslant 2\big(\nu^2 + 2\nu\big)^{\nu} l t^{l-1} |\lambda|^{-\nu-1} \tag{13.49}$$

for all $t \geqslant 3l/|\lambda|$.

Proof. First, observe that if

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{\eta=0}^{l}\beta_\eta \mathrm{e}^{\lambda,\eta}(t)\right)=\sum_{\eta=0}^{l}\gamma_\eta \mathrm{e}^{\lambda,\eta}(t), \quad t\in\mathbb{R}^1,$$

for some $\beta_\eta,\gamma_\eta\in\mathbb{C}$, then $\beta_l=\gamma_l/(\mathrm{i}\lambda)$ and

$$\lambda\beta_\eta+(\eta+1)\beta_{\eta+1}=-\mathrm{i}\gamma_\eta \quad \text{for all } \eta\in\{0,\dots,l-1\}.$$

Hence, $\alpha_l=(\mathrm{i}\lambda)^{-\nu}$, and the estimate

$$|\alpha_\eta|\leqslant(l-\eta+\nu+1)^\nu l^{l-\eta}|\lambda|^{\eta-l-\nu}, \quad \eta\in\{0,\dots,l-1\}, \tag{13.50}$$

can be easily proved by induction on ν. Using (13.50), one has

$$\sum_{\eta=0}^{l-1}|\alpha_\eta|t^\eta\leqslant\frac{t^l}{|\lambda|^\nu}\sum_{k=1}^{l}\psi(k), \tag{13.51}$$

where the function $\psi\colon[0,+\infty)\to(0,+\infty)$ is defined by

$$\psi(x)=(x+\nu+1)^\nu\left(\frac{l}{t|\lambda|}\right)^x.$$

If $t\geqslant 3l/|\lambda|$, we obtain $\psi'(x)<0$ for all $x\geqslant 0$. Therefore,

$$\sum_{k=1}^{l}\psi(k)\leqslant\psi(1)+\int_1^l\psi(x)\,\mathrm{d}x\leqslant\psi(1)+(\nu+2)^\nu\int_0^\infty x^\nu\left(\frac{l}{t|\lambda|}\right)^x\mathrm{d}x.$$

This, together with (13.51), implies (13.49). □

Proof of Theorem 13.19. Owing to Corollary 13.6, we can suppose that

$$\inf_{\lambda\in\mathcal{Z}(\widehat{T})}\frac{\operatorname{Im}\lambda}{\log(2+|\lambda|)}>-\infty. \tag{13.52}$$

By assumption and (8.4), there exists a sequence $\{\zeta_k\}_{k=1}^\infty$ of zeros of $\widehat{T}$ satisfying the following conditions:

(a) $\lim_{k\to\infty}q_k=+\infty$, where

$$q_k=\frac{l_k}{\max\{\operatorname{Im}\zeta_k,\log(2+|\zeta_k|)\}} \quad \text{and} \quad l_k=m(\zeta_k,T);$$

(b) $|\zeta_k|>1$ and $R|\zeta_k|>6l_k$ for each k;
(c) $q_k^2<q_{k+1}$ and $1\leqslant l_k\leqslant l_{k+1}/q_{k+1}$ for all k.

Consider the function

$$f(t)=\sum_{k=1}^{\infty}\gamma_k\left(\frac{\mathrm{i}t}{R}\right)^{l_k}\mathrm{e}^{\mathrm{i}\zeta_k t},\quad t\in(-R,R),\tag{13.53}$$

where

$$\gamma_k=\exp\left(l_k q_k^{-1/3}\right).$$

Because of (a), (c), and (13.52), the series in (13.53) converges in $C^{\infty}(-R,R)$, whence $f\in C_T^{\infty}(-R,R)$. Assume that for some $\varepsilon\in(0,R)$, there exists $F\in\mathcal{D}'(R-\varepsilon,R+\varepsilon)$ such that $F=f$ on $(R-\varepsilon,R)$. Then there exists $u\in C(R-\varepsilon/2,R+\varepsilon/2)$ such that

$$F=\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\nu}u\quad \text{on } (R-\varepsilon/2,R+\varepsilon/2)$$

for some $\nu\in\mathbb{N}$. According to (13.53), we can write

$$u(t)=p(t)+\sum_{k=1}^{\infty}\frac{\gamma_k}{R^{l_k}}u_k(t),\quad t\in(R-\varepsilon/2,R),\tag{13.54}$$

where p is a polynomial,

$$u_k(t)=\mathrm{e}^{\mathrm{i}\zeta_k t}\sum_{\eta=0}^{l_k}\alpha_{\eta,k}(\mathrm{i}t)^{\eta},$$

and $\alpha_{\eta,k}\in\mathbb{C}$ are defined by

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\nu}\sum_{\eta=0}^{l_k}\alpha_{\eta,k}(\mathrm{i}t)^{\eta}\mathrm{e}^{\mathrm{i}\zeta_k t}=(\mathrm{i}t)^{l_k}\mathrm{e}^{\mathrm{i}\zeta_k t}.\tag{13.55}$$

Using (b), we find from (13.55) and Lemma 13.1 that

$$\left|u_k(t)-\frac{(\mathrm{i}t)^{l_k}}{(\mathrm{i}\zeta_k)^{\nu}}\mathrm{e}^{\mathrm{i}\zeta_k t}\right|\leqslant 2\left(\nu^2+2\nu\right)^{\nu}l_k t^{l_k-1}|\zeta_k|^{-\nu-1}\mathrm{e}^{-(\operatorname{Im}\zeta_k)t}\tag{13.56}$$

for all $t\in(R-\varepsilon/2,R)$, $k\in\mathbb{N}$. Let $s\in\mathbb{N}$, $s\geqslant 2$. Together, (a)–(c), (13.52), and (13.56) give the estimate

$$\sum_{k=1}^{s-1}\frac{\gamma_k}{R^{l_k}}|u_k(t)|\leqslant \mathrm{e}^{c_1 l_{s-1}},\quad t\in(R-\varepsilon/2,R),\tag{13.57}$$

with $c_1>0$ independent of s,t. Next, if $q_s^{-1/2}<\varepsilon/2$, we have by a similar way

$$\frac{\gamma_k}{R^{l_k}}\left|u_k\left(R-q_s^{-1/2}\right)\right|\leqslant \exp\left(l_k\left(q_k^{-1/3}-q_s^{-1/2}+c_2 q_k^{-1}\right)\right),$$

where $c_2 > 0$ is independent of s. Now condition (c) yields

$$\sum_{k=s+1}^{\infty} \frac{\gamma_k}{R^{l_k}} \left| u_k\left(R - q_s^{-1/2}\right)\right| = O(1) \quad \text{as } s \to +\infty. \tag{13.58}$$

In addition, for sufficiently large s, relations (13.56) and (8.4) imply that

$$\left| u_s\left(R - q_s^{-1/2}\right)\right| > R^{l_s} \exp\left(-2l_s q_s^{-1/2}\right). \tag{13.59}$$

Putting estimates (13.57)–(13.59) together then leads to the conclusion that

$$\left| u\left(R - q_s^{-1/2}\right)\right| \to +\infty \quad \text{as } s \to +\infty$$

(see (a), (c), and (13.54)). But this is impossible, since $u \in C(R - \varepsilon/2, R + \varepsilon/2)$. Thus, the function f satisfies all the requirements of the theorem. □

Theorems 13.16, 13.18, and 13.19 admit the following important consequences.

Corollary 13.7. *If* $T \in (\mathrm{Inv}_+ \cap \mathrm{Inv}_-)(\mathbb{R}^1)$, *then*

$$m(\lambda, T) + |\mathrm{Im}\,\lambda| \leqslant c \log(2 + |\lambda|)$$

for all $\lambda \in \mathcal{Z}(\widehat{T})$, *where the constant* $c > 0$ *is independent of* λ.

Proof. Combining Theorems 13.16(i), 13.18, and Corollary 13.6, we arrive at the inequality

$$|\mathrm{Im}\,\lambda| \leqslant c_1 \log(2 + |\lambda|), \quad \lambda \in \mathcal{Z}\left(\widehat{T}\right),$$

with $c_1 > 0$ independent of λ. The same estimate for $m(\lambda, T)$ now follows from Theorems 13.16(i) and 13.19. □

Corollary 13.8. *Let* $T \in \mathcal{E}'_\natural(\mathbb{R}^1)$, $R > r(T)$, *and assume that*

$$\sup_{\lambda \in \mathcal{Z}(\widehat{T})} \frac{m(\lambda, T) + |\mathrm{Im}\,\lambda|}{\log(2 + |\lambda|)} = +\infty.$$

Then there exists an even function $g \in C_T^{\infty}(-R, R)$ *such that if* $\varepsilon \in (0, R)$ *and* $G \in \mathcal{D}'(R - \varepsilon, R + \varepsilon)$, *then* $G|_{(R-\varepsilon,R)} \neq g|_{(R-\varepsilon,R)}$.

Proof. If condition (13.46) is fulfilled, then it is enough to put

$$g(t) = f(t) + f(-t),$$

where f is the function defined in the proof of Theorem 13.18. Therefore, it remains to consider the case where (13.48) holds. By Theorem 13.19 there exists $f \in C_T^{\infty}(-2R, 2R)$ such that for all $\varepsilon \in (0, R)$, $F \in \mathcal{D}'(2R - \varepsilon, 2R + \varepsilon)$, we have that $F \neq f$ on $(2R - \varepsilon, 2R)$. Then the function

$$g(t) = f(R + t) + f(R - t)$$

satisfies all the required conditions. □

We complement Theorems 13.18 and 13.19 by the following result.

Theorem 13.20. *There exists $T \in \mathcal{D}_\natural(\mathbb{R}^1)$ such that the following statements hold.*

(i) $r(T) > 0$, $\mathcal{Z}(\widehat{T}) \subset \mathbb{R}^1$, *and* $m(\lambda, T) = 0$ *for all* $\lambda \in \mathcal{Z}(\widehat{T})$.
(ii) *For each* $R > r(T)$, *there is an even function* $g \in C_T^\infty(-R, R)$ *such that if* $\varepsilon \in (0, R)$ *and* $G \in \mathcal{D}'(R-\varepsilon, R+\varepsilon)$, *then* $G|_{(R-\varepsilon,R)} \neq g|_{(R-\varepsilon,R)}$.

The proof starts with the following lemma.

Lemma 13.2. *Let* $n \in \mathbb{N}$, $\varepsilon > 0$, $p \in \mathbb{Z}_+$, $z \in \mathbb{C}$, *and let* $w(z) = (e^{i\varepsilon z} - 1)^n$. *Then the following estimates are valid.*

(i) $|w(z)| \leqslant (\varepsilon|z|(1 + \varepsilon e^{|z|}))^n$.
(ii) *If* $\varepsilon < e^{-|z|}$ *then* $|w(z)| \geqslant (\varepsilon|z|(1 - \varepsilon e^{|z|}))^n$.
(iii) $|w^{(p)}(z)| \leqslant (n\varepsilon)^p (1 + e^{\varepsilon|\mathrm{Im}\, z|})^n$.
(iv) $|w^{(p)}(z)| \leqslant (np)^p \varepsilon^n (|z| + 1/n)^n (1 + \varepsilon e^{|z|+1})^n$.

Proof. To prove (i) and (ii) it is enough to consider the Taylor expansion of $e^{i\varepsilon z} - 1$ at origin. Part (iii) is obvious from the relation

$$w^{(p)}(z) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (ik\varepsilon)^p e^{i\varepsilon k z}.$$

Finally, estimate (iv) is a consequence of (i) and the Cauchy formula

$$w^{(p)}(z) = \int_{|\zeta - z| = 1/n} w(\zeta)(\zeta - z)^{-1-p}\, d\zeta.$$

□

Proof of Theorem 13.20. For $k \in \mathbb{N}$, we define

$$m_k = 2^{2^k}, \quad \zeta_k = \exp\left(m_k^{2/3}\right), \quad \gamma_k = \exp\left(m_k^{4/5}\right), \quad \varepsilon_k = 1/(4m_k),$$

and

$$w_k(z) = \left(e^{i\varepsilon_k z} - 1\right)^{m_k} \varepsilon_k^{-m_k}, \quad z \in \mathbb{C}.$$

Let $a_1, a_2, \ldots$ be the set of all numbers of the form $\zeta_k + l\varepsilon_k$, $k \in \mathbb{N}$, $l \in \{0, \ldots, m_k\}$, arranged in ascending order of magnitude. Define $T \in \mathcal{D}_\natural(\mathbb{R}^1)$ by the formula

$$\widehat{T}(z) = \prod_{n=1}^{\infty} \frac{\sin(z/a_n)}{z/a_n}, \quad z \in \mathbb{C}$$

(see Theorem 6.3). By Lindemann's theorem (see, for instance, [196], Chap. 2, Sect. 7) one sees that a_i/a_j is irrational for $i \neq j$. Thus, statement (i) in Theorem 13.20 holds.

Let $R > r(T)$ and

$$f(t) = \sum_{k=1}^{\infty} \gamma_k \frac{e^{i\zeta_k t}}{(2R)^{m_k}} w_k(t), \quad t \in (-2R, 2R). \tag{13.60}$$

Lemma 13.2(iv) and Theorem 13.14 show that $f \in C_T^{\infty}(-2R, 2R)$. Suppose now that for some $\varepsilon \in (0, R)$, there is $F \in \mathcal{D}'(2R - \varepsilon, 2R + \varepsilon)$ such that $F|_{(2R-\varepsilon,2R)} = f|_{(2R-\varepsilon,2R)}$. Then

$$F = \left(\frac{d}{dt}\right)^{\nu} u \quad \text{on } (2R - \varepsilon/2, 2R + \varepsilon/2)$$

for some $u \in C(2R - \varepsilon/2, 2R + \varepsilon/2)$, $\nu \in \mathbb{N}$. Bearing (13.60) in mind, we obtain

$$u(t) = q(t) + \sum_{k=1}^{\infty} \gamma_k \frac{e^{i\zeta_k t}}{(2R)^{m_k}} u_k(t), \quad t \in (2R - \varepsilon/2, 2R), \tag{13.61}$$

where q is a polynomial, and

$$u_k(t) = \frac{1}{\varepsilon_k^{m_k}} \sum_{l=1}^{m_k} (-1)^{m_k - l} \binom{m_k}{l} \frac{e^{il\varepsilon_k t}}{(i(\zeta_k + l\varepsilon_k))^{\nu}}. \tag{13.62}$$

Writing the function $(1+z)^{-\nu}$ as a power series, we get

$$\big(i(\zeta_k + l\varepsilon_k)\big)^{-\nu} = \frac{1}{(i\zeta_k)^{\nu}} \sum_{p=0}^{\infty} \xi_{p,\nu} \left(\frac{l\varepsilon_k}{\zeta_k}\right)^{p},$$

where

$$\xi_{p,\nu} = (-1)^p \binom{\nu + p - 1}{p}.$$

Notice that

$$|\xi_{p,\nu}| \leqslant 2^{\nu+p-1} \quad \text{for all } p \in \mathbb{Z}_+. \tag{13.63}$$

Relation (13.62) can be rewritten as

$$u_k(t) = \sum_{p=0}^{\infty} \xi_{p,\nu} \frac{w_k^{(p)}(t)}{(i\zeta_k)^{p+\nu}}, \quad t \in (2R - \varepsilon/2, 2R). \tag{13.64}$$

For $s \in \mathbb{N}$, $s \geqslant 2$, we put $\theta_s = m_s^{-1/4}$. Assume now that s is large enough so that $\theta_s < \varepsilon/2$. Using Lemma 13.2(iii), one has

$$\left|w_k^{(p)}(2R - \theta_s)\right| \leqslant 4^{-p} \exp(c_1 m_k \log m_k), \quad k \in \mathbb{N}, p \in \mathbb{Z}_+, \tag{13.65}$$

where $c_1 > 0$ is independent of k, p, s. Taking (13.63) and (13.64) into account, we conclude that

$$\sum_{k=1}^{s-1} \frac{\gamma_k}{(2R)^{m_k}} |u_k(2R-\theta_s)| \leqslant \exp(c_2 m_{s-1} \log m_{s-1}), \tag{13.66}$$

where $c_2 > 0$ is independent of s. Next, let

$$N_k = m_k^{1/2} \quad \text{and} \quad \tau_{p,k,s} = \left|w_k^{(p)}(2R-\theta_s)\xi_{p,\nu}\right| \zeta_k^{-p}.$$

Then (13.65) and (13.63) imply the estimate

$$\sum_{p=N_k+1}^{\infty} \tau_{p,k,s} \leqslant \zeta_k^{-N_k} \exp(c_1 m_k \log m_k). \tag{13.67}$$

Suppose now that $k \geqslant s$. Lemma 13.2(iv) yields

$$\left|w_k^{(p)}(2R-\theta_s)\right| \leqslant c_3 (2R)^{m_k} (pm_k)^p \exp\bigl(-m_k\theta_s/(2R)\bigr),$$

where $c_3 > 0$ is independent of p, k, s. For $k > s$, this, together with (13.63), implies that

$$\sum_{p=0}^{N_k} \tau_{p,k,s} \leqslant \exp\left(c_4 m_k^{1/2} \log m_k - m_k\theta_s/(2R)\right), \tag{13.68}$$

where $c_4 > 0$ is independent of s, k. Estimates (13.67) and (13.68) lead to the conclusion that

$$\sum_{k=s+1}^{\infty} \frac{\gamma_k}{(2R)^{m_k}} |u_k(2R-\theta_s)| = O(1) \quad \text{as } s \to +\infty. \tag{13.69}$$

Using now (13.62) and (13.67), we see that

$$\left|u_s(2R-\theta_s) - \xi_{0,\nu} \frac{w_s(2R-\theta_s)}{(\mathrm{i}\zeta_s)^\nu}\right|$$
$$\leqslant \sum_{p=1}^{\infty} \tau_{p,s,s}\zeta_s^{-\nu} \leqslant \frac{\exp(c_1 m_s \log m_s)}{\zeta_s^{N_s}} + \sum_{p=1}^{N_s} \frac{|\xi_{p,\nu} w_s^{(p)}(2R-\theta_s)|}{\zeta_s^{\nu+1}}.$$

Hence,

$$|u_s(2R-\theta_s)| \geqslant \frac{(2R)^{m_s}}{\zeta_s^\nu} \exp\bigl(-m_s\theta_s/(4R)\bigr)$$

if s is sufficiently large (see Lemma 13.2(ii)). Combining this with (13.61), (13.66), and (13.69), we deduce that

$$|u(2R-\theta_s)| \to +\infty \quad \text{as } s \to +\infty.$$

This contradicts the fact that $u \in C(2R-\varepsilon/2, 2R+\varepsilon/2)$. Thus, the function $g(t) = f(R+t) + f(R-t)$ satisfies (ii), and we are done. □

We end our considerations with the proof of the following related result.

Theorem 13.21. *Let $T \in \mathcal{E}'(\mathbb{R}^1)$ and suppose that*

$$\sup_{\lambda \in \mathcal{Z}(\widehat{T})} \frac{m(\lambda, T)}{1+|\operatorname{Im}\lambda|} = +\infty.$$

Then for each $R > r(T)$, there exists $f \in C_T(-R, R)$ such that $f|_{(0,R)} \notin L^1(0, R)$ and $f|_{(-R,0)} \notin L^1(-R, 0)$. In particular, the function f does not admit continuous extension to $[-R, R]$.

Proof. The hypothesis upon T implies that there is a sequence $\{\zeta_k\}_{k=1}^{\infty}$ of zeros of $\widehat{T}$ with the following properties:

(a) $\lim_{k\to\infty} q_k = +\infty$, where $q_k = l_k(1+|\operatorname{Im}\zeta_k|)^{-1}$ and $l_k = m(\zeta_k, T)$;
(b) $q_{k+1} > q_k^3 > 1$ and $q_{k+1} > l_k^3/q_k$ for all $k \in \mathbb{N}$.

For $R > r(T)$, now define

$$f(t) = \sum_{k=1}^{\infty} \gamma_k \left(\frac{t}{R}\right)^{l_k} e^{i\zeta_k t}, \quad t \in (-R, R), \tag{13.70}$$

where $\gamma_k = \exp(l_k q_k^{-1/3})$. It follows by (a), (13.70), and Theorems 13.9(ii) that $f \in C_T(-R, R)$. Let $s \in \mathbb{N}$, $s \geqslant 2$, $q_s^{-1/2} + q_s^{-1} < R$, and let $\theta \in (q_s^{-1/2} - q_s^{-1}, q_s^{-1/2} + q_s^{-1})$. It is not difficult to see that

$$\sum_{k=1}^{s-1} \gamma_k e^{R|\operatorname{Im}\zeta_k|} \leqslant \exp\left(c_1 l_{s-1} q_{s-1}^{-1/3}\right), \tag{13.71}$$

where $c_1 > 0$ is independent of s. In addition, owing to (b),

$$\sum_{k=s+1}^{\infty} \gamma_k (1-\theta/R)^{l_k} e^{R|\operatorname{Im}\zeta_k|} \leqslant c_2, \tag{13.72}$$

where $c_2 > 0$ is independent of s and θ. Using now (13.70)–(13.72), we conclude from (a) and (b) that

$$|f(R-\theta)| > \frac{1}{2}\gamma_s (1-\theta/R)^{l_s} \exp\left((\theta - R)\operatorname{Im}\zeta_s\right)$$

if s is large enough. Setting $E_s = (R - q_s^{-1/2} - q_s^{-1}, R - q_s^{-1/2} + q_s^{-1})$, we infer from the previous estimate that

$$\int_{E_s} |f(t)|\,\mathrm{d}t \to +\infty \quad \text{as } s \to +\infty.$$

Hence, $f|_{(0,R)} \notin L^1(0, R)$. By a similar way it can be shown that $f|_{(-R,0)} \notin L^1(-R, 0)$. This completes the proof. □

13.6 One-Sided Liouville's Property

As before, throughout the section we suppose that

$$T \in \mathcal{E}'(\mathbb{R}^1), \quad T \neq 0 \quad \text{and} \quad \operatorname{supp} T \subset [-r(T), r(T)].$$

Let $f \in (\mathcal{D}'_T \cap L^{1,\mathrm{loc}})(0, +\infty)$. According to results in Sect. 13.3, we can associate with f the series

$$f \sim \sum_{\lambda \in \mathcal{Z}(\widehat{T})} \sum_{\eta=0}^{m(\lambda,T)} c_{\lambda,\eta}(T, f) \mathrm{e}^{\lambda,\eta}. \tag{13.73}$$

In this section we will specify the form of series (13.73) for the case where f satisfies some growth restrictions at infinity.

The following result is an analog of the classical Liouville theorem for entire functions.

Theorem 13.22. *Let $f \in (\mathcal{D}'_T \cap L^{1,\mathrm{loc}})(0, +\infty)$ and assume that*

$$\liminf_{R \to +\infty} R^{-\alpha} \mathrm{e}^{\beta R} \int_{R-\xi}^{R+\xi} |f(t)|\,\mathrm{d}t = 0 \tag{13.74}$$

for some $\alpha \geqslant 0$, $\beta \in \mathbb{R}^1$, and $\xi > r(T)$. Then

$$c_{\lambda,\eta}(T, f) = 0,$$

provided that $\operatorname{Im} \lambda < \beta$. The same is true if $\operatorname{Im} \lambda = \beta$ and $\eta \geqslant \alpha$. In particular, if (13.74) is valid for each $\beta > 0$ and $\alpha = 0$, then $f = 0$.

Proof. Let $\lambda \in \mathcal{Z}(\widehat{T})$, $\nu \in \{0, \dots, m(\lambda, T)\}$, $\varepsilon \in (0, \xi - r(T))$, and let $\gamma = \xi - \varepsilon - r(T)$, $R > \xi$. For any $u \in \mathcal{D}(-\varepsilon, \varepsilon)$, we have

$$\int_{R-\gamma}^{R+\gamma} |(f * u * T_{\lambda,\nu})(t)|\,\mathrm{d}t \leqslant \int_{R-\xi}^{R+\xi} |f(t)|\,\mathrm{d}t \int_{\gamma-\xi}^{\xi-\gamma} |(u * T_{\lambda,\nu})(t)|\,\mathrm{d}t.$$

Now relations (13.74) and (13.21) show that there is an increasing infinite sequence $\{R_n\}_{n=1}^{\infty}$ of positive numbers such that $R_1 > \xi$ and

$$\lim_{n\to\infty} R_n^{-\alpha} e^{\beta R_n} \int_{R-\gamma}^{R+\gamma} \left| \sum_{\mu=0}^{m(\lambda,T)-\nu} c_{\lambda,\nu+\mu}(T, f*u) \binom{\nu+\mu}{\nu} e^{\lambda,\eta}(t) \right| dt = 0.$$

If $\operatorname{Im}\lambda < \beta$, this implies $c_{\lambda,\eta}(T, f*u) = 0$ for each $\eta \in \{0, \dots, m(\lambda, T)\}$. Similarly, we have

$$c_{\lambda,\eta}(T, f*u) = 0,$$

provided that $\operatorname{Im}\lambda = \beta$, $\eta \geqslant \alpha$. Since $u \in \mathcal{D}(-\varepsilon, \varepsilon)$ could be arbitrary, this, together with Proposition 13.5(iii), brings us to the assertion of the theorem. □

Remark 13.5. Assumption (13.74) in Theorem 13.22 cannot be replaced by

$$\int_{R-\xi}^{R+\xi} |f(t)|\, dt = O\big(R^{\alpha} e^{-\beta R}\big) \quad \text{as } R \to +\infty.$$

We may come to this conclusion having regarded the function $f = e^{\lambda,\eta}$ for $\lambda \in \mathcal{Z}(\widehat{T})$ and $\eta \in \{0, \dots, m(\lambda, T)\}$. Also, Theorem 13.22 fails in general with $\xi \in (0, r(T))$. In fact, assume that T is the characteristic function of the interval $(-\gamma, \gamma)$ for some $\gamma > 0$, and let f be a γ-periodic nonzero function in $C^{\infty}(\mathbb{R}^1)$ vanishing in $(-\xi, \xi)$ with

$$\int_{-\gamma}^{\gamma} f(t)\, dt = 0.$$

Then $f \in C_T^{\infty}(\mathbb{R}^1)$ and

$$\int_{R-\xi}^{R+\xi} |f(t)|\, dt = 0$$

for $R = 2m\gamma = 2mr(T)$, $m \in \mathbb{Z}$.

We conclude this section with the following result.

Theorem 13.23. *Let $\{T_\nu\}_{\nu=1}^m$ be a family of nonzero distributions in $\mathcal{E}'(\mathbb{R}^1)$, let*

$$\xi > \sum_{\nu=1}^{m} r(T_\nu),$$

and suppose that the set $\{1, \dots, m\}$ is represented as a union of disjoint sets $A_1, \dots, A_l$ such that the sets

$$\bigcup_{\nu\in A_s} \mathcal{Z}(\widehat{T_\nu}), \qquad s = 1, \dots, l,$$

are also disjoint. Assume that $f_\nu \in (\mathcal{D}'_{T_\nu} \cap L^{1,\mathrm{loc}})(0, +\infty)$ and

$$\liminf_{R\to+\infty} e^{\beta R} \int_{R-\xi}^{R+\xi} \left| \sum_{\nu=1}^{m} f_\nu(t) \right| dt = 0$$

for each $\beta > 0$. *Then*

$$\sum_{\nu \in A_s} f_\nu = 0$$

for all $s \in \{1, \dots, l\}$.

Proof. Let

$$f = \sum_{\nu=1}^{m} f_\nu \quad \text{and} \quad T = T_1 * \dots * T_m.$$

Then $f \in (\mathcal{D}'_T \cap L^{1,\mathrm{loc}})(0, +\infty)$, $\xi > r(T)$, and (13.74) is satisfied for each $\beta > 0$ and $\alpha = 0$. Theorem 13.22 yields $f = 0$. For $s \in \{1, \dots, l\}$, now define $F_s \in \mathcal{D}'(0, \infty)$ and $\Phi_s, \Psi_s \in \mathcal{E}'(\mathbb{R}^1)$ by letting

$$F_s = \sum_{\nu \in A_s} f_\nu, \qquad \widehat{\Phi}_s = \prod_{\nu \in A_s} \widehat{T}_\nu, \qquad \widehat{\Psi}_s = \prod_{\substack{\nu=1 \\ \nu \notin A_s}}^{m} \widehat{T}_\nu.$$

Then $F_s \in (\mathcal{D}'_{\Phi_s} \cap \mathcal{D}'_{\Psi_s})(0, +\infty)$ and $\mathcal{Z}(\widehat{\Phi}_s) \cap \mathcal{Z}(\widehat{\Psi}_s) = \varnothing$. Proposition 13.8 shows that

$$c_{\lambda,\eta}(\Phi_s, F_s) = 0$$

for all $\lambda \in \mathcal{Z}(\widehat{\Phi}_s)$ and $\eta \in \{0, \dots, m(\lambda, \Phi_s)\}$. In order to complete the proof one needs only Proposition 13.6(ii) . ◻

Chapter 14
Mean Periodic Functions on Multidimensional Domains

The theory developed in Chap. 13 leads to a number of interesting problems for mean periodic functions on domains in $\mathbb{R}^n$, $n \geqslant 2$. Here, we consider some of them.

In Sect. 14.1 we gather definitions, notation, and technical tools needed in the rest of the chapter. The unifying theme of the results that we present in Sects. 14.2 and 14.3 is the John support theorem. Fritz John [129, Chap. 6] proved the global uniqueness for integrals of f over spheres of radius 1 in the standard measure under the assumption that supp f is disjoint from $|x| \leqslant 1$. Section 14.2 describes similar but much more complex results concerning mean periodic functions. We find the exact dependence between the order of smoothness of functions satisfying John-type conditions and the set of nonzero coefficients in their Fourier expansions with respect to spherical harmonics. It follows that if $f \in C^\infty(B_R)$ is mean periodic with respect to $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$ $(0 < r(T) < R)$ and $f = 0$ in $B_{r(T)}$, then $f = 0$ in B_R. Theorem 14.3 in Sect. 14.2 shows that the smoothness condition in the above statement can be weakened. This so-called "hemisphere theorem" is based on the completeness of the system $T^{\lambda,\eta}$ (see Theorem 9.9). In Sect. 14.3 we define multidimensional analogues of the distribution ζ_T (see Sect. 8.4). This enables us to establish the sharpness of the main results of Sect. 14.2.

The aim of Sects. 14.4 and 14.5 is to expand mean periodic functions locally into Taylor- and Laurent-type series. First, the case of a ball is considered; it turns out that the functions $\Phi_{\lambda,\eta,k,j}$ introduced in Part II play an analogous role as the powers of the complex variable z do in the classical Taylor series for holomorphic functions. We then study the structure of a mean periodic function in an annular domain. This leads to a Laurent-type series expansion where the negative powers of z in the classical case are replaced by the functions $\Psi_{\lambda,\eta,k,j}$. In these contexts estimates of the coefficients are discussed, as well as convergence theorems for various classes of mean periodic functions. As a consequence, some mean periodic extendability results are obtained.

The results of Sect. 14.6 are of the following character: if f is mean periodic in $\mathbb{R}^n$ with respect to $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$, then $f = 0$ provided that f enjoys certain growth conditions. The sharpness of the conditions is shown, and spectral versions of these results are established. We also study similar questions for mean periodic functions

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on unbounded domains in $\mathbb{R}^n$. The growth of such functions is determined by the behavior of the corresponding eigenfunctions of the Laplacian (see Theorem 14.34).

The final Sect. 14.7 is devoted to the problem of approximation on domains in $\mathbb{R}^n$ of solutions of the homogeneous equation $f * T = 0$ ($T \in \mathcal{E}'(\mathbb{R}^n)\backslash\{0\}$) by exponential solutions. Hörmander's approximation theorem (Theorem 14.35) shows that in the case of convex domains this question is solved positively. We consider some analogues of Hörmander's result for domains without the convexity condition.

14.1 General Properties

Throughout this chapter we assume that $n \geqslant 2$ and shall preserve the notation from Chap. 9.

Let $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$, let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$, and let

$$\mathcal{O}_T = \left\{x \in \mathbb{R}^n : \mathring{B}_{r(T)}(x) \subset \mathcal{O}\right\}. \tag{14.1}$$

Throughout we suppose that $T \neq 0$ and $\mathcal{O}_T \neq \varnothing$. Let us consider the convolution equation

$$f * T = 0 \quad \text{in } \mathcal{O}_T \tag{14.2}$$

with unknown $f \in \mathcal{D}'(\mathcal{O})$. In this chapter we shall investigate various classes of solutions of (14.2).

By analogy with Sect. 13.1 denote by $\mathcal{D}'_T(\mathcal{O})$ the set of all $f \in \mathcal{D}'(\mathcal{O})$ satisfying (14.2). Now define

$$\begin{aligned} C^m_T(\mathcal{O}) &= \big(\mathcal{D}'_T \cap C^m\big)(\mathcal{O}) \quad \text{for } m \in \mathbb{Z}_+ \text{ or } m = \infty, \\ C_T(\mathcal{O}) &= C^0_T(\mathcal{O}), \qquad \mathrm{QA}_T(\mathcal{O}) = \big(\mathcal{D}'_T \cap \mathrm{QA}\big)(\mathcal{O}), \\ \mathrm{RA}_T(\mathcal{O}) &= \big(\mathcal{D}'_T \cap \mathrm{RA}\big)(\mathcal{O}), \\ G^\alpha_T(\mathcal{O}) &= \big(\mathcal{D}'_T \cap G^\alpha\big)(\mathcal{O}) \quad \text{for } \alpha > 0. \end{aligned}$$

It is obvious that if T is the Dirac measure δ_0, then $\mathcal{D}'_T(\mathcal{O}) = \{0\}$. We shall now show that for $T \neq \delta_0$, the class $\mathcal{D}'_T(\mathcal{O})$ is broad enough.

Proposition 14.1. *Let $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$, $\lambda = \sqrt{\zeta_1^2 + \dots + \zeta_n^2}$, and*

$$f(x) = p(x)u_\zeta(x), \quad x \in \mathbb{R}^n,$$

where p is a polynomial, and $u_\zeta(x) = e^{i\langle\zeta,x\rangle_{\mathbb{C}}}$. Then the following items are equivalent.

(i) $f \in \mathcal{D}'_T(\mathbb{R}^n)$.

(ii) $\left(\left(\frac{\partial}{\partial x}\right)^\alpha p\right)\left(-i\frac{\partial}{\partial \zeta_1}, \dots, -i\frac{\partial}{\partial \zeta_n}\right)\widehat{T}(\zeta) = 0$ *for all $\alpha \in \mathbb{Z}^n_+$.*

(iii) $\lambda \in \mathcal{Z}_T$ *and* $\deg P \leqslant n_\lambda(\widetilde{T}) - 1$.
(iv) $(\Delta + \lambda^2)^{n(\lambda,T)+1} f = 0$.

Proof. Since $u_\zeta * T = \widehat{T}(\zeta)u_\zeta$, the equation $f * T = 0$ is equivalent to

$$p\left(-\mathrm{i}\frac{\partial}{\partial \zeta_1}, \dots, -\mathrm{i}\frac{\partial}{\partial \zeta_n}\right)\big(\widehat{T}(\zeta)u_\zeta(x)\big) = 0, \quad x \in \mathbb{R}^n.$$

Applying the Leibniz rule, we get the equivalence (i)⇔(ii). Next, if (ii) is true, for $\lambda \neq 0$, one has

$$\widetilde{T}^{(\eta)}(\lambda) = 0 \quad \text{for all } \eta \in \{0, \dots, \deg P\}$$

(see (9.41) and (9.42)). If $\lambda = 0$, the same result follows by (9.41), (9.42), and the Taylor expansions of $\widetilde{T}$ and $\widehat{T}$ at the origin. This argument may be reversed, and so (ii)⇔(iii). For $\lambda \in \mathcal{Z}_T$, we now define $T_1 = (\Delta + \lambda^2)^{n(\lambda,T)+1}\delta_0 \in \mathcal{E}'_\natural(\mathbb{R}^n)$. Then $\lambda \in \mathcal{Z}_{T_1}$ and $n_\lambda(\widetilde{T}_1) = n_\lambda(\widetilde{T})$, and the equivalence (iii)⇔(i) leads to (iii)⇔(iv), proving the desired statement. □

For $a, b \in \mathbb{Z}_+$, $a \geqslant b$, and $\lambda \in \mathbb{C}$, we set

$$\binom{a}{b}_\lambda = \begin{cases} \dfrac{a!}{b!(a-b)!} & \text{if } \lambda \neq 0, \\ \dfrac{(2a)!}{(2b)!(2a-2b)!} & \text{if } \lambda = 0. \end{cases} \tag{14.3}$$

Proposition 14.2.

(i) *Let p be a polynomial such that the function* $\widetilde{T}(z)/p(-z^2)$ *is entire, and suppose that* $f \in \mathcal{D}'(\mathcal{O})$ *and* $p(\Delta)f = 0$ *in* $\mathcal{O}$. *Then* $f \in \mathrm{RA}_T(\mathcal{O})$.
(ii) *Let* $\lambda \in \mathbb{C}$, $\eta, k \in \mathbb{Z}_+$, *and* $j \in \{1, \dots, d(n,k)\}$. *Then*

$$\Phi_{\lambda,\eta,k,j} * T = \sum_{\nu=0}^{\eta} \binom{\eta}{\nu}_\lambda \widetilde{T}^{\langle \eta-\nu \rangle}(\lambda)\Phi_{\lambda,\nu,k,j} \quad \text{in } \mathbb{R}^n \tag{14.4}$$

and

$$\Psi_{\lambda,\eta,k,j} * T = \sum_{\nu=0}^{\eta} \binom{\eta}{\nu}_\lambda \widetilde{T}^{\langle \eta-\nu \rangle}(\lambda)\Psi_{\lambda,\nu,k,j} \quad \text{in } \mathbb{R}^n \backslash \dot{B}_{r(T)}. \tag{14.5}$$

In particular, if $\lambda \in \mathcal{Z}_T$ *and* $\eta \in \{0, \dots, n(\lambda, T)\}$, *then* $\Phi_{\lambda,\eta,k,j} \in \mathrm{RA}_T(\mathbb{R}^n)$ *and* $\Psi_{\lambda,\eta,k,j} \in \mathrm{RA}_T(\mathbb{R}^n \backslash \{0\})$.

Proof. To prove (i) observe that there exists $\psi \in \mathcal{E}'_\natural(\mathbb{R}^n)$ such that $p(\Delta)\psi = T$ and $r(\psi) = r(T)$ (see Theorem 9.2(ii) and (9.44)). Then

$$f * T = f * p(\Delta)\psi = p(\Delta)f * \psi = 0$$

in $\mathcal{O}_T$. In view of ellipticity of the operator $p(\Delta)$ this brings us to (i).

Next, equalities (14.4) and (14.5) with $\eta = 0$ are consequences of (1.60) and Corollary 9.1. Going now to the definition of $\Phi_{\lambda,\eta,k,j}$ and $\Psi_{\lambda,\eta,k,j}$, we obtain (ii) in the general case. □

We now establish some elementary properties of the class $\mathcal{D}'_T(\mathcal{O})$.

Proposition 14.3.

(i) *If* $f \in \mathcal{D}'_T(\mathcal{O})$, *then* $\frac{\partial f}{\partial x_m} \in \mathcal{D}'_T(\mathcal{O})$ *for each* $m \in \{1, \dots, n\}$.
(ii) *If* $f \in \mathcal{D}'_T(\mathcal{O})$, $\varphi \in \mathcal{E}'_\natural(\mathbb{R}^n)$, *and*

$$\left\{x \in \mathbb{R}^n : \dot{B}_{r(T)+r(\varphi)}(x) \subset \mathcal{O}\right\} \neq \varnothing,$$

then $f * \varphi \in \mathcal{D}'_T(\mathcal{O}_\varphi)$.

Proof. This is immediate from the definition of $\mathcal{D}'_T(\mathcal{O})$. □

Proposition 14.4. *Let* $f \in \mathcal{D}'_T(\mathcal{O})$, *and let* $\lambda \in \mathcal{Z}_T$. *Then the following results are true.*

(i) $(\Delta + \lambda^2) f * T_{\lambda,n(\lambda,T)} = 0$ *in* $\mathcal{O}_T$.
(ii) *If* $\lambda \neq 0$ *and* $n(\lambda, T) \geqslant 1$, *then*

$$\left(\Delta + \lambda^2\right) f * T_{\lambda,n(\lambda,T)-1} + 2\lambda n(\lambda, T) f * T_{\lambda,n(\lambda,T)} = 0 \quad \textit{in } \mathcal{O}_T.$$

(iii) *If* $\lambda \neq 0$ *and* $n(\lambda, T) \geqslant 2$, *then*

$$\left(\Delta+\lambda^2\right) f * T_{\lambda,\eta} + 2\lambda(\eta+1) f * T_{\lambda,\eta+1} + (\eta+2)(\eta+1) f * T_{\lambda,\eta+2} = 0 \quad \textit{in } \mathcal{O}_T$$

for all $\eta \in \{0, \dots, n(\lambda, T) - 2\}$.
(iv) *If* $0 \in \mathcal{Z}_T$ *and* $n(0, T) \geqslant 1$, *then*

$$\Delta f * T_{0,\eta} + (2\eta + 2)(2\eta + 1) f * T_{0,\eta+1} = 0 \quad \textit{in } \mathcal{O}_T$$

for all $\eta \in \{0, \dots, n(0, T) - 1\}$.

Proof. We recall from Sect. 9.5 that $r(T_{\lambda,\eta}) = r(T)$ for all $\lambda \in \mathcal{Z}_T$ and $\eta \in \{0, \dots, n(\lambda, T)\}$. The required results now follow at once from Proposition 9.13. □

For the rest of the section, we assume that the set $\mathcal{O}$ is O(n)-invariant, that is,

$$\tau\mathcal{O} = \mathcal{O} \quad \text{for each } \tau \in \mathrm{O}(n).$$

We put

$$\mathcal{D}'_{T,\natural}(\mathcal{O}) = \left(\mathcal{D}'_T \cap \mathcal{D}'_\natural\right)(\mathcal{O})$$

and

$$C^m_{T,\natural}(\mathcal{O}) = \left(C^m_T \cap \mathcal{D}'_\natural\right)(\mathcal{O}) \quad \text{for } m \in \mathbb{Z}_+ \text{ or } m = \infty.$$

Proposition 14.5. *Let $f \in \mathcal{D}'(\mathcal{O})$. Then in order that $f \in \mathcal{D}'_T(\mathcal{O})$, it is necessary and sufficient that $f^{k,j} \in \mathcal{D}'_T(\mathcal{O})$ for all $k \in \mathbb{Z}_+$, $j \in \{1, \dots, d(n,k)\}$.*

Proof. First, assume that $f \in \mathcal{D}'_T(\mathcal{O})$. Then relation (9.10) implies that $f^{k,j} \in \mathcal{D}'_T(\mathcal{O})$ for all k, j. The converse statement is obvious from Proposition 9.1(vi). □

Proposition 14.6. *Let $l \in \mathbb{N}$, $f \in (\mathcal{D}'_T \cap L^{1,\mathrm{loc}})(\mathcal{O})$ and assume that f has the form $f(x) = u(\varrho)Y(\sigma)$ for some $Y \in \mathcal{H}_1^{n,k}$. There holds:*

(i) $u(\varrho)Y_j^{(k)}(\sigma) \in (\mathcal{D}'_T \cap L^{1,\mathrm{loc}})(\mathcal{O})$ *for all $j \in \{1, \dots, d(n,k)\}$.*
(ii) *If $f \in C_T^l(\mathcal{O})$, then $(u'(\varrho) - ku(\varrho)\varrho^{-1})Y_j^{(k+1)}(\sigma) \in C_T^{l-1}(\mathcal{O})$ for all $j \in \{1, \dots, d(n,k+1)\}$.*
(iii) *If $f \in C_T^l(\mathcal{O})$ and $k \geqslant 1$, then*

$$\left(u'(\varrho) + \frac{n+k-2}{\varrho}u(\varrho)\right)Y_j^{(k-1)}(\sigma) \in C_T^{l-1}(\mathcal{O})$$

for all $j \in \{1, \dots, d(n,k-1)\}$.

The proof of Proposition 14.6 can be found in [225, Part I, Propositions 5.6, 5.7 and 5.8].

Proposition 14.7. *Let $k \in \mathbb{Z}_+$, $j \in \{1, \dots, d(n,k)\}$, $R > r(T)$, and let $f \in \mathcal{D}'_{k,j}(B_R)$. Then the following assertions hold.*

(i) $f \in \mathcal{D}'_T(B_R)$ *if and only if* $\mathcal{A}_j^k(f) \in \mathcal{D}'_{T,\natural}(B_R)$.
(ii) $f \in \mathcal{D}'_T(B_R)$ *if and only if* $\mathfrak{A}_{k,j}(f) \in \mathcal{D}'_{\Lambda(T),\natural}(-R, R)$.

Proof. The first part follows from Theorem 9.7(ii), (iv), while the second part follows from Theorem 9.3(i), (ii). □

Let us now describe the class $\mathcal{D}'_T(\mathcal{O})$ for the case $r(T) = 0$.

Proposition 14.8. *Let $r(T) = 0$, assume that $\mathcal{O}$ is connected, and let $f \in \mathcal{D}'(\mathcal{O})$. Then $f \in \mathcal{D}'_T(\mathcal{O})$ if and only if for all $k \in \mathbb{Z}_+$, $j \in \{1, \dots, d(n,k)\}$, the following relation is valid:*

$$f^{k,j} = \sum_{\lambda \in \mathcal{Z}_T} \sum_{\eta=0}^{n(\lambda,T)} a_{\lambda,\eta,k,j}\Phi_{\lambda,\eta,k,j} + b_{\lambda,\eta,k,j}\Psi_{\lambda,\eta,k,j},$$

where $a_{\lambda,\eta,k,j}, b_{\lambda,\eta,k,j} \in \mathbb{C}$, and $b_{\lambda,\eta,k,j} = 0$, provided that $0 \in \mathcal{O}$. In addition, if $f \in \mathcal{D}'_T(\mathcal{O})$, then the coefficients $a_{\lambda,\eta,k,j}$ and $b_{\lambda,\eta,k,j}$ are determined uniquely by f.

Proof. The required conclusion follows from [225, Part III, Lemmas 2.10 and 2.13] and Proposition 9.3. □

Analogues of Proposition 14.8 for $r(T) > 0$ will be given in Sect. 14.5. We shall now establish the following result, which will be used later.

Proposition 14.9. *Let $k \in \mathbb{Z}_+$, $j \in \{1, \dots, d(n,k)\}$, and let E be an infinite subset of $\mathbb{C}$ such that $E \cap \mathcal{Z}(\widetilde{T}) = \varnothing$ and $(-z) \in E$ for each $z \in E$. Assume that $\mathcal{O}$ is bounded and connected. Suppose that $\mathring{B}_{r(T)+\varepsilon}(x) \subset \mathcal{O}$ for some $\varepsilon > 0$ and $x \in \mathbb{R}^n$. Let $f \in (\mathcal{D}'_T \cap \mathcal{D}'_{k,j})(\mathcal{O})$, and let*

$$\mathcal{U} = \left\{x \in \mathbb{R}^n \colon \mathring{B}_\varepsilon(x) \subset \mathcal{O}\right\}.$$

Then for each $m \in \mathbb{Z}_+$, there exists $g \in (C_T^m \cap \mathcal{D}'_{k,j})(\mathcal{U})$ such that $p(\Delta)g = f$ in $\mathcal{U}$ for some polynomial p. Moreover, all the zeroes of the polynomial $q(z) = p(-z^2)$ are simple, and $\mathcal{Z}(q) \subset E$.

Proof. We define

$$R = \sup\{r > 0 \colon S_r \subset \mathcal{U}\}$$

and assume that $m \in \mathbb{Z}_+$. If $\eta \in \mathcal{D}_\natural(\mathcal{O})$ and $\eta = 1$ in $\mathcal{U}$, then $f\eta \in \mathcal{E}'_{k,j}(\mathbb{R}^n) \cap \mathcal{D}'_T(\mathcal{U})$. Theorem 9.3(v) implies that $\mathfrak{A}_{k,j}(f\eta)$ is an even distribution of finite order on $(-R, R)$. Let s be the order of $\mathfrak{A}_{k,j}(f\eta)$ on $(-R, R)$, and let p be a polynomial such that $\mathcal{Z}(q) \subset E$ and $\mathcal{Z}(q) \cap \mathcal{Z}(q') = \varnothing$, where $q(z) = p(-z^2)$. If the degree of p is large enough, then the equation

$$p\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right)\Phi = \mathfrak{A}_{k,j}(f\eta)$$

has a solution $\Phi \in C_\natural^{m+2}(-R, R)$. Setting $F = \mathfrak{A}_{k,j}^{-1}(\Phi)$, we conclude from Theorem 9.5(iv) that $F \in C_{k,j}^m(\mathcal{U})$ and

$$F * T_1 = f \quad \text{in } \mathcal{U},$$

where $T_1 = p(\Delta)\delta_0$. Therefore, $F * T * T_1 = 0$ in $\mathcal{U}_T$, which means

$$F * T = \sum_{\lambda \in \mathcal{Z}_{T_1}} \sum_{\eta=0}^{n(\lambda, T_1)} a_{\lambda,\eta}\Phi_{\lambda,\eta,k,j} + b_{\lambda,\eta}\Psi_{\lambda,\eta,k,j} \quad \text{in } \mathcal{U}_T, \tag{14.6}$$

where $a_{\lambda,\eta}, b_{\lambda,\eta} \in \mathbb{C}$, and $b_{\lambda,\eta} = 0$ if $0 \in \mathcal{U}_T$ (see Proposition 14.8). Due to (14.4), (14.5), and (14.6), there exist constants $a'_{\lambda,\eta}, b'_{\lambda,\eta} \in \mathbb{C}$ such that the function

$$g = F - \sum_{\lambda \in \mathcal{Z}_{T_1}} \sum_{\eta=0}^{n(\lambda, T_1)} a'_{\lambda,\eta}\Phi_{\lambda,\eta,k,j} + b'_{\lambda,\eta}\Psi_{\lambda,\eta,k,j}$$

is in the class $C_T^m(\mathcal{U})$, and $b'_{\lambda,\eta} = 0$, provided that $0 \in \mathcal{U}$. This completes the proof. □

14.2 Modern Versions of the John Theorem. Connections with the Spectrum. The Hemisphere Theorem

The classical John's theorem asserts that if $f \in C^{\infty}(\mathbb{R}^n)$ and

$$\int_{\mathbb{S}^{n-1}} f(y+\sigma)\,\mathrm{d}\omega(\sigma) = 0 \quad \text{for all } y \in \mathbb{R}^n,$$

then $f = 0$, provided that $B_1 \cap \operatorname{supp} f = \varnothing$. In this section we shall obtain far-reaching generalizations and refinements of this statement.

For $\nu \in \mathbb{Z}$, we set $\mathcal{M}^{\nu}_{\natural}(\mathbb{R}^n) = (\mathcal{M}^{\nu} \cap \mathcal{E}'_{\natural})(\mathbb{R}^n)$ (see Sect. 1.2). Our first result is as follows.

Theorem 14.1. *Let $\nu, m \in \mathbb{Z}$, $m \geqslant \max\{0, 2[(1-\nu)/2]\}$, $T \in \mathcal{M}^{\nu}_{\natural}(\mathbb{R}^n)$, and let $R > r(T) > 0$. Assume that $f \in \mathcal{D}'_T(B_R)$ and $f = 0$ in $B_{r(T)}$. Then the following assertions hold.*

(i) *If $f \in L^{1,\mathrm{loc}}_m(B_R)$, then $f^{k,j} = 0$ in B_R for $k \leqslant m+\nu+1$, $j \in \{1,\dots,d(n,k)\}$.*
(ii) *If $f \in C^m(B_R)$, then $f^{k,j} = 0$ in B_R for $k \leqslant m+\nu+2$, $j \in \{1,\dots,d(n,k)\}$.*

Notice that assumptions of this theorem cannot be weakened in the general case (see Theorem 14.9 below). To prove the theorem a couple of lemmas will be needed.

Lemma 14.1. *Let $T \in \mathcal{M}^{0}_{\natural}(\mathbb{R}^n)$, $R > r(T) > 0$, $f \in (\mathcal{D}'_T \cap L^{1,\mathrm{loc}}_{\natural})(B_R)$, and suppose $f = 0$ in $B_{r(T)}$. Then $f = 0$ in B_R.*

Proof. Consider a sequence of functions $f_q \in L^{1,\mathrm{loc}}_{\natural}(B_R)$, $q = 1, 2, \dots$, such that

$$f_1 = f, \quad f_{q+1}(x) = \int_0^{|x|} t^{1-n}\,\mathrm{d}t \int_0^t u^{n-1} g_q(u)\,\mathrm{d}u, \tag{14.7}$$

where $g_q : [0, R) \to \mathbb{C}$ is defined by

$$g_q(|x|) = f_q(x).$$

It is easy to verify that $\Delta f_{q+1} = f_q$ in B_R and $f_q = 0$ in $B_{r(T)}$. In addition, (14.7) shows that $f_q \in C^{2q-3}(B_R)$ for $q \geqslant 2$. Now define

$$w_q(x) = (f_q * T)(x), \quad x \in B_{R-r(T)}.$$

Using induction on q, we now prove that $w_q = 0$ for all q. For $q = 1$, this follows from the hypothesis of the lemma. Next, if $w_q = 0$ for some $q \in \mathbb{N}$, then $\Delta w_{q+1} = 0$. Since $T \in \mathcal{M}^{0}_{\natural}(\mathbb{R}^n)$, we conclude that $w_{q+1} \in C_{\natural}(B_{R-r(T)})$. Hence, w_{q+1} is a radial harmonic function in $B_{R-r(T)}$ such that $w_{q+1}(0) = 0$. This yields $w_{q+1} = 0$ in $B_{R-r(T)}$, as required. Thus, $f_q \in C^{2q-3}_T(B_R)$ for all $q \geqslant 2$. Assume now that $2q \geqslant n+5$. Using [225, Part III, Lemma 2.8], one has $f_q = 0$ in B_R. Consequently, $f = \Delta^{q-1} f_q = 0$ in B_R, as we wished to prove. □

Lemma 14.2. *Let* $T \in \mathcal{M}_{\natural}^{\nu}(\mathbb{R}^n)$ *for some* $\nu \in \{0, 1\}$, *and let* $R > r(T) > 0$. *Suppose that* $f \in \mathcal{D}'_T(B_R)$ *and* $f = 0$ *in* $B_{r(T)}$. *Then assertions* (i) *and* (ii) *of Theorem 14.1 are valid.*

Proof. We shall prove assertion (ii) by induction on m. The proof of (i) is quite similar to that of (ii).

Let $m = 0$. This means that $f \in C(B_R)$. Without loss of generality we can assume that f has the form

$$f(x) = \varphi(\varrho)(\sigma_1 + \mathrm{i}\sigma_2)^k,$$

where $k \in \{1, 2, 3\}$ (see Lemma 14.1 and Propositions 14.5 and 14.6(i)). We set

$$U_k(x) = u_k(\varrho), \quad x \in B_R,$$

where

$$u_1(\varrho) = \int_0^{\varrho} \varphi(\xi)\, \mathrm{d}\xi, \tag{14.8}$$

$$u_2(\varrho) = \int_0^{\varrho} \eta \int_0^{\eta} \frac{\varphi(\xi)}{\xi}\, \mathrm{d}\xi\, \mathrm{d}\eta, \tag{14.9}$$

$$u_3(\varrho) = \int_0^{\varrho} \zeta \int_0^{\zeta} \eta \int_0^{\eta} \frac{\varphi(\xi)}{\xi^2}\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}\zeta. \tag{14.10}$$

Also let

$$V_k(x) = (U_k * T)(x), \quad x \in B_{R-r(T)}.$$

Formulae (14.8)–(14.10) show that $V_k \in C_{\natural}^k(B_{R-r(T)})$ and

$$\left(\frac{\partial}{\partial x_1} + \mathrm{i}\frac{\partial}{\partial x_2}\right)^k V_k = 0.$$

This yields

$$V_k(x) = \sum_{s=0}^{k-1} c_s \varrho^{2s}, \quad c_s \in \mathbb{C}. \tag{14.11}$$

Since $U_k \in C^k(B_R)$, $U_k = 0$ in $B_{r(T)}$, and $T \in \mathcal{M}_{\natural}^1(\mathbb{R}^n)$, we obtain

$$\big(\Delta^s V_k\big)(0) = 0, \quad s \in \{0, \ldots, k-1\}.$$

This, together with (14.11), gives $V_k = 0$. By the definition of V_k and Lemma 14.1 we have $U_k = 0$. Then (14.8)–(14.10) imply that $f = 0$.

Assume now that assertion (ii) of Theorem 14.1 is valid for all $m \in \{0, \ldots, l-1\}$ and prove it for $m = l$. Then it is enough to consider the case where $f \in C^l(B_R)$ has the form $f(x) = \varphi(\varrho)(\sigma_1 + \mathrm{i}\sigma_2)^{l+3}$ (see Propositions 14.5 and 14.6(i)). By

Proposition 14.6(iii), the function

$$g(x) = \left(\varphi'(\varrho) + \frac{n+l+1}{\varrho}\varphi(\varrho)\right)(\sigma_1 + \mathrm{i}\sigma_2)^{l+2} \tag{14.12}$$

is in the class $C^{l-1}(B_R)$, and $g = 0$ in $B_{r(T)}$. In view of the induction hypothesis, $g = 0$ in B_R. Since $\varphi = 0$ on $[0, r(T)]$, the desired result is now obvious from (14.12). □

Proof of Theorem 14.1. Let $k \in \mathbb{Z}_+$ and $j \in \{1, \dots, d(n,k)\}$. Then by assumption on f and Proposition 14.5 we deduce that $f^{k,j} \in \mathcal{D}'_T(B_R)$ and $f^{k,j} = 0$ in $B_{r(T)}$. First, assume that $\nu \in \mathbb{Z}$, $\nu < 0$. By the definition of $\mathcal{M}^\nu(\mathbb{R}^n)$ we have

$$T = p(\Delta)T_1$$

for some $T_1 \in \mathcal{M}^0(\mathbb{R}^n)$, where p is a polynomial of degree at most $[(1-\nu)/2]$. Moreover, if ν is odd, we can suppose that $T_1 \in \mathcal{M}^1(\mathbb{R}^n)$. It follows from Theorem 9.2 that $T_1 \in \mathcal{E}'_\natural(\mathbb{R}^n)$ and $r(T_1) = r(T)$. Putting

$$F = p(\Delta)\big(f^{k,j}\big),$$

we obtain $F \in \mathcal{D}'_{T_1}(B_R)$ and $F = 0$ in $B_{r(T)}$. In addition, if $f \in L^{1,\mathrm{loc}}_m(B_R)$ (respectively $f \in C^m(B_R)$), then $F \in L^{1,\mathrm{loc}}_{m-2[(1-\nu)/2]}(B_R)$ (respectively $F \in C^{m-2[(1-\nu)/2]}(B_R)$). Bearing in mind that $f^{k,j} = 0$ in $B_{r(T)}$ and using Lemma 14.2 and Proposition 14.8, we arrive at assertions (i) and (ii) of Theorem 14.1 for $\nu < 0$.

It remains to consider the case $\nu > 0$. In this case $\Delta^{[\nu/2]}T \in \mathcal{M}^0(\mathbb{R}^n)$. Moreover, if ν is odd, we can suppose $\Delta^{[\nu/2]}T \in \mathcal{M}^1(\mathbb{R}^n)$ (see the definition of $\mathcal{M}^\nu(\mathbb{R}^n)$ for $\nu > 0$). The proof of Lemma 1.12 in [225, Part II] shows that there exists $\Phi \in \mathcal{D}'_{k,j}(B_R)$ such that

$$\Delta^{[\nu/2]}\Phi = f^{k,j} \quad \text{in } B_R$$

and $\Phi = 0$ in $B_{r(T)}$. Furthermore, if $f \in L^{1,\mathrm{loc}}_m(B_R)$ (respectively $f \in C^m(B_R)$), then $\Phi \in L^{1,\mathrm{loc}}_{m+2[\nu/2]}(B_R)$ (respectively $\Phi \in C^{m+2[\nu/2]}(B_R)$). Setting

$$T_2 = \Delta^{[\nu/2]}T,$$

we see that $r(T_2) = r(T)$ and $\Phi \in \mathcal{D}'_{T_2}(B_R)$. As before, using Lemma 14.2, we obtain (i) and (ii) with $\nu > 0$. This completes the proof. □

As a consequence of Theorem 14.1, we obtain the following statement.

Corollary 14.1. *Let $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$, $R > r(T) > 0$, and*

$$\big|\widetilde{T}(z)\big| \leqslant \gamma_1(1+|z|)^{\gamma_2}\mathrm{e}^{r(T)|\mathrm{Im}\, z|}, \quad z \in \mathbb{C},$$

where $\gamma_1 > 0$ *and* $\gamma_2 \in \mathbb{R}^1$ *are independent of* z. *Assume that* $f \in C_T^m(B_R)$ *for some integer*

$$m \geqslant \max\left\{0, 2\left[\frac{1-[-\gamma_2-(n+3)/2]}{2}\right]\right\}$$

and $f = 0$ *in* $B_{r(T)}$. *Then*

$$f^{k,j} = 0 \quad \text{in } B_R \quad \text{for } k \leqslant m+1+[-\gamma_2-(n+1)/2],\ j \in \{1,\dots,d(n,k)\}.$$

Proof. By Theorem 9.2 and (9.44) we have $T \in \mathcal{M}^{\nu}(\mathbb{R}^n)$, where $\nu = [-\gamma_2 - (n+3)/2]$. The required result now follows from Theorem 14.1(ii). □

We now state and prove the following uniqueness result.

Theorem 14.2. *Let* $T \in \mathcal{E}'_{\natural}(\mathbb{R}^n)$ *with* $r(T) > 0$, *and let* $\mathcal{O}$ *be a* ζ *domain in* $\mathbb{R}^n$ *with* $\zeta = r(T)$ *such that* $\mathring{B}_{r(T)} \subset \mathcal{O}$ *(see Definition 1.1). Assume that* $f \in \mathcal{D}'_T(\mathcal{O})$ *and* $f = 0$ *in* $B_{r(T)}$. *Then the following statements are valid.*

(i) *If* $f = 0$ *in* $B_{r(T)+\varepsilon}$ *for some* $\varepsilon > 0$, *then* $f = 0$ *in* $\mathcal{O}$.
(ii) *If* $f \in C_T^{\infty}(\mathcal{O})$, *then* $f = 0$ *in* $\mathcal{O}$.
(iii) *If* $T = T_1 + T_2$, *where* $T_1 \in \mathcal{D}_{\natural}(\mathbb{R}^n)$, $T_2 \in \mathcal{E}'_{\natural}(\mathbb{R}^n)$, *and* $r(T_2) < r(T)$, *then* $f = 0$ *in* $\mathcal{O}$.

The results in Sect. 14.3 show that the assumptions in this theorem cannot be considerably weakened.

Proof of Theorem 14.2. Definition 1.1 shows that it is enough to prove statements (i)–(iii) for the case where $\mathcal{O} = B_R$, $R \in (r(T), +\infty]$. The first assertion of the theorem can easily be derived from its second assertion by means of the standard smoothing procedure (see Theorem 6.1(i)), but if $f \in C_T^{\infty}(B_R)$, then by Corollary 14.1 $f^{k,j} = 0$ in B_R for all k, j. In view of Proposition 9.1(vi), this gives (ii).

To prove (iii), first observe that $r(T) = r(T_1)$ and $f \in \mathcal{D}'_{T_1}(B_{r(T)+\varepsilon})$ for some $\varepsilon \in (0, 2r(T) - r(T_2))$. This, together with Proposition 14.7(ii) and Theorem 9.3(ii), implies that

$$\mathfrak{A}_{k,j}\big(f^{k,j}\big) \in \mathcal{D}'_{\Lambda(T_1)}\big(-r(T)-\varepsilon, r(T)+\varepsilon\big)$$

and

$$\mathfrak{A}_{k,j}\big(f^{k,j}\big) = 0 \quad \text{in } \big(-r(T), r(T)\big)$$

for all k, j. By assumption on T_1 and Theorem 6.3 we conclude that $\Lambda(T_1) \in \mathcal{D}_{\natural}(\mathbb{R}^1)$. Then Corollary 13.2 shows that $\mathfrak{A}_{k,j}(f^{k,j})$ vanishes on $(-r(T)-\varepsilon, r(T)+\varepsilon)$. Hence, $f = 0$ in $B_{r(T)+\varepsilon}$ (see Theorem 9.3(ii) and Proposition 9.1(vi)). Assertion (iii) is now obvious from (i). □

To continue, for $r > 0$, let

$$S_r^+ = \big\{x \in \mathbb{R}^n : |x| = r \text{ and } x_1 \geqslant 0\big\}.$$

The following "hemisphere theorem" is a refinement of Theorem 14.2(ii).

Theorem 14.3. *Let $T \in \mathcal{E}'_{\natural}(\mathbb{R}^n)$ with $r(T) > 0$, let $\mathcal{O}$ be a ζ domain in $\mathbb{R}^n$ with $\zeta = r(T)$ containing the ball $\dot{B}_{r(T)}$, and let $f \in \mathcal{D}'_T(\mathcal{O})$. Assume that $f = 0$ in $B_{r(T)}$ and $f \in C^\infty(\mathcal{O}_1)$ for some open subset $\mathcal{O}_1$ of $\mathcal{O}$ such that $S^+_{r(T)} \subset \mathcal{O}_1$. Then $f = 0$ in $\mathcal{O}$.*

We note that the set $S^+_{r(T)}$ in this theorem cannot be decreased in general (see Theorem 14.8(i)).

Proof of Theorem 14.3. By assumption, $B_{r(T)+\varepsilon} \subset \mathcal{O}$ for some $\varepsilon > 0$. According to Theorem 14.2(i), (ii) it is enough to prove that $f \in C^\infty(B_{r(T)+\varepsilon})$. First, suppose that $T \in (\mathcal{E}'_{\natural} \cap C^{n-1})(\mathbb{R}^n)$. Then $T^{\lambda,\eta} \in (\mathcal{E}'_{\natural} \cap C)(\mathbb{R}^n)$ and $r(T^{\lambda,\eta}) = r(T)$ for all $\lambda \in \mathcal{Z}_T$ and $\eta \in \{0, \dots, n(\lambda, T)\}$ (see (9.95) and Proposition 9.12(ii)). In addition, the convolution $F = f * T^{\lambda,\eta}$ satisfies

$$\left(\Delta + \lambda^2\right)^{\eta+1} F = 0$$

in B_ε. By the ellipticity of the operator $(\Delta + \lambda^2)^{\eta+1}$ we have $F \in \mathrm{RA}(B_\varepsilon)$. Let $\mathbf{e}_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$. It follows from the hypothesis of the theorem that there exists $\varepsilon_1 > 0$ such that

$$\lim_{t \to +0} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m F(tx) = 0 \quad \text{when } |x - \mathbf{e}_1| < \varepsilon_1, m \in \mathbb{Z}_+.$$

Since $F \in \mathrm{RA}(B_\varepsilon)$, this yields $F = 0$ in B_ε. Using now Theorem 9.9, we obtain $f = 0$ in $B_{r(T)+\varepsilon}$.

In the general case there exists $T_1 \in (\mathcal{E}'_{\natural} \cap C^{n-1})(\mathbb{R}^n)$ such that $r(T_1) = r(T)$ and $p(\Delta) T_1 = T$ for some polynomial p. This shows that

$$p(\Delta) f \in \mathcal{D}'_{T_1}(B_{r(T)+\varepsilon}), \quad p(\Delta) f = 0 \quad \text{in } B_{r(T_1)}, \quad \text{and} \quad p(\Delta) f \in C^\infty(\mathcal{O}_1).$$

As above, we infer that $p(\Delta) f = 0$ in $B_{r(T)+\varepsilon}$. Thus, $f \in \mathrm{RA}(B_{r(T)+\varepsilon})$, and the theorem is completely proved. □

One corollary is worth recording.

Corollary 14.2. *Let $T \in \mathcal{E}'_{\natural}(\mathbb{R}^n)$ with $r(T) > 0$, and assume that $p(\Delta)T = 0$ in $B_{r(T)}$ for some nonzero polynomial p. Suppose that $R > r(T)$ and $\varepsilon \in (0, R - r(T))$. Then the following results are true.*

(i) *Let $f \in \mathcal{D}'_T(B_R)$, $f = 0$ in $B_{r(T)-\varepsilon, r(T)}$, and $f \in C^\infty(\mathcal{O})$, where $\mathcal{O}$ is an open subset of B_R such that $S^+_{r(T)} \subset \mathcal{O}$. Then $f = 0$ in $B_{r(T)-\varepsilon, r(T)+\varepsilon}$.*

(ii) *Let $f \in C^\infty_T(B_R)$ and $f = 0$ in $B_{r(T), r(T)+\varepsilon}$. Then $f = 0$ in $B_{r(T)-\varepsilon, r(T)+\varepsilon}$.*

Proof. Setting $\psi = p(\Delta)T$, we see that $\psi \in \mathcal{E}'_{\natural}(\mathbb{R}^n)$, $\operatorname{supp} \psi = S_{r(T)}$, and $\mathcal{D}'_T(B_R) \subset \mathcal{D}'_\psi(B_R)$. To prove (i) it is enough to consider the case $\varepsilon \leqslant r(T)$ (see Theorem 14.3). We define $g \in \mathcal{D}'(B_{r(T)+\varepsilon})$ by letting $g = f$ on $B_{r(T)-\varepsilon, r(T)+\varepsilon}$ and $g = 0$ in $B_{r(T)}$. Then $g \in \mathcal{D}'_\psi(B_{r(T)+\varepsilon})$, and part (i) must be valid because of Theorem 14.3.

Regarding (ii), define $\varphi \in \mathcal{D}_\natural(\mathbb{R}^n)$ by letting $\varphi = f^{0,1}$ on $B_{r(T)+\varepsilon}$ and $\varphi = 0$ in $B_{r(T),\infty}$. By our hypothesis and Proposition 14.5, $\psi \in \mathcal{D}'_\varphi(B_{r(T)+\varepsilon})$. Since $\psi = 0$ in $B_{r(T)}$, Theorem 14.2(iii) yields $f^{0,1} = 0$ in $B_{r(T)-\varepsilon,r(T)+\varepsilon}$. Owing to Lemma 9.4 and Theorem 9.7(iv), $f^{k,j} = 0$ in $B_{r(T)-\varepsilon,r(T)+\varepsilon}$ for all $k \in \mathbb{Z}_+$ and $j \in \{1,\dots,d(n,k)\}$. Combining this with Proposition 9.1(vi), we arrive at (ii). □

We shall now consider the case where $\mathcal{O} = \mathbb{R}^n$.

Theorem 14.4. *Let $T = T_1 * T_2$, where $T_1, T_2 \in \mathcal{E}'_\natural(\mathbb{R}^n)$ and $r(T_2) > 0$. Assume that $f \in \mathcal{D}'_T(\mathbb{R}^n)$ and $f = 0$ in $B_{r(T)}$. Then*

(i) *If $T_2 \in \mathrm{Inv}(\mathbb{R}^n)$ and*

$$\frac{|\mathrm{Im}\,\lambda|}{\log(2+|\lambda|)} \to +\infty \quad \text{as } \lambda \to \infty, \lambda \in \mathcal{Z}_{T_2},$$

then $f = 0$.

(ii) *If $T_2 \in \mathfrak{E}(\mathbb{R}^n)$ and f is of finite order, then $f = 0$.*

Proof. Let $k \in \mathbb{Z}_+$, $j \in \{1,\dots,d(n,k)\}$, and let

$$F = f^{k,j} * T_1.$$

Then $F \in \mathcal{D}'_{T_2}(\mathbb{R}^n)$ and $F = 0$ in $B_{r(T_2)}$. Assumptions in (i) and (ii) show that $F \in C^\infty(\mathbb{R}^n)$ (see Hörmander [126], Theorem 16.6.5, and Theorem 14.18(ii) below). Because of Corollary 14.1, $F = 0$ in $\mathbb{R}^n$, which means that $f^{k,j} \in \mathcal{D}'_{T_1}(\mathbb{R}^n)$. Bearing in mind that $r(T_1) = r(T) - r(T_2) < r(T)$ and using Theorem 14.2(i) and Proposition 14.8, we obtain $f^{k,j} = 0$ in $\mathbb{R}^n$. Assertions (i) and (ii) now follow from Proposition 9.1(vi). □

Theorem 14.5. *Let $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$ with $r(T) > 0$. Then the following assertions are true.*

(i) *If $T \notin \mathrm{Inv}_+(\mathbb{R}^n)$, $f \in \mathcal{D}'_T(\mathbb{R}^n)$, and $f = 0$ in $B_{r(T)}$, then $f = 0$ in $\mathbb{R}^n$.*

(ii) *If $T \in \mathrm{Inv}_+(\mathbb{R}^n)$, then there exists nonzero $f \in (\mathcal{D}'_T \cap \mathcal{D}'_\natural)(\mathbb{R}^n)$ vanishing in the ball $B_{r(T)}$.*

We point out that the description of the set

$$\left\{f \in \mathcal{D}'_T(B_R)\colon\ f = 0 \text{ in } B_{r(T)}\right\}$$

for $T \in \mathrm{Inv}_+(\mathbb{R}^n)$, $R \in (r(T), +\infty]$ can be found in the following section (see Theorem 14.10).

Proof of Theorem 14.5. It is enough to prove (i) for the case where $f \in \mathcal{D}'_{k,j}(\mathbb{R}^n)$ with $k \in \mathbb{Z}_+$, $j \in \{1,\dots,d(n,k)\}$ (see Propositions 14.5 and 9.1(vi)). Proposition 14.7(ii) and Theorem 9.3(ii) show that $\mathfrak{A}_{k,j}(f) \in \mathcal{D}'_{\Lambda(T)}(\mathbb{R}^1)$ and

$$\mathfrak{A}_{k,j}(f) = 0 \quad \text{on } \bigl(-r(T),\ r(T)\bigr).$$

Since $\Lambda(T) \notin (\mathrm{Inv}_- \cap \mathrm{Inv}_+)(\mathbb{R}^1)$, Theorem 13.3(i) yields $\mathfrak{A}_{k,j}(f) = 0$ in $\mathbb{R}^1$. This, together with Theorem 9.3(ii), gives (i).

As for (ii), observe that there exists nonzero $g \in \mathcal{D}'_{\Lambda(T),\natural}(\mathbb{R}^1)$ such that $g = 0$ on $(-r(T), r(T))$ (see Proposition 13.1(i) and Theorem 13.3(ii)). Now define $f = \mathfrak{A}_{0,1}^{-1}(g)$. By Corollary 9.2 and Theorem 9.5(i) we infer that f is nonzero and $f \in \mathcal{D}'_{T,\natural}(\mathbb{R}^n)$. In addition, $f = 0$ in $B_{r(T)}$, which brings us to the desired result.

□

14.3 Multidimensional Analogues of the Distribution ζ_T. Mean Periodic Functions with Support in Exterior of a Ball. Exactness of Uniqueness Theorems

Our main purpose in this section is to show that the results in Sect. 14.2 are precise. First of all, we prove that these results fail in general without the assumption that T is a radial distribution.

Theorem 14.6. *Let F be a finite subgroup of the orthogonal group $\mathrm{O}(n)$. Assume that a distribution $T \in \mathcal{E}'(\mathbb{R}^n)$ has the form*

$$T = T_1 T_2,$$

where $T_1 \in \mathcal{E}'_\natural(\mathbb{R}^n)$, $T_1 \neq 0$, and $T_2 \colon \mathbb{R}^n \to \mathbb{R}^1$ is an F-invariant homogeneous harmonic polynomial with $\deg T_2 \geqslant 1$. Then for each $R > 0$, there exists a nonzero F-invariant function $f_R \in C_T^\infty(\mathbb{R}^n)$ such that $f_R = 0$ in B_R.

Proof. First, note that there exists $\eta \in \mathbb{R}^n$, $\eta \neq 0$, such that $T_2(\lambda\eta) = 0$ for all $\lambda \in \mathbb{R}^1$. This follows from the assumption on T_2 and from the fact that harmonic polynomials of different degrees are orthogonal in the space $L^2(\mathbb{S}^{n-1})$ (see Lemma 4.2). Let $R > 0$, and let $u \in C^\infty(\mathbb{R}^1)$ be a nonzero function such that $u = 0$ on $(-\infty, R]$ and $u > 0$ on $(R, +\infty)$. Now define

$$f_R(x) = \sum_{\alpha \in F} u(\langle \alpha x, \eta \rangle_{\mathbb{R}}), \qquad x \in \mathbb{R}^n. \tag{14.13}$$

Then f_R is a nonzero F-invariant function in the class $C^\infty(\mathbb{R}^n)$, and $f_R = 0$ in B_R. Next, since T_1 is radial, for $x \in \mathbb{R}^n$, one has

$$(f_R * T)(x) = \langle T_1(y), T_2(y) f_R(x-y) \rangle = \left\langle T_1(y), \int_{\mathrm{O}(n)} T_2(\tau y) f_R(x - \tau y)\, \mathrm{d}\tau \right\rangle,$$

where $\mathrm{d}\tau$ is the Haar measure on $\mathrm{O}(n)$ normalized by

$$\int_{\mathrm{O}(n)} \mathrm{d}\tau = 1.$$

Using now (1.55) with $G = \mathrm{O}(n)$, $G/K = \mathbb{S}^{n-1}$, from (14.13) and the Funk–Hecke theorem (see [225, Part I, Theorem 5.1]) we conclude that $f_R \in C_T^\infty(\mathbb{R}^n)$. Hence the theorem. □

Assume now that $T \in \mathrm{Inv}_+(\mathbb{R}^n)$. Recall from Propositions 8.20(ii) that the distribution $\zeta_{\Lambda(T)}$ is odd and for $k \in \mathbb{Z}_+$, $j \in \{1, \dots, d(n,k)\}$, define $\zeta_{T,k,j} \in \mathcal{D}'_{k,j}(\mathbb{R}^n)$ by the formula

$$\zeta_{T,k,j} = -\mathfrak{A}_{k,j}^{-1}\big(\zeta'_{\Lambda(T)}\big). \tag{14.14}$$

We now establish basic properties of $\zeta_{T,k,j}$.

Theorem 14.7. *For $T \in \mathrm{Inv}_+(\mathbb{R}^n)$, the following statements are valid.*

(i) $\zeta_{T,k,j} \in (\mathcal{D}'_T \cap \mathcal{D}'_{k,j})(\mathbb{R}^n)$ *and* $\mathcal{A}_j^k(\zeta_{T,k,j}) = \zeta_{T,0,1}$.
(ii) *If* $r(T) > 0$, *then* $\zeta_{T,k,j} = 0$ *in* $B_{r(T)}$ *and* $S_{r(T)} \subset \operatorname{supp} \zeta_{T,k,j}$.
(iii) *If* $R > r(T)$, $x \in \mathbb{R}^n$, $u \in \mathcal{E}'_\natural(\mathbb{R}^n)$, *and* $\zeta_{T,k,j} * u = 0$ *in* $B_R(x)$, *then*

$$u = T * v$$

for some $v \in \mathcal{E}'_\natural(\mathbb{R}^n)$.
(iv) *If* $T \in \mathfrak{M}(\mathbb{R}^n)$, *then*

$$\zeta_{T,k,j} = \sum_{\eta=0}^{n(0,T)} \frac{a^{0,0}_{2(n(0,T)-\eta)}(\widetilde{T})}{(2\eta)!}\Phi_{0,\eta,k,j} + 2\sum_{\lambda\in\mathcal{Z}_T\setminus\{0\}}\ \sum_{\eta=0}^{n(\lambda,T)} a^{\lambda,\eta}_{n(\lambda,T)}\big(\widetilde{T}\big)(\eta\Phi_{\lambda,\eta-1,k,j} + \lambda\Phi_{\lambda,\eta,k,j}), \tag{14.15}$$

where the series converges in $\mathcal{D}'(\mathbb{R}^n)$, *and the first sum is set to be equal to zero if* $0 \notin \mathcal{Z}_T$.
(v) *If* $T \in \mathfrak{M}(\mathbb{R}^n)$, $m \in \mathbb{Z}_+$, *and* $R \in (0, +\infty)$, *then there exists a positive constant* $\sigma = \sigma(m, R, T)$ *such that* $\zeta_{T,k,j} \in C^m(B_R)$, *provided that* $k > \sigma$. *In addition, if*

$$\frac{n(\lambda,T) + |\mathrm{Im}\,\lambda|}{\log(2 + |\lambda|)} \to 0 \quad \text{as } \lambda \to \infty, \lambda \in \mathcal{Z}_T, \tag{14.16}$$

then the same is true with $R = +\infty$.

Proof. First, observe that

$$\mathfrak{A}_{k,j}(\zeta_{T,k,j} * u) = -\zeta'_{\Lambda(T)} * \Lambda(u) \tag{14.17}$$

for each $u \in \mathcal{E}'_\natural(\mathbb{R}^n)$ (see (14.14) and Theorem 9.3(i)). Putting $u = T$ and using Theorem 9.3(ii) and Proposition 8.20(i), we have $\zeta_{T,k,j} \in \mathcal{D}'_T(\mathbb{R}^n)$. This, together with (14.14) and Theorem 9.3(i), implies (i).

Let us prove (ii). Theorem 9.3(ii) and Proposition 8.20(v) show that $\zeta_{T,k,j} = 0$ in $B_{r(T)}$ and $S_{r(T)} \cap \operatorname{supp} \zeta_{T,k,j} \neq \varnothing$. Since $\zeta_{T,k,j} \in \mathcal{D}'_{k,j}(\mathbb{R}^n)$, it follows by Proposition 9.1(v) that $S_{r(T)} \subset \operatorname{supp} \zeta_{T,k,j}$. Thus, (ii) is established.

Next, if $u \in \mathcal{E}'_\natural(\mathbb{R}^n)$, then $\zeta_{T,k,j} * u \in \mathcal{D}'_T(\mathbb{R}^n)$. Theorem 14.2(i) ensures us that if assumptions of (iii) are satisfied, then $\zeta_{T,k,j} * u = 0$ in $\mathbb{R}^n$. Applying now Proposition 8.20(iv), we see from (14.17) that

$$\big(\Lambda(u)\big)' = \Lambda(T) * \Psi'$$

for some $\Psi \in \mathcal{E}'_\natural(\mathbb{R}^1)$. Since $\Lambda(u)$, $\Lambda(T)$, $\Psi \in \mathcal{E}'(\mathbb{R}^1)$, this gives (iii). Part (iv) is a consequence of (14.14), Theorem 8.5, and (9.60). As for (v), observe that if $m \in \mathbb{Z}_+$, $R \in (0, +\infty)$ are fixed, and k is large enough, then the series in (14.15) converges in $C^m(\mathcal{K})$ for each nonempty compact set $\mathcal{K} \subset B_R$ (see Proposition 9.7 and (9.90)). In particular this yields $\zeta_{T,k,j} \in C^m(B_R)$. For the case where $R = +\infty$ and (14.16) is fulfilled, the arguments are similar. This concludes the proof. □

Theorem 14.8. *Let $T \in \mathrm{Inv}_+(\mathbb{R}^n)$, $r(T) > 0$. Then the following assertions hold.*

(i) *There exists nonzero $f \in \mathcal{D}'_T(\mathbb{R}^n)$ such that $f = 0$ in the strip $\{x \in \mathbb{R}^n : |x_1| < r(T)\}$.*
(ii) *For each $\varepsilon \in (0, r(T))$, there exists nonzero $f \in C^\infty_{T,\natural}(\mathbb{R}^n)$ such that $f = 0$ in $B_{r(T)-\varepsilon}$.*
(iii) *If $T \in \mathfrak{M}(\mathbb{R}^n)$, $m \in \mathbb{Z}_+$, and F is a finite subgroup of $\mathrm{O}(n)$, then for each $R \in (r(T), +\infty)$, there exists nonzero F-invariant $f \in \mathcal{D}'_T(\mathbb{R}^n)$ such that $f = 0$ in $B_{r(T)}$ and $f \in C^m_T(B_R)$. Moreover, if (14.16) is satisfied, then the same is valid with $R = +\infty$.*

We note that (i) is no longer valid for the class $(\mathcal{D}'_T \cap L^{1,\mathrm{loc}})(\mathbb{R}^n)$, provided that $T \in L^1(\mathbb{R}^n)$ (see Theorem 14.2(i)). In addition, the assumption that $T \in \mathrm{Inv}_+(\mathbb{R}^n)$ in (ii) is not necessary (see the proof of Theorem 13.3(iv)).

It is convenient to prove first the following simple lemma.

Lemma 14.3. *For each $m > 0$, there exist $k \in \mathbb{Z}_+$ and $j \in \{1, \dots, d(n,k)\}$ such that $k > m$ and*

$$\sum_{\alpha \in F} Y_j^k(\alpha\sigma) \neq 0 \quad \textit{for some } \sigma \in \mathbb{S}^{n-1}.$$

Proof. Take $g \in C^\infty(\mathbb{S}^{n-1})$ such that the function

$$f(\sigma) = \sum_{\alpha \in F} g(\alpha\sigma), \quad \sigma \in \mathbb{S}^{n-1},$$

is nonzero and $\operatorname{supp} f \neq \mathbb{S}^{n-1}$. Writing the Fourier series of f into spherical harmonics (see (9.3) with $\varrho = 1$) and using (9.1), we arrive at the desired result. □

Proof of Theorem 14.8. First, define $f \in \mathcal{D}'_T(\mathbb{R}^n)$ by letting

$$f(\cdot) = \zeta_{\Lambda(T)}(\langle \cdot, \mathbf{e}_1 \rangle_{\mathbb{R}^n}),$$

where $\mathbf{e}_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$. Then f satisfies all the requirements of (i) (see Proposition 8.20). Assertion (ii) can be obtained from Theorem 14.5(ii) by means of the standard smoothing trick. Turning to (iii), we set

$$f(x)=\sum_{\alpha\in F}\zeta_{T,k,j}(\alpha x),\quad x\in\mathbb{R}^n.$$

Then f is F-invariant and $f\in\mathcal{D}'_T(\mathbb{R}^n)$. In addition, Theorem 14.7(ii), (v) and Lemma 14.3 show that for some $k\in\mathbb{Z}_+$ and $j\in\{1,\dots,d(n,k)\}$, the distribution f satisfies all the requirements of (iii). Hence the theorem. □

Next, for $\alpha>-1$ and $r>0$, we define the function $\Upsilon_{\alpha,r}\in L^1(\mathbb{R}^n)$ by letting

$$\Upsilon_{\alpha,r}(x)=\begin{cases}\left(r^2-|x|^2\right)^{\alpha} & \text{if } |x|<r,\\ 0 & \text{if } |x|\geqslant r.\end{cases}$$

Also let $\Upsilon_{-1,r}\in\mathcal{E}'_\natural(\mathbb{R}^n)$ be the distribution with action on $\mathcal{E}(\mathbb{R}^n)$ described by the formula

$$\langle\Upsilon_{-1,r},u\rangle=\int_{S_r}u(x)\,\mathrm{d}\omega(x),\quad u\in\mathcal{E}\left(\mathbb{R}^n\right),$$

where $\mathrm{d}\omega$ is the surface measure on S_r. A calculation yields

$$\widetilde{\Upsilon}_{\alpha,r}(z)=\begin{cases}2^{\frac{n}{2}+\alpha}\pi^{\frac{n}{2}}\Gamma(1+\alpha)r^{n+\alpha}\mathbf{I}_{\frac{n}{2}+\alpha}(rz) & \text{if } \alpha>-1,\\ (2\pi)^{\frac{n}{2}}r^{n-1}\mathbf{I}_{\frac{n}{2}-1}(rz) & \text{if } \alpha=-1\end{cases}\tag{14.18}$$

(see (5.2) and [225, Part I, Example 6.1]).

The following result shows that Theorem 14.1 cannot be reinforced for a broad class of distributions T.

Theorem 14.9. *Let $\alpha\geqslant-1$, $r>0$, and let $T=\Upsilon_{\alpha,r}+T_1$ where $T_1\in C^m_\natural(\mathbb{R}^n)$, $m\geqslant\alpha+2$, and* supp $T_1\subset\mathring{B}_r$. *Assume that $k\in\mathbb{Z}_+$ and $j\in\{1,\dots,d(n,k)\}$. Then $T\in(\mathcal{M}^{1+[\alpha]}\cap\mathfrak{N})(\mathbb{R}^n)$, and the following assertions hold.*

(i) *If $\alpha\notin\mathbb{Z}$, then $\zeta_{T,k,j}\in L^{p,\mathrm{loc}}_{[k-\alpha-2]}(\mathbb{R}^n)$, provided that $k>\alpha+2$, $p=(1-\{k-\alpha-2\})^{-1}$, and $\zeta_{T,k,j}\in C^{[k-\alpha-3]}(\mathbb{R}^n)$ for $k>\alpha+3$.*

(ii) *If $\alpha\in\mathbb{Z}$, then $\zeta_{T,k,j}\in L^{p,\mathrm{loc}}_{k-\alpha-3}(\mathbb{R}^n)$, provided that $k\geqslant\alpha+3$, $p\in[1,+\infty)$, and $\zeta_{T,k,j}\in C^{k-\alpha-4}(\mathbb{R}^n)$ for $k\geqslant\alpha+4$.*

It can be shown that the assumptions on p in this theorem cannot be weakened for a broad class of distributions T (see Zaraisky [275] and Theorem 14.7).

Proof of Theorem 14.9. By the definitions of T and $\mathcal{M}^{\nu}(\mathbb{R}^n)$ we see that $T\in\mathcal{M}^{1+[\alpha]}(\mathbb{R}^n)$. Using now (14.18), (7.3), and Theorem 9.2(i), we obtain

$$\widetilde{T}(z)=c_1\mathbf{I}_{\frac{n}{2}+\alpha}(rz)+O\left(\frac{\mathrm{e}^{|\mathrm{Im}\,z|r}}{|z|^{(n-1)/2+m}}\right)\tag{14.19}$$

and

$$\widetilde{T}'(z) = c_2 z \mathbf{I}_{\frac{n}{2}+\alpha+1}(rz) + O\left(\frac{e^{|\operatorname{Im} z|r}}{|z|^{(n-1)/2+m}}\right) \tag{14.20}$$

as $z \to \infty$, $\operatorname{Re} z \geqslant 0$, where $c_1, c_2 \in \mathbb{C}\backslash\{0\}$ are independent of z. Let $\{\lambda_m\}_{m=1}^{\infty}$ be the sequence of the zeros of $\widetilde{T}$ in the half-plane $\operatorname{Re} z \geqslant 0$ arranged according to increasing modulus. We write each zero as many times as its multiplicity, and for zeros with equal absolute values, the numbering is chosen arbitrarily. By (14.19) and (7.8),

$$\lambda_m = \frac{\pi}{r}\left(m + \frac{n-1+2\alpha}{4} + l_{\alpha,n}\right) + O\left(m^{-1}\right) \quad \text{as } m \to \infty, \tag{14.21}$$

where $l_{\alpha,n} \in \mathbb{Z}$ is independent of m. Combining this with (14.20) and (7.8), we find

$$\widetilde{T}'(\lambda_m) = c_3 \frac{(-1)^m}{m^{(n+1)/2+\alpha}} + O\left(m^{-(\frac{n+3}{2}+\alpha)}\right) \quad \text{as } m \to \infty, \tag{14.22}$$

where $c_3 \in \mathbb{C}\backslash\{0\}$ is independent of m. In particular, all the zeros of $\widetilde{T}$ with sufficiently large absolute values are simple, and $T \in \mathfrak{N}(\mathbb{R}^n)$. Let $k > \alpha + 2$, and let $s \in \mathbb{Z}_+$, $s < k - \alpha - 2$. Using (14.15), (14.21), (14.22), and Proposition 7.1, we deduce

$$\left(\frac{d}{d\varrho}\right)^s \left(\zeta_{T,k,j}(\varrho\sigma)\right) = Y_j^{(k)}(\sigma)\Bigg(u_1(\varrho) + u_2(\varrho)\sum_{m=1}^{\infty}\frac{\cos(\pi m(1+\varrho/r))}{m^{k-\alpha-2-s}}$$
$$+ u_3(\varrho)\sum_{m=1}^{\infty}\frac{\sin(\pi m(1+\varrho/r))}{m^{k-\alpha-2-s}}\Bigg),$$

where $\varrho > r/2$, $u_1 \in C(r/2, +\infty)$, $u_2, u_3 \in C^{\infty}(r/2, +\infty)$. Since $\zeta_{T,k,j} = 0$ in B_r, this, together with Edwards [60, Sect. 7.3.5(ii)], brings us to (i) and (ii). □

For the final step, we characterize the set

$$\left\{f \in \mathcal{D}'_T(B_R) \colon f = 0 \quad \text{in } B_r\right\},$$

where $0 < r \leqslant r(T) < R \leqslant +\infty$. According to Proposition 14.5, it is enough to describe distributions in the class $(\mathcal{D}'_T \cap \mathcal{D}'_{k,j})(B_R)$ vanishing on B_r.

Theorem 14.10. *Let $T \in \operatorname{Inv}_+(\mathbb{R}^n)$ with $r(T) > 0$. Then the following assertions hold.*

(i) *If $0 < r \leqslant r(T) < R \leqslant +\infty$ and $f \in (\mathcal{D}'_T \cap \mathcal{D}'_{k,j})(B_R)$, then in order that $f = 0$ in B_r, it is necessary and sufficient that*

$$f = \zeta_{T,k,j} * U \quad \textit{in } B_R \tag{14.23}$$

for some $U \in \mathcal{E}'_{\natural}(\mathbb{R}^n)$ with $\operatorname{supp} U \subset \mathring{B}_{r(T)-r}$.

(ii) *If* $R \in (r(T), +\infty]$ *and* $f \in (C_T^\infty \cap \mathcal{D}'_{k,j})(B_R)$, *then in order that* $(Df)(0) = 0$ *for each differential operator* D, *it is necessary and sufficient that relation* (14.23) *is satisfied for some* $U \in \mathcal{D}_\natural(\mathbb{R}^n)$ *with* supp $U \subset \mathring{B}_{r(T)}$.

Proof. To prove (i), first assume that $f \in (\mathcal{D}'_T \cap \mathcal{D}'_{k,j})(B_R)$. By Proposition 14.7(ii) we have $\mathfrak{A}_{k,j}(f) \in \mathcal{D}'_{\Lambda(T),\natural}(-R, R)$. Theorem 9.3(ii) shows that $\mathfrak{A}_{k,j}(f) = 0$ in $(-r, r)$ if and only if $f = 0$ in B_r. Because of Remark 13.1, in order that $\mathfrak{A}_{k,j}(f) = 0$ in $(-r, r)$, it is necessary and sufficient that

$$\mathfrak{A}_{k,j}(f) = \zeta'_{\Lambda(T)} * u \quad \text{in } (-R, R)$$

for some $u \in \mathcal{E}'_\natural(\mathbb{R}^1)$ with supp $u \subset [r - r(T), r(T) - r]$. Using now (14.14) and Theorem 9.5(i), we arrive at (i).

The same argument works for (ii) except that we now apply Theorems 13.4(ii), 9.3(vi) and 9.5(vi). □

14.4 Analogues of the Taylor and the Laurent Expansions for Mean Periodic Functions. Estimates of the Coefficients

Throughout the section we assume that $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$, $T \neq 0$, and that $\mathcal{O}$ is an $\mathrm{O}(n)$-invariant domain in $\mathbb{R}^n$ such that $\mathcal{O}_T \neq \varnothing$ (see (14.1)).

Let

$$\lambda \in \mathcal{Z}_T, \quad \eta \in \{0, \dots, n(\lambda, T)\}, \quad k \in \mathbb{Z}_+, \quad j \in \{1, \dots, d(n, k)\}.$$

For each $f \in \mathcal{D}'_T(\mathcal{O})$, we define the coefficients $a_{\lambda,\eta,k,j}(T, f)$ and $b_{\lambda,\eta,k,j}(T, f)$ as follows. Bearing in mind that $f^{k,j} \in (\mathcal{D}'_T \cap \mathcal{D}'_{k,j})(\mathcal{O})$ (see Proposition 14.5), we derive from Proposition 14.4

$$\left(\Delta + \lambda^2\right)^{n(\lambda,T)+1}\left(f^{k,j} * T_{\lambda,0}\right) = 0 \quad \text{in } \mathcal{O}_T.$$

Because of Proposition 14.8, there exist complex constants $a_{\lambda,\eta,k,j}(T, f)$ and $b_{\lambda,\eta,k,j}(T, f)$ such that

$$f^{k,j} * T_{\lambda,0} = \sum_{\eta=0}^{n(\lambda,T)} a_{\lambda,\eta,k,j}(T, f)\Phi_{\lambda,\eta,k,j} + b_{\lambda,\eta,k,j}(T, f)\Psi_{\lambda,\eta,k,j} \quad \text{in } \mathcal{O}_T \tag{14.24}$$

and $b_{\lambda,\eta,k,j}(T, f) = 0$, provided that $0 \in \mathcal{O}$.

By definition,

$$a_{\lambda,\eta,k,j}(T, f) = a_{\lambda,\eta,k,j}\left(T, f^{k,j}\right) \tag{14.25}$$

and

$$b_{\lambda,\eta,k,j}(T, f) = b_{\lambda,\eta,k,j}\left(T, f^{k,j}\right). \tag{14.26}$$

Next, for all $f_1, \dots, f_m \in \mathcal{D}'_T(\mathcal{O})$ and $c_1, \dots, c_m \in \mathbb{C}$,

$$a_{\lambda,\eta,k,j}\left(T, \sum_{\nu=1}^{m} c_\nu f_\nu\right) = \sum_{\nu=1}^{m} c_\nu a_{\lambda,\eta,k,j}(T, f_\nu) \tag{14.27}$$

and

$$b_{\lambda,\eta,k,j}\left(T, \sum_{\nu=1}^{m} c_\nu f_\nu\right) = \sum_{\nu=1}^{m} c_\nu b_{\lambda,\eta,k,j}(T, f_\nu). \tag{14.28}$$

Assume now that $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$, where $\mathcal{O}_1$ and $\mathcal{O}_2$ are $\mathrm{O}(n)$-invariant domains in $\mathbb{R}^n$ such that $\mathring{B}_{r(T)}(x) \subset \mathcal{O}_1 \cap \mathcal{O}_2$ for some $x \in \mathbb{R}^n$. According to what has been said above, for $f \in \mathcal{D}'_T(\mathcal{O})$, we can write

$$a_{\lambda,\eta,k,j}(T, f|_{\mathcal{O}_1}) = a_{\lambda,\eta,k,j}(T, f|_{\mathcal{O}_2}) = a_{\lambda,\eta,k,j}(T, f). \tag{14.29}$$

In addition,

$$b_{\lambda,\eta,k,j}(T, f|_{\mathcal{O}_1}) = b_{\lambda,\eta,k,j}(T, f|_{\mathcal{O}_2}) = b_{\lambda,\eta,k,j}(T, f) \tag{14.30}$$

(see Proposition 9.3).

Other general properties of $a_{\lambda,\eta,k,j}(T, f)$ and $b_{\lambda,\eta,k,j}(T, f)$ are contained in the following propositions.

Proposition 14.10.

(i) *If* $\nu \in \{0, \dots, n(\lambda, T)\}$ *and* $f \in \mathcal{D}'_T(\mathcal{O})$, *then*

$$f^{k,j} * T_{\lambda,\nu} = \sum_{\mu=0}^{n(\lambda,T)-\nu} \binom{\nu+\mu}{\nu}_{\lambda} \big(a_{\lambda,\nu+\mu,k,j}(T, f)\Phi_{\lambda,\mu,k,j} + b_{\lambda,\nu+\mu,k,j}(T, f)\Psi_{\lambda,\mu,k,j}\big) \quad \text{in } \mathcal{O}_T \quad (\text{see } (14.3)). \tag{14.31}$$

(ii) *Let* $f_m \in \mathcal{D}'_T(\mathcal{O})$, $m = 1, 2, \dots$, *and suppose that* $f_m \to f$ *in* $\mathcal{D}'(\mathcal{O})$ *as* $m \to \infty$. *Then* $a_{\lambda,\eta,k,j}(T, f_m) \to a_{\lambda,\eta,k,j}(T, f)$ *and* $b_{\lambda,\eta,k,j}(T, f_m) \to b_{\lambda,\eta,k,j}(T, f)$ *as* $m \to \infty$.

(iii) *Let* $f \in \mathcal{D}'_T(\mathcal{O})$, $u \in \mathcal{E}'_\natural(\mathbb{R}^n)$, *and assume that* $\mathcal{O}_{T*u} \neq \varnothing$. *Then*

$$a_{\lambda,\eta,k,j}(T, f * u) = \sum_{\nu=\eta}^{n(\lambda,T)} a_{\lambda,\nu,k,j}(T, f) \binom{\nu}{\eta}_{\lambda} \widetilde{u}^{\langle \nu-\eta \rangle}(\lambda)$$

and

$$b_{\lambda,\eta,k,j}(T, f * u) = \sum_{\nu=\eta}^{n(\lambda,T)} b_{\lambda,\nu,k,j}(T, f) \binom{\nu}{\eta}_{\lambda} \widetilde{u}^{\langle \nu-\eta \rangle}(\lambda).$$

In particular, for each polynomial p,

$$a_{\lambda,\eta,k,j}(T, p(\Delta)f) = \sum_{\nu=\eta}^{n(\lambda,T)} a_{\lambda,\nu,k,j}(T, f)\binom{\nu}{\eta}_{\lambda} q^{\langle \nu-\eta\rangle}(\lambda)$$

and

$$b_{\lambda,\eta,k,j}(T, p(\Delta)f) = \sum_{\nu=\eta}^{n(\lambda,T)} b_{\lambda,\nu,k,j}(T, f)\binom{\nu}{\eta}_{\lambda} q^{\langle \nu-\eta\rangle}(\lambda),$$

where $q(z) = p(-z^2)$.

Proof. The proof of (i) can be obtained by induction on ν using (14.24), Theorem 9.1, and Proposition 14.4. To verify (ii) it is enough to use Proposition 9.1(i) and apply (i) repeatedly for $\nu = n(\lambda, T), \dots, 0$. Part (iii) follows from (14.24), (9.10), and Proposition 14.2(ii). □

Proposition 14.11.

(i) *If* $\mu \in \mathcal{Z}_T$ *and* $\nu \in \{0, \dots, n(\mu, T)\}$, *then*

$$a_{\lambda,\eta,k,j}(T, \Phi_{\mu,\nu,k,j}) = \delta_{\lambda,\mu}\delta_{\eta,\nu}$$

and

$$b_{\lambda,\eta,k,j}(T, \Phi_{\mu,\nu,k,j}) = 0.$$

If, in addition, $0 \notin \mathcal{O}$, *then*

$$a_{\lambda,\eta,k,j}(T, \Psi_{\mu,\nu,k,j}) = 0$$

and

$$b_{\lambda,\eta,k,j}(T, \Psi_{\mu,\nu,k,j}) = \delta_{\lambda,\mu}\delta_{\eta,\nu}.$$

(ii) *Suppose that* $k \in \mathbb{Z}_+$ *and* $j \in \{1, \dots, d(n, k)\}$ *are fixed and let*

$$a_{\lambda,\eta,k,j}(T, f) = b_{\lambda,\eta,k,j}(T, f) = 0$$

for all $\lambda \in \mathcal{Z}_T$ *and* $\eta \in \{0, \dots, n(\lambda, T)\}$. *Then* $f^{k,j} = 0$ *in* $\mathcal{O}$.

(iii) *If* $0 \in \mathcal{O}$ *and* $f \in C_T^m(\mathcal{O})$, $m = [n/2] + k - 1 + \operatorname{ord} T$, *then*

$$a_{\lambda,\eta,k,j}(T, f) = \langle T_{\lambda,\eta,k,j}, f^{k,j}\rangle. \tag{14.32}$$

Proof. Part (i) follows from (14.24), Proposition 14.2(ii), and Proposition 9.11(i). Next, using Proposition 14.10(i) and Theorem 9.9, we obtain (ii). To prove (iii), first assume that $f \in C_T^\infty(\mathcal{O})$. Identity (14.31) yields

$$\left(\mathcal{A}_j^k(f^{k,j}) * T_{\lambda,\nu}\right)(x) = \sum_{\mu=0}^{n(\lambda,T)-\nu} a_{\lambda,\nu+\mu,k,j}(T, f)\binom{\nu+\mu}{\nu}_{\lambda} \Phi_{\lambda,\mu,0,1}(x)$$

(see Theorem 9.7(iv) and (9.85)). Putting $x = 0$ and using (9.16), we find

$$a_{\lambda,\nu,k,j}(T, f) = \big\langle T_{\lambda,\nu}, \mathcal{A}_j^k\big(f^{k,j}\big)\big\rangle.$$

In combination with (9.98) this gives (14.32) for $f \in C_T^\infty(\mathcal{O})$. Next, let $f \in C_T^m(\mathcal{O})$, $m = [n/2] + k - 1 + \operatorname{ord} T$, $\varphi \in \mathcal{D}_\natural(\mathbb{R}^n)$, and assume that $\mathcal{O}_{T*\varphi \neq \varnothing}$. The previous arguments show that

$$a_{\lambda,\eta,k,j}(T, f * \varphi) = \langle T_{\lambda,\eta,k,j}, f * \varphi\rangle. \tag{14.33}$$

Now it is easy to deduce (iii) in the general case from (14.33) and Proposition 14.10(ii) with the help of the standard smoothing method (see (9.93)). This completes the proof. □

Corollary 14.3. *Let $\lambda, \mu \in \mathcal{Z}_T$, $f \in \mathcal{D}'(\mathcal{O})$, and*

$$\big(\Delta + \mu^2\big)^{n(\mu,T)+1} f = 0 \quad \textit{in } \mathcal{O}.$$

Then

$$f * T_{\lambda,0} = \begin{cases} 0 & \textit{if } \lambda \neq \mu, \\ f & \textit{if } \lambda = \mu. \end{cases}$$

The proof follows immediately from Proposition 14.8, (14.24), and Proposition 14.11.

Proposition 14.12. *Let $f \in \mathcal{D}_T'(\mathcal{O})$, and let*

$$T = \big(\Delta + \lambda_1^2\big)^{s_1} \cdots \big(\Delta + \lambda_l^2\big)^{s_l} Q, \tag{14.34}$$

where $Q \in \mathcal{E}_\natural'(\mathbb{R}^n)$, $s_1, \dots, s_l \in \mathbb{N}$, and $\{\lambda_1, \dots, \lambda_l\}$ is a set of distinct complex numbers in $\mathcal{Z}_T$. Then $r(Q) = r(T)$, $s_m \leqslant n(\lambda_m, T) + 1$ for all $m \in \{1, \dots, l\}$, and the distribution

$$g = f - \sum_{m=1}^{l} \sum_{\eta=n(\lambda_m,T)+1-s_m}^{n(\lambda_m,T)} a_{\lambda_m,\eta,k,j}(T, f)\Phi_{\lambda_m,\eta,k,j} + b_{\lambda_m,\eta,k,j}(T, f)\Psi_{\lambda_m,\eta,k,j}$$

is in the class $\mathcal{D}_Q'(\mathcal{O})$. In addition, if $\lambda \in \mathcal{Z}_Q$, then $n(\lambda, Q) \leqslant n(\lambda, T)$, and

$$a_{\lambda,\eta,k,j}(Q, g) = a_{\lambda,\eta,k,j}(T, f), \tag{14.35}$$

$$b_{\lambda,\eta,k,j}(Q, g) = b_{\lambda,\eta,k,j}(T, f) \tag{14.36}$$

for all $\eta \in \{0, \dots, n(\lambda, Q)\}$.

Proof. The proof of Proposition 13.7 is applicable with minor modifications. We can suppose that $l = s_1 = 1$. It follows from the hypothesis that

$$\widetilde{T}(z) = \big(\lambda_1^2 - z^2\big)\widetilde{Q}(z).$$

Therefore,

$$s_1 \leqslant n(\lambda_1, T) + 1, \quad n(\lambda, Q) \leqslant n(\lambda, T)$$

for all $\lambda \in \mathcal{Z}_Q$, and Theorem 9.2 yields $r(Q) = r(T)$. Next,

$$Q = -T_{\lambda_1, n(\lambda_1, T)} / b_{n(\lambda_1, T)}^{\lambda_1, n(\lambda_1, T)}$$

(see Proposition 9.13(ii)). Using now (14.31), Proposition 14.2(ii), and Proposition 9.11(i), we infer that $g \in \mathcal{D}'_Q(\mathcal{O})$. Relations (14.35) and (14.36) are now obvious from (14.24) and Proposition 9.11(iv). □

As we already know from Proposition 9.1(vi), for every $f \in \mathcal{D}'_T(\mathcal{O})$, the Fourier series

$$f = \sum_{k=0}^{\infty} \sum_{j=1}^{d(n,k)} f^{k,j}$$

unconditionally converges to f in $\mathcal{D}'(\mathcal{O})$. To any $f^{k,j}$ we now assign the series

$$f^{k,j} \sim \sum_{\lambda \in \mathcal{Z}_T} \sum_{\eta=0}^{n(\lambda,T)} a_{\lambda,\eta,k,j}(T, f) \Phi_{\lambda,\eta,k,j} + b_{\lambda,\eta,k,j}(T, f)\, \Psi_{\lambda,\eta,k,j}. \tag{14.37}$$

This series is an analog of the classical Laurent expansion for holomorphic functions. The functions $\Phi_{\lambda,\eta,k,j}$ (respectively $\Psi_{\lambda,\eta,k,j}$) play an analogous role as the nonnegative (respectively negative) powers of the complex variable z. If $\mathcal{O}$ is a ball, we have an analog of the Taylor expansion.

Theorem 14.11.

(i) *If $f \in \mathcal{D}'_T(\mathcal{O})$ and the series in (14.37) converges in $\mathcal{D}'(\mathcal{O})$, then its sum is equal to $f^{k,j}$.*

(ii) *Let $f \in \mathcal{D}'(\mathcal{O})$ and assume that for all $k \in \mathbb{Z}_+$, $j \in \{1, \dots, d(n,k)\}$,*

$$f^{k,j} = \sum_{\lambda \in \mathcal{Z}_T} \sum_{\eta=0}^{n(\lambda,T)} \alpha_{\lambda,\eta,k,j} \Phi_{\lambda,\eta,k,j} + \beta_{\lambda,\eta,k,j} \Psi_{\lambda,\eta,k,j}, \quad \alpha_{\lambda,\eta,k,j}, \beta_{\lambda,\eta,k,j} \in \mathbb{C},$$

where $\beta_{\lambda,\eta,k,j}$ are set to be equal to zero if $0 \in \mathcal{O}$, and the series converges in $\mathcal{D}'(\mathcal{O})$. Then $f \in \mathcal{D}'_T(\mathcal{O})$ and

$$\alpha_{\lambda,\eta,k,j} = a_{\lambda,\eta,k,j}(T, f), \quad \beta_{\lambda,\eta,k,j} = b_{\lambda,\eta,k,j}(T, f).$$

Proof. The proof proceeds as that of Theorem 13.9 except that we now apply Proposition 14.10(ii) and Proposition 14.11(i), (ii). □

To utilize this result we need the following estimates of $a_{\lambda,\eta,k,j}(T, f)$ and $b_{\lambda,\eta,k,j}(T, f)$.

Theorem 14.12. *Let $r(T) > 0$. Then the following results are true.*

(i) *Let $f \in \mathcal{D}'_T(\mathcal{O})$, and let p be a polynomial. Then there exists $c_1 > 0$ independent of f such that for all $\lambda \in \mathcal{Z}_T$, $|\lambda| > c_1$, the following estimates hold:*

$$\max_{0 \leqslant \eta \leqslant n(\lambda,T)} |a_{\lambda,\eta,k,j}(T, f)| \leqslant \frac{c_2}{|p(-\lambda^2)|} \max_{0 \leqslant \eta \leqslant n(\lambda,T)} \big|a_{\lambda,\eta,k,j}\big(T, p(\Delta)f\big)\big|,$$

$$\max_{0 \leqslant \eta \leqslant n(\lambda,T)} |b_{\lambda,\eta,k,j}(T, f)| \leqslant \frac{c_2}{|p(-\lambda^2)|} \max_{0 \leqslant \eta \leqslant n(\lambda,T)} \big|b_{\lambda,\eta,k,j}\big(T, p(\Delta)f\big)\big|$$

with the constant $c_2 > 0$ independent of λ, f.

(ii) *Assume that $R \in (r(T), +\infty]$, let $m \in \mathbb{Z}_+$, and let $f \in C^{2m}_T(B_R)$. Then there exist $c_3, c_4, c_5, c_6 > 0$ independent of f, m such that for all $\lambda \in \mathcal{Z}_T$, $|\lambda| > c_3$,*

$$\max_{0 \leqslant \eta \leqslant n(\lambda,T)} |a_{\lambda,\eta,k,j}(T, f)| \leqslant \sigma_\lambda(\widetilde{T}) c_4^{m+1} |\lambda|^{c_5 - 2m} \times \left(\int_{B_{r(T)}} \big|\Delta^m \big(f^{k,j}\big)(x)\big| \,\mathrm{d}x + c_6^m c_7 \right),$$

where $c_7 > 0$ is independent of m, λ.

(iii) *Let $f \in \mathcal{D}'_T(\mathbb{R}^n)$ and assume that* ord $f < +\infty$. *Then for each $\alpha > 0$,*

$$|a_{\lambda,\eta,k,j}(T, f)| \leqslant \sigma_\lambda\big(\widetilde{T}\big)(2 + |\lambda|)^{c_8} c_9^{n(\lambda,T)} n(\lambda, T)! \mathrm{e}^{-\alpha |\mathrm{Im}\, \lambda|},$$

where $c_8, c_9 > 0$ are independent of λ, η.

Proof. Arguing in the same way as in the proof of Theorem 13.10(i) and using Proposition 14.10(iii), we arrive at (i).

To show (ii) we define $Q \in (\mathcal{E}'_\natural \cap C^{k+n+1})(\mathbb{R}^n)$ by (14.34) so that $\lambda_p \notin \mathcal{Z}(\widetilde{Q})$ for each $p \in \{1, \dots, l\}$ (see Theorem 9.2). Because of Proposition 14.12, for all $\lambda \in \mathcal{Z}_T \setminus \{\lambda_1, \dots, \lambda_l\}$ and $\eta \in \{0, \dots, n(\lambda, T)\}$,

$$a_{\lambda,\eta,k,j}(T, f) = a_{\lambda,\eta,k,j}(Q, f - h), \tag{14.38}$$

where

$$h = \sum_{p=1}^{l} \sum_{\eta = n(\lambda_p,T)+1-s_p}^{n(\lambda_p,T)} a_{\lambda_p,\eta,k,j}(T, f) \Phi_{\lambda_p,\eta,k,j}. \tag{14.39}$$

Then it follows from (14.32), Proposition 9.12(i), and Proposition 6.6(iv) that

$$|a_{\lambda,\eta,k,j}(Q, f - h)| \leqslant c_{10} \sigma_\lambda\big(\widetilde{T}\big)(2 + |\lambda|)^{c_{11}} \int_{B_{r(T)}} \big|f^{k,j}(x) - h(x)\big| \,\mathrm{d}x,$$

where $\lambda \in \mathcal{Z}_T \setminus \{\lambda_1, \dots, \lambda_l\}$, $\eta \in \{0, \dots, n(\lambda, T)\}$, and $c_{10}, c_{11} > 0$ are independent of λ, η, f. This, together with (14.38), (14.39), and (i), gives (ii).

Turning to (iii), we use the same arguments as in the beginning of the proof (ii). However, we now choose $Q \in (\mathcal{E}'_\natural \cap C^q)(\mathbb{R}^n)$, where $q = k + n + 1 + \mathrm{ord}\, f$. Let

$\lambda \in \mathcal{Z}_T \setminus \{\lambda_1, \dots, \lambda_l\}$ and $\eta \in \{0, \dots, n(\lambda, T)\}$. Propositions 9.1(ii) and 9.12(i) show that

$$\left|\left(\left(f^{k,j} - h\right) * Q_{\lambda,\eta}\right)(x)\right| \leqslant c_{12}\sigma_\lambda\left(\widetilde{Q}\right), \quad x \in \mathbb{R}^n,$$

where $c_{12} > 0$ is independent of λ, η. Hence,

$$|a_{\lambda,n(\lambda,T),k,j}(Q, f - h)\Phi_{\lambda,0,k,j}(x)| \leqslant c_{12}\sigma_\lambda\left(\widetilde{Q}\right) \tag{14.40}$$

(see (14.31)). Moreover, if $n(\lambda, T) > 1$ and $\nu \in \{0, \dots, n(\lambda, Q) - 1\}$, then

$$\begin{aligned} &|a_{\lambda,\nu,k,j}(Q, f - h)\Phi_{\lambda,0,k,j}(x)| \\ &\leqslant c_{12}\sigma_\lambda\left(\widetilde{Q}\right) + \sum_{\mu=1}^{n(\lambda,Q)-\nu} \binom{\mu+\nu}{\nu}_\lambda |a_{\lambda,\nu+\mu,k,j}(Q, f - h)\Phi_{\lambda,\mu,k,j}(x)|. \end{aligned} \tag{14.41}$$

By induction on ν we see from (14.40), (14.41), and Proposition 9.4 that for all $\alpha > 0$ and $\nu \in \{0, \dots, n(\lambda, T)\}$,

$$|a_{\lambda,\nu,k,j}(Q, f - h)| \leqslant c_{13}\frac{\sigma_\lambda(\widetilde{Q})n(\lambda, Q)!c_{14}^{n(\lambda,T)-\nu}(2 + |\lambda|)^{c_{15}}}{\nu!e^{\alpha|\mathrm{Im}\,\lambda|}},$$

where $c_{13}, c_{14}, c_{15} > 0$ are independent of λ, ν. Using now (14.38) and Proposition 6.6(iv), we obtain (iii). □

Corollary 14.4. *Let $R \in (r(T), +\infty]$. Then the following statements are valid.*

(i) *Let $f \in \mathcal{D}'_T(B_R)$. Then*

$$|a_{\lambda,\eta,k,j}(T, f)| \leqslant (2 + |\lambda|)^{c_1}\sigma_\lambda\left(\widetilde{T}\right),$$

where $c_1 > 0$ is independent of λ, η. In particular, if $T \in \mathfrak{M}(\mathbb{R}^n)$, then

$$|a_{\lambda,\eta,k,j}(T, f)| \leqslant (2 + |\lambda|)^{c_2},$$

where $c_2 > 0$ is independent of λ, η.

(ii) *If $T \in \mathfrak{M}(\mathbb{R}^n)$ and $f \in C_T^\infty(B_R)$, then for each $\alpha > 0$,*

$$|a_{\lambda,\eta,k,j}(T, f)| \leqslant c_3(2 + |\lambda|)^{-\alpha},$$

where $c_3 > 0$ is independent of λ, η.

(iii) *If $\alpha > 0$, $T \in \mathfrak{G}_\alpha(\mathbb{R}^n)$, $r(T) > 0$, and $f \in C_T^\infty(B_R) \cap G^\alpha(\mathring{B}_{r(T)})$, then*

$$|a_{\lambda,\eta,k,j}(T, f)| \leqslant c_4 \exp\left(-c_5|\lambda|^{1/\alpha}\right),$$

where $c_4, c_5 > 0$ are independent of λ, η.

(iv) *If $T \in \mathfrak{M}(\mathbb{R}^n)$, $r(T) > 0$ and $f \in C_T^\infty(B_R) \cap \mathrm{QA}(\mathring{B}_{r(T)})$, then*

$$\max_{0\leqslant\eta\leqslant n(\lambda,T)} |a_{\lambda,\eta,k,j}(T, f)| \leqslant M_{q,k,j}(1 + |\lambda|)^{-2q}$$

for all $\lambda \in \mathcal{Z}_T$ *and* $q \in \mathbb{N}$*, where the constants* $M_{q,k,j} > 0$ *are independent of* λ*, and*

$$\sum_{\nu=1}^{\infty} \frac{1}{\inf_{q \geqslant \nu} M_{q,k,j}^{1/2q}} = +\infty. \tag{14.42}$$

(v) *Let* $T \in \mathfrak{E}(\mathbb{R}^n)$*,* $f \in \mathcal{D}'_T(\mathbb{R}^n)$*, and suppose that* f *is of finite order in* $\mathbb{R}^n$*. Then for each* $\alpha > 0$*,*

$$|a_{\lambda,\eta,k,j}(T, f)| \leqslant c_6 \mathrm{e}^{-\alpha|\mathrm{Im}\,\lambda|},$$

where $c_6 > 0$ *is independent of* λ, η*.*

Proof. It can be supposed that $r(T) > 0$ (see Proposition 14.8). Part (i) now follows from Proposition 14.9, Theorem 14.12(ii), and Proposition 14.10(iii) exactly as in the proof of Corollary 13.3(i). The second statement is clear from Theorem 14.12(ii). Applying now (9.6) and Theorem 14.12(ii) with $m = [\gamma|\lambda|^{1/\alpha}]$, $\gamma \in (0, 1)$, $\lambda \in \mathcal{Z}_T$, where $|\lambda|$ is large enough, we arrive at (iii). The proof of (iv) is derived from the definition of $\mathrm{QA}(\mathring{B}_{r(T)})$ by making use of Theorem 14.12(ii), (9.6), and Lemma 8.1(i). Part (v) follows directly from Theorem 14.12(iii). □

The following is an analog of Theorem 14.12 for the case of spherical annulus.

Theorem 14.13. *Assume that*

$$0 \leqslant r < r' < R' < R \leqslant +\infty, \quad R' - r' = 2r(T). \tag{14.43}$$

Suppose that $m \in \mathbb{Z}_+$ *and let* $f \in C_T^{2m}(B_{r,R})$*. Then there exist* $c_1, c_2, c_3, c_4, c_5 > 0$ *independent of* f, m *such that for all* $\lambda \in \mathcal{Z}_T$*,* $|\lambda| > c_1$*,*

$$\begin{aligned}
&\max_{0 \leqslant \eta \leqslant n(\lambda,T)} \left(|a_{\lambda,\eta,k,j}(T, f)| + |b_{\lambda,\eta,k,j}(T, f)|\right) \\
&\quad \leqslant \sigma_\lambda\big(\widetilde{T}\big) c_2^{m+1} |\lambda|^{c_3 - 2m} \mathrm{e}^{c_4(|\mathrm{Im}\,\lambda|+1)(n(\lambda,T)+1)} n(\lambda, T)! \\
&\quad \times \left(c_5^m c_6 + \int_{B_{r',R'}} \left|\Delta^m\big(f^{k,j}\big)(x)\right| \mathrm{d}x\right),
\end{aligned}$$

where $c_6 > 0$ *is independent of* m, λ*.*

Proof. Let $Q \in (\mathcal{E}'_\natural \cap C^{k+n+2})(\mathbb{R}^n)$ be defined just as in the proof of Theorem 14.12, and let

$$h = \sum_{p=1}^{l} \sum_{\eta=0}^{n(\lambda_p,T)} a_{\lambda_p,\eta,k,j}(T, f) \Phi_{\lambda_p,\eta,k,j} + b_{\lambda_p,\eta,k,j}(T, f) \Psi_{\lambda_p,\eta,k,j}.$$

Then $\mathcal{Z}_Q \subset \mathcal{Z}_T$, and the set $\mathcal{Z}_T \setminus \mathcal{Z}_Q$ is finite. Assume that $\lambda \in \mathcal{Z}_Q$ and $\eta \in \{0, \ldots, n(\lambda, Q)\}$. By Proposition 14.12, $n(\lambda, T) = n(\lambda, Q)$, and relations (14.35), (14.36) hold with $g = f - h$. For brevity, we set

$$a_{\lambda,\eta} = a_{\lambda,\eta,k,j}(Q, f - h), \quad b_{\lambda,\eta} = b_{\lambda,\eta,k,j}(Q, f - h).$$

Choose $y \in \mathbb{R}^n$ so that $|y| = (r' + R')/2$ and $Y_j^{(k)}(y/|y|) \neq 0$. For $t \in (r - |y| + r(T), R - |y| - r(T))$, now define

$$\begin{aligned} u_{\lambda,\eta}(t) &= \Phi_{\lambda,\eta,k,j}\big(y(1 + t/|y|)\big), \\ v_{\lambda,\eta}(t) &= \Psi_{\lambda,\eta,k,j}\big(y(1 + t/|y|)\big), \\ w_{\lambda,\eta}(t) &= \big(\big(f^{k,j} - h\big) * Q_{\lambda,\eta}\big)\big(y(1 + t/|y|)\big). \end{aligned}$$

Owing to Proposition 9.1(iii) and Proposition 9.12(i),

$$|w_{\lambda,\eta}(0)| + \big|w'_{\lambda,\eta}(0)\big| \leqslant c_7 \sigma_\lambda\big(\widetilde{Q}\big) \int_{B_r(y)} \big|f^{k,j}(x) - h(x)\big| \, \mathrm{d}x, \tag{14.44}$$

where c_7 is independent of λ, η, f. By virtue of (14.31) we have

$$a_{\lambda,n(\lambda,Q)} u_{\lambda,0}(0) + b_{\lambda,n(\lambda,Q)} v_{\lambda,0}(0) = w_{\lambda,n(\lambda,Q)}(0)$$

and

$$a_{\lambda,n(\lambda,Q)} u'_{\lambda,0}(0) + b_{\lambda,n(\lambda,Q)} v'_{\lambda,0}(0) = w'_{\lambda,n(\lambda,Q)}(0).$$

Because of Proposition 9.2(ii), (14.44), and Proposition 9.5(i), these equalities yield

$$|a_{\lambda,n(\lambda,Q)}| + |b_{\lambda,n(\lambda,Q)}| \leqslant c_8 \sigma_\lambda\big(\widetilde{Q}\big)(1 + |\lambda|)^{c_9} \mathrm{e}^{c_{10}|\mathrm{Im}\,\lambda|} \int_{B_r(y)} \big|f^{k,j}(x) - h(x)\big| \, \mathrm{d}x, \tag{14.45}$$

where $c_8, c_9, c_{10} > 0$ are independent of λ, f. Suppose now that $n(\lambda, Q) > 1$ and let $\nu \in \{0, \ldots, n(\lambda, Q) - 1\}$. By (14.31),

$$\begin{aligned} &a_{\lambda,\nu} u_{\lambda,0}(0) + b_{\lambda,\nu} v_{\lambda,0}(0) \\ &\quad = w_{\lambda,\nu}(0) - \sum_{\mu=1}^{n(\lambda,Q)-\nu} \binom{\nu+\mu}{\nu}_\lambda \big(a_{\lambda,\nu+\mu} u_{\lambda,\mu}(0) + b_{\lambda,\nu+\mu} v_{\lambda,\mu}(0)\big) \end{aligned}$$

and

$$\begin{aligned} &a_{\lambda,\nu} u'_{\lambda,0}(0) + b_{\lambda,\nu} v'_{\lambda,0}(0) \\ &\quad = w'_{\lambda,\nu}(0) - \sum_{\mu=1}^{n(\lambda,Q)-\nu} \binom{\nu+\mu}{\nu}_\lambda \big(a_{\lambda,\nu+\mu} u'_{\lambda,\mu}(0) + b_{\lambda,\nu+\mu} v'_{\lambda,\mu}(0)\big). \end{aligned}$$

Using now (14.45), Proposition 9.2(ii), and Proposition 9.5(i), we infer by induction on ν that

$$\begin{aligned} |a_{\lambda,\nu}| + |b_{\lambda,\nu}| \leqslant{} & \frac{c_{11} \sigma_\lambda(\widetilde{Q})(1 + |\lambda|)^{c_{12}} n(\lambda, Q)! \mathrm{e}^{c_{13}|\mathrm{Im}\,\lambda|(1+n(\lambda,Q))}}{c_{14}^{\nu-n(\lambda,Q)} \nu!} \\ & \times \int_{B_r(y)} \big|f^{k,j}(x) - h(x)\big| \, \mathrm{d}x, \end{aligned}$$

where $c_{11}, c_{12}, c_{13}, c_{14} > 0$ are independent of λ, ν, f.

Since $B_{r(T)}(y) \subset B_{r',R'}$, this, together with Theorem 14.12(i), Proposition 6.6(iv), and Proposition 9.5(ii), gives us the desired conclusion. □

Corollary 14.5. *Assume that (14.43) is fulfilled. Then the following assertions hold.*

(i) *Let $f \in \mathcal{D}'_T(B_{r,R})$. Then*

$$|a_{\lambda,\eta,k,j}(T,f)| + |b_{\lambda,\eta,k,j}(T,f)| \leqslant (2+|\lambda|)^{c_1} e^{c_2(|\mathrm{Im}\,\lambda|+1)(n(\lambda,T)+1)} n(\lambda,T)!\,\sigma_\lambda\big(\widetilde{T}\big),$$

where $c_1, c_2 > 0$ are independent of λ, η. In particular, if $T \in \mathfrak{I}(\mathbb{R}^n)$, then

$$|a_{\lambda,\eta,k,j}(T,f)| + |b_{\lambda,\eta,k,j}(T,f)| \leqslant (2+|\lambda|)^{c_3},$$

where $c_3 > 0$ are independent of λ, η.

(ii) *If $T \in \mathfrak{I}(\mathbb{R}^n)$ and $f \in C^\infty_T(B_{r,R})$, then for each $\alpha > 0$,*

$$|a_{\lambda,\eta,k,j}(T,f)| + |b_{\lambda,\eta,k,j}(T,f)| \leqslant c_4(2+|\lambda|)^{-\alpha},$$

where $c_4 > 0$ are independent of λ, η.

(iii) *If $\alpha > 0$, $T \in \mathfrak{I}_\alpha(\mathbb{R}^n)$, and $f \in C^\infty_T(B_{r,R}) \cap G^\alpha(\dot{B}_{r',R'})$, then*

$$|a_{\lambda,\eta,k,j}(T,f)| + |b_{\lambda,\eta,k,j}(T,f)| \leqslant c_5 \exp\big(-c_6|\lambda|^{1/\alpha}\big),$$

where $c_5, c_6 > 0$ are independent of λ, η.

(iv) *If $T \in \mathfrak{I}(\mathbb{R}^n)$ and $f \in C^\infty_T(B_{r,R}) \cap \mathrm{QA}(\dot{B}_{r',R'})$, then*

$$\max_{0\leqslant\eta\leqslant n(\lambda,T)} \big(|a_{\lambda,\eta,k,j}(T,f)| + |b_{\lambda,\eta,k,j}(T,f)|\big) \leqslant M_{q,k,j}(1+|\lambda|)^{-2q}$$

for all $\lambda \in \mathcal{Z}_T$ and $q \in \mathbb{N}$, where the constants $M_{q,k,j} > 0$ are independent of λ and satisfy (14.42).

Proof. According to Proposition 14.8, it is enough to consider the case $r(T) > 0$. The proof then proceeds as that of Corollary 14.4 except that we now use Theorem 14.13. □

Remark 14.1. Natural analogues of Theorem 14.12(iii) and Corollary 14.4(v) for the case where $\mathcal{O} = B_{r,R}$ are no longer valid in general even if $r = 0$, $R = +\infty$. We may come to this conclusion having regarded the function $f \in C_T(\mathbb{R}^n \setminus \{0\})$ of the form

$$f = \sum_{\lambda \in \mathcal{Z}_T} \Phi_{\lambda,0,k,j} + \mathrm{i}\Psi_{\lambda,0,k,j},$$

where $T \in \mathfrak{E}(\mathbb{R}^n)$ (see Proposition 9.4).

We are now in a position to prove the following multidimensional analog of Theorem 13.11.

Theorem 14.14. *Let U be an* $\mathrm{O}(n)$*-invariant compact subset of* $\mathcal{O}$ *such that* $\dot{B}_{r(T)}(x) \subset U$ *for some* $x \in \mathbb{R}^n$. *Let* $f \in C_T^\infty(\mathcal{O})$ *and assume that there exist* $\alpha > 0$, $\beta \geqslant 0$ *such that*

$$\liminf_{q\to+\infty} \alpha^{-2q} q^{-\beta} \int_U \left|\left(\Delta^q f\right)(x)\right| \mathrm{d}x = 0. \tag{14.46}$$

Then

$$a_{\lambda,\eta,k,j}(T, f) = b_{\lambda,\eta,k,j}(T, f) = 0,$$

provided that $|\lambda| > \alpha$, *and the same is true if* $|\lambda| = \alpha$, $\eta \geqslant \beta$. *In particular, if* $\alpha \leqslant \min_{\lambda\in\mathcal{Z}_T} |\lambda|$ *and* $\beta = 0$, *then* $f = 0$.

This is a best possible result, as, for example,

$$f = c_1\Phi_{\lambda,\eta,k,j} + c_2\Psi_{\lambda,\eta,k,j}, \quad c_1, c_2 \in \mathbb{C},$$

shows (see Proposition 9.6).

Proof of Theorem 14.14. Relation (14.46) and Theorems 14.12(ii) and 14.13 imply that

$$a_{\lambda,\eta,k,j}(T, f) = b_{\lambda,\eta,k,j}(T, f) = 0$$

if $|\lambda|$ is sufficiently large. In order to complete the proof, one need only (14.46), Proposition 9.6, and Proposition 14.11(ii). □

Another uniqueness result is as follows.

Theorem 14.15. *Let* $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$, $T \neq 0$, *and let* (9.110) *hold. Assume that* $0 \leqslant r < R \leqslant +\infty$ *and*

$$f = \sum_{\lambda\in\mathcal{Z}_T} \sum_{\eta=0}^{n(\lambda,T)} a_{\lambda,\eta}\Phi_{\lambda,\eta,k,j} + b_{\lambda,\eta}\Psi_{\lambda,\eta,k,j} \quad \textit{in } B_{r,R}, \tag{14.47}$$

where $a_{\lambda,\eta}, b_{\lambda,\eta} \in \mathbb{C}$, *and conditions* (9.115) *and* (9.116) *are fulfilled with* $c_{\lambda,\eta} = a_{\lambda,\eta}, b_{\lambda,\eta}$. *Suppose that* $f = 0$ *in* B_{r_1,r_2} *for some* $r_1 > r$, $r_2 < R$, $r_1 < r_2$. *Then* $a_{\lambda,\eta} = b_{\lambda,\eta} = 0$ *for all* λ, η.

Proof. It is enough to consider the case where $r = 0$, $R = +\infty$ (see Propositions 9.19 and 9.21(iii)). We define $f_1 \in C^\infty(\mathbb{R}^n)$ by letting $f_1(x) = 0$ if $|x| < r_2$ and $f_1(x) = f(x)$ if $|x| \geqslant r_2$. Then $\mathfrak{A}_{k,j}(f_1) = 0$ in B_{r_2} in view of Theorem 9.3(ii). Using Theorem 9.4, Lemma 8.1(ii), and Proposition 9.5(ii), we deduce from (14.47) that $\mathfrak{A}_{k,j}(f_1) \in \mathrm{QA}(\mathbb{R}^1)$. This, together with Theorems 8.1(i) and 9.3(ii), gives $f_1 = 0$. The desired result is now obvious from Theorem 14.11(ii). □

In the final of this section we give the following analog of Proposition 13.8.

Proposition 14.13. *Let* $H \in \mathcal{E}'_\natural(\mathbb{R}^n)$, $H \neq 0$, *and* $R = r(T) + r(H)$. *Assume that* $\lambda \in \mathcal{Z}_T$, $n(\lambda, T) > n(\lambda, H)$, *and* $\eta \in \{n(\lambda, H) + 1, \dots, n(\lambda, T)\}$, *where the number* $n(\lambda, H)$ *is set to be equal to* -1 *for* $\lambda \notin \mathcal{Z}_H$. *Then the following results are true.*

(i) *If* $\mathring{B}_R(x) \subset \mathcal{O}$ *for some* $x \in \mathbb{R}^n$ *and* $f \in (\mathcal{D}'_T \cap \mathcal{D}'_H)(\mathcal{O})$*, then*

$$a_{\lambda,\eta,k,j}(T, f) = b_{\lambda,\eta,k,j}(T, f) = 0. \tag{14.48}$$

(ii) *If* $B_R(x) \subset \mathcal{O}$ *for some* $x \in \mathbb{R}^n$ *and* $f \in C_T^\infty(\mathcal{O} \cup B_{0,+\infty}) \cap C_H^\infty(\mathcal{O})$*, then* (14.48) *is valid.*

(iii) *If* $B_R(x) \subset \mathcal{O}$ *for some* $x \in \mathbb{R}^n$*,* $H \in \mathcal{D}_\natural(\mathbb{R}^n)$*, and* $f \in \mathcal{D}'_T(\mathcal{O} \cup B_{0,+\infty}) \cap \mathcal{D}'_H(\mathcal{O})$*, then* (14.48) *holds.*

Proof. The proof of (i) is analogous to the proof of Proposition 13.8(i), only instead of Proposition 13.5(iii) one applies Proposition 14.10(iii). To prove (ii) and (iii) take $f \in \mathcal{D}'_T(\mathcal{O} \cup B_{0,+\infty}) \cap \mathcal{D}'_H(\mathcal{O})$. Assumptions in (ii) and (iii) show that the convolution $F = f * H$ is in the class $C_T^\infty(\mathcal{U})$, where

$$\mathcal{U} = \left\{x \in \mathbb{R}^n : \mathring{B}_{r(H)} \subset \mathcal{O} \cup B_{0,+\infty}\right\}.$$

In addition, $F = 0$ in $\{x \in \mathbb{R}^n : \mathring{B}_{r(H)} \subset \mathcal{O}\}$. By virtue of Theorem 14.2, $F = 0$ in $\mathcal{U}$, and hence $f \in (\mathcal{D}'_T \cap \mathcal{D}'_H)(\mathcal{U})$. Now part (i) is applicable, and the proof is complete. □

14.5 Convergence Theorems. Extendability and Nonextendability Results

The results in the previous section make it possible to obtain some convergence theorems for the series in (14.37) and to characterize some classes of solutions of convolution equations on domains with spherical symmetry. Our first result in this direction is as follows.

Theorem 14.16. *Let* $T \in \mathrm{Inv}_+(\mathbb{R}^n)$*,* $R \in (r(T), +\infty]$*, and* $f \in \mathcal{D}'(B_R)$*. Then in order that* $f \in \mathcal{D}'_T(B_R)$*, it is necessary and sufficient that for all* $k \in \mathbb{Z}_+$*,* $j \in \{1, \dots, d(n,k)\}$*, the following equality holds*

$$f^{k,j} = \zeta_{T,k,j} * u_{k,j} \quad \text{in } B_R \tag{14.49}$$

for some $u_{k,j} \in \mathcal{E}'_\natural(\mathbb{R}^n)$ *with* supp $u_{k,j} \subset \mathring{B}_{r(T)}$.

Proof. First, suppose that $f \in \mathcal{D}'_T(B_R)$. Then for all $k \in \mathbb{Z}_+$, $j \in \{1, \dots, d(n,k)\}$, the distribution $\mathfrak{A}_{k,j}(f^{k,j})$ is in $\mathcal{D}'_{\Lambda(T),\natural}(-R, R)$ (see Propositions 14.5 and 14.7(ii)). Using Theorem 13.13 and Remark 13.1, we obtain

$$\mathfrak{A}_{k,j}\left(f^{k,j}\right) = \zeta'_{\Lambda(T)} * v_{k,j} \quad \text{in } (-R, R)$$

for some $v_{k,j} \in \mathcal{E}'_\natural(\mathbb{R}^1)$ with supp $v_{k,j} \subset [-r(T), r(T)]$. Define $u_{k,j} \in \mathcal{E}'_\natural(\mathbb{R}^n)$ with supp $u_{k,j} \subset \mathring{B}_{r(T)}$ by letting

$$\Lambda(u_{k,j}) = -v_{k,j}.$$

Applying (14.14) and Theorem 9.5(i), we arrive at (14.49).

Conversely, if (14.49) holds, then $f^{k,j} \in \mathcal{D}'_T(B_R)$ for all k, j (see Theorem 14.7). Now Proposition 14.5 tells us that $f \in \mathcal{D}'_T(B_R)$, and the proof is complete. $\square$

We now focus on the case $T \in \mathfrak{M}(\mathbb{R}^n)$.

Theorem 14.17. *Let $T \in \mathfrak{M}(\mathbb{R}^n)$ and $R \in (r(T), +\infty]$. Then the following assertions hold.*

(i) *Let $f \in \mathcal{D}'(B_R)$. Then $f \in \mathcal{D}'_T(B_R)$ if and only if for all $k \in \mathbb{Z}_+$, $j \in \{1, \dots, d(n,k)\}$, the following relation holds:*

$$f^{k,j} = \sum_{\lambda \in \mathcal{Z}_T} \sum_{\eta=0}^{n(\lambda,T)} \alpha_{\lambda,\eta,k,j} \Phi_{\lambda,\eta,k,j}, \tag{14.50}$$

where $\alpha_{\lambda,\eta,k,j} \in \mathbb{C}$, and the series converges in $\mathcal{D}'(B_R)$.

(ii) *Let $f \in C^\infty(B_R)$. Then $f \in C^\infty_T(B_R)$ if and only if for all k, j, decomposition (14.50) holds with the series converging in $\mathcal{E}(B_R)$.*

(iii) *Let $f \in \mathcal{D}'_{k,j}(B_R)$ for some $k \in \mathbb{Z}_+$, $j \in \{1, \dots, d(n,k)\}$. Then $f \in \mathrm{QA}_T(B_R)$ if and only if relation (14.50) is satisfied with the series converging in $\mathcal{E}(B_R)$,*

$$\max_{0 \leqslant \eta \leqslant n(\lambda,T)} |\alpha_{\lambda,\eta,k,j}| \leqslant M_{q,k,j}(1+|\lambda|)^{-2q} \tag{14.51}$$

for all $\lambda \in \mathcal{Z}_T$ and $q \in \mathbb{N}$, and the constants $M_{q,k,j} > 0$ are independent of λ and satisfy (14.42).

(iv) *Let $r(T) > 0$ and $f \in C^\infty_T(B_R) \cap \mathrm{QA}(\mathring{B}_{r(T)})$. Then $f^{k,j} \in \mathrm{QA}_T(B_R)$ for all k, j.*

The proof of this theorem is clear from Theorem 14.11, Corollary 14.4(i), (ii), (iv) and Propositions 9.18 and 9.19.

It can be shown that the assumption on T in Theorem 14.17 cannot be considerably weakened (see V.V. Volchkov [225], Part III, Theorem 2.5).

Theorem 14.18.

(i) *Let $\alpha > 0$, $T \in \mathfrak{G}_\alpha(\mathbb{R}^n)$, $r(T) > 0$, $R \in (r(T), +\infty]$, and $f \in C^\infty(B_R) \cap G^\alpha(\mathring{B}_{r(T)})$. Then $f \in C^\infty_T(B_R)$ if and only if for all k, j, equality (14.50) is satisfied with the series converging in $\mathcal{E}(B_R)$. In addition, if $f \in C^\infty_T(B_R) \cap G^\alpha(\mathring{B}_{r(T)})$, then $f^{k,j} \in G^\alpha_T(B_R)$ for all k, j.*

(ii) *Let $T \in \mathfrak{E}(\mathbb{R}^n)$, $f \in \mathcal{D}'(\mathbb{R}^n)$, and assume that f is of finite order in $\mathbb{R}^n$. Then $f \in \mathcal{D}'_T(\mathbb{R}^n)$ if and only if for all k, j, relation (14.50) holds with the series converging in $\mathcal{E}(\mathbb{R}^n)$. In particular, if $f \in \mathcal{D}'_T(\mathbb{R}^n)$, then $f^{k,j} \in C^\infty_T(\mathbb{R}^n)$ for all k, j.*

The proof follows at once from Theorem 14.11, Corollary 14.4(iii), (v), and Propositions 9.18(ii) and 9.20.

As an application, we now establish the following result.

Theorem 14.19. *Let $T \in \mathrm{Inv}_+(\mathbb{R}^n)$, $R > r(T)$, and assume that $k \in \mathbb{Z}_+$, $j \in \{1, \dots, d(n,k)\}$. Then for each $f \in (\mathcal{D}'_T \cap \mathcal{D}'_{k,j})(B_R)$, there exists a unique $F \in (\mathcal{D}'_T \cap \mathcal{D}'_{k,j})(\mathbb{R}^n)$ such that $F = f$ in B_R. Moreover, if $T \in \mathfrak{M}(\mathbb{R}^n)$, then the following statements are valid.*

(i) *If $f \in C^\infty(B_R)$, then $F \in C^\infty(\mathbb{R}^n)$.*
(ii) *If $f \in C^\infty(B_R) \cap \mathrm{QA}(\dot{B}_{r(T)})$, then $F \in \mathrm{QA}(\mathbb{R}^n)$.*

Proof. According to Proposition 14.8, it is enough to consider the case where $r(T) > 0$. By Theorem 14.16,

$$f = \zeta_{T,k,j} * u \quad \text{in } B_R$$

for some $u \in \mathcal{E}'_\natural(\mathbb{R}^n)$. Therefore, F can be defined by letting

$$F = \zeta_{T,k,j} * u.$$

Using now Theorem 14.2(i), we see that this extension of f is unique. Finally, by Theorem 14.17(ii)–(iv) and Propositions 9.18 by 9.19, we conclude that (i) and (ii) hold. □

Remark 14.2. Theorem 14.18(i) ensures us that if $\alpha > 0$, $T \in \mathfrak{G}_\alpha(\mathbb{R}^n)$, $R > r(T) > 0$, and

$$f \in \left(\mathcal{D}'_{k,j} \cap C^\infty_T\right)(B_R) \cap G^\alpha\left(\dot{B}_{r(T)}\right),$$

then there exists a unique $F \in (\mathcal{D}'_{k,j} \cap C^\infty_T)(\mathbb{R}^n)$ such that $F = f$ in B_R. In fact, F can be defined by formula (14.50) (see Corollary 14.4(iii) and Proposition 9.20).

Assume now that $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$, $R > r(T) > 0$, and $f \in (C^\infty_T \cap C^\infty_{k,j})(B_R)$ for some $k \in \mathbb{Z}_+$ and $j \in \{1, \dots, d(n,k)\}$. Then Theorem 9.7 implies that

$$\left\langle T, \mathcal{A}^k_j\left(\Delta^\nu f\right)\right\rangle = 0 \quad \text{for all } \nu \in \mathbb{Z}_+. \tag{14.52}$$

In this case Theorem 14.19 and Remark 14.2 admit the following refinement.

Theorem 14.20. *Let $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$, $r(T) > 0$, $k \in \mathbb{Z}_+$, $j \in \{1, \dots, d(n,k)\}$, let $f \in C^\infty_{k,j}(\dot{B}_{r(T)})$, and suppose that (14.52) holds. Then the following statements are valid.*

(i) *If $T \in \mathfrak{M}(\mathbb{R}^n)$, then there exists a unique $F \in (C^\infty_T \cap C^\infty_{k,j})(\mathbb{R}^n)$ such that $F = f$ in $\dot{B}_{r(T)}$. Furthermore, if $f \in \mathrm{QA}(\dot{B}_{r(T)})$, then $F \in \mathrm{QA}_T(\mathbb{R}^n)$.*
(ii) *If $\alpha > 0$, $T \in \mathfrak{G}_\alpha(\mathbb{R}^n)$, and $f \in G^\alpha(\dot{B}_{r(T)})$, then there exists a unique $F \in (G^\alpha_T \cap C^\infty_{k,j})(\mathbb{R}^n)$ such that $F = f$ in $\dot{B}_{r(T)}$.*

Proof. Combine Propositions 9.18–9.20 and Theorems 9.10 and 14.11 with Corollaries 14.4(iii), (iv), 9.7, 9.8. □

The following theorems show that the assumptions on T in Theorem 14.19 and Remark 14.2 cannot be considerably relaxed.

Theorem 14.21. *Let $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$, $T \neq 0$, $R > r(T)$, and assume that*

$$\sup_{\lambda \in \mathcal{Z}_T} \frac{n(\lambda, T) + |\mathrm{Im}\,\lambda|}{\log(2 + |\lambda|)} = +\infty.$$

Then for all $k \in \mathbb{Z}_+$, $j \in \{1, \dots, d(n,k)\}$, there exists $f \in (C_T^\infty \cap \mathcal{D}'_{k,j})(B_R)$ such that if $\varepsilon > 0$ and $F \in \mathcal{D}'_{k,j}(B_{R+\varepsilon})$, then

$$F|_{B_R} \neq f.$$

Proof. Assume the contrary. Then there exist k, j such that each $f \in (C_T^\infty \cap \mathcal{D}'_{k,j})(B_R)$ extends to a distribution in $\mathcal{D}'_T(B_{R+\varepsilon})$ for some $\varepsilon > 0$. Hence, the distribution $\mathfrak{A}_{k,j}(f) \in C^\infty_{\Lambda(T)}(-R, R)$ admits extension to a distribution in $\mathcal{D}'_{\Lambda(T)}(-R - \varepsilon, R + \varepsilon)$ (see Proposition 14.7(ii) and Corollary 9.2). According to Proposition 14.7(ii) and Corollary 9.2, this contradicts Corollary 13.8 proving the theorem. □

Theorem 14.22. *There exists $T \in \mathcal{D}_\natural(\mathbb{R}^n)$ such that the following statements hold.*

(i) $r(T) > 0$, $\mathcal{Z}(\widetilde{T}) \subset \mathbb{R}^1$, *and* $n(\lambda, T) = 0$ *for all* $\lambda \in \mathcal{Z}_T$.
(ii) *For all $k \in \mathbb{Z}_+$, $j \in \{1, \dots, d(n,k)\}$, and $R > r(T)$, there exists $f \in (C_T^\infty \cap \mathcal{D}'_{k,j})(B_R)$ such that if $\varepsilon > 0$ and $F \in \mathcal{D}'_{k,j}(B_{R+\varepsilon})$, then $F|_{B_R} \neq f$.*

Proof. Once Theorem 13.20 has been established, the proof of our theorem is similar to that of Theorem 14.21. □

Theorem 14.23. *Let $\alpha > 0$, $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$, $T \neq 0$, and*

$$\sup_{\lambda \in \mathcal{Z}_T} \frac{|\mathrm{Im}\,\lambda|}{(1 + |\lambda|)^{1/\alpha}} = +\infty.$$

Then for all $k \in \mathbb{Z}_+$, $j \in \{1, \dots, d(n,k)\}$, and $R > r(T)$, there exists $f \in (C_T^\infty \cap G^\alpha)(B_R)$ such that if $\varepsilon > 0$ and $F \in \mathcal{D}'_{k,j}(B_{R+\varepsilon})$, then $F|_{B_R} \neq f$.

Proof. No change is required in the proof of Theorem 14.21, except that we now apply Remark 13.4 and Corollary 9.3. □

We now present analogs of Theorems 14.17 and 14.18(i) for a spherical annulus.

Theorem 14.24. *Let $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$, $T \neq 0$, let $0 \leqslant r < R \leqslant +\infty$, and suppose that r' and R' satisfy* (14.43). *Then the following assertions hold.*

(i) *Let* $T \in \mathfrak{I}(\mathbb{R}^n)$ *and* $f \in \mathcal{D}'(B_{r,R})$. *Then* $f \in \mathcal{D}'_T(B_{r,R})$ *if and only if for all* $k \in \mathbb{Z}_+$ *and* $j \in \{1, \dots, d(n,k)\}$,

$$f^{k,j} = \sum_{\lambda \in \mathcal{Z}_T} \sum_{\eta=0}^{n(\lambda,T)} \alpha_{\lambda,\eta,k,j} \Phi_{\lambda,\eta,k,j} + \beta_{\lambda,\eta,k,j} \Psi_{\lambda,\eta,k,j}, \tag{14.53}$$

where $\alpha_{\lambda,\eta,k,j}, \beta_{\lambda,\eta,k,j} \in \mathbb{C}$, *and the series converges in* $\mathcal{D}'(B_{r,R})$.

(ii) *Let* $T \in \mathfrak{I}(\mathbb{R}^n)$ *and* $f \in C^\infty(B_{r,R})$. *Then* $f \in C^\infty_T(B_{r,R})$ *if and only if for all* k, j, *relation* (14.53) *holds with the series converging in* $\mathcal{E}(B_{r,R})$.

(iii) *Let* $\alpha > 0$, $T \in \mathfrak{I}_\alpha(\mathbb{R}^n)$, *and* $f \in C^\infty(B_R) \cap G^\alpha(\dot{B}_{r',R'})$. *Then* $f \in C^\infty_T(B_{r,R})$ *if and only if for all* k, j, *decomposition* (14.53) *holds with the series converging in* $\mathcal{E}(B_{r,R})$. *In addition, if* $f \in C^\infty_T(B_{r,R}) \cap G^\alpha(\dot{B}_{r',R'})$, *then* $f^{k,j} \in G^\alpha_T(B_{r,R})$ *for all* k, j.

(iv) *Let* $T \in \mathfrak{I}(\mathbb{R}^n)$ *and suppose that* $f \in \mathcal{D}'_{k,j}(B_{r,R})$ *for some* k, j. *Then* $f \in \mathrm{QA}_T(B_{r,R})$ *if and only if equality* (14.53) *is satisfied with the series converging in* $\mathcal{E}(B_{r,R})$,

$$\max_{0 \leqslant \eta \leqslant n(\lambda,T)} (|\alpha_{\lambda,\eta,k,j}| + |\beta_{\lambda,\eta,k,j}|) \leqslant M_{q,k,j}(1+|\lambda|)^{-2q}$$

for all $\lambda \in \mathcal{Z}_T$ *and* $q \in \mathbb{N}$, *and the constants* $M_{q,k,j} > 0$ *are independent of* λ *and satisfy* (14.42).

(v) *Let* $T \in \mathfrak{I}(\mathbb{R}^n)$, *and let* $f \in C^\infty_T(B_{r,R}) \cap \mathrm{QA}(\dot{B}_{r',R'})$. *Then* $f^{k,j} \in \mathrm{QA}_T(B_{r,R})$ *for all* k, j.

The proof immediately follows from Theorem 14.11, Corollary 14.5, and Propositions 9.18–9.21.

Theorem 14.25. *Let* $T \in \mathfrak{I}(\mathbb{R}^n)$, $0 \leqslant r < R \leqslant +\infty$, $R - r > 2r(T)$, *and assume that* $k \in \mathbb{Z}_+$ *and* $j \in \{1, \dots, d(n,k)\}$. *Then for each* $f \in (\mathcal{D}'_T \cap \mathcal{D}'_{k,j})(B_{r,R})$, *there exists a unique* $F \in (\mathcal{D}'_T \cap \mathcal{D}'_{k,j})(\mathbb{R}^n \setminus \{0\})$ *such that* $F = f$ *in* $B_{r,R}$ *and the following statements are valid.*

(i) *If* $f \in C^\infty(B_{r,R})$, *then* $F \in C^\infty(\mathbb{R}^n \setminus \{0\})$.

(ii) *If* r' *and* R' *satisfy* (14.43) *and* $f \in C^\infty(B_{r,R}) \cap \mathrm{QA}(\dot{B}_{r',R'})$, *then* $F \in \mathrm{QA}(\mathbb{R}^n \setminus \{0\})$.

Proof. Theorems 14.24 and 14.11 and Corollary 14.5 imply that F can be defined by decomposition (14.53). Then (i) and (ii) follow by Propositions 9.18, 9.19, and 9.21(ii), (iii). It remains to observe that a desired extension of f is unique by Theorem 14.2(ii). □

Remark 14.3. Let $\alpha > 0$, $T \in \mathfrak{I}_\alpha(\mathbb{R}^n)$, let $0 \leqslant r < R \leqslant +\infty$, and assume that r' and R' satisfy (14.43). Suppose that $k \in \mathbb{Z}_+$ and $j \in \{1, \dots, d(n,k)\}$. Then for each $f \in (\mathcal{D}'_{k,j} \cap C^\infty_T)(B_{r,R}) \cap G^\alpha(\dot{B}_{r',R'})$, there exists a unique $F \in (G^\alpha_T \cap \mathcal{D}'_{k,j})(\mathbb{R}^n \setminus \{0\})$ such that

$$F = f \quad \text{in } B_{r,R}.$$

This can be proved by obvious modifying of the proof of Theorem 14.25 (see Corollary 14.5(iii) and Proposition 9.21(iv)).

As another application of Theorems 14.17 and 14.24, we now establish the following theorem on a removable singularity.

Theorem 14.26. *Let* $T \in \mathfrak{I}(\mathbb{R}^n)$*,* $R \in (2r(T), +\infty]$*, and let* $f \in (\mathcal{D}'_T \cap \mathcal{D}'_{k,j})(B_R)$ *for some* $k \in \mathbb{Z}_+$ *and* $j \in \{1, \ldots, d(n,k)\}$*. Suppose that* $0 < \alpha < \beta < R$ *and* $\beta - \alpha > 2r(T)$*. Then the following assertions are true.*

(i) *If* $f \in C^\infty(B_{\alpha,\beta})$*, then* $f \in C^\infty(B_R)$*.*
(ii) *If* $f \in \mathrm{QA}(B_{\alpha,\beta})$*, then* $f \in \mathrm{QA}(B_R)$*.*

We note that $\beta - \alpha < 2r(T)$ is not admissible in the general case (see [225], Part III, Remark 14.6).

Proof of Theorem 14.26. It follows by the hypothesis and Theorem 14.17(i) that $f = f^{k,j}$ in B_R, and equality (14.50) holds, where the series converges in $\mathcal{D}'(B_R)$. Let $f \in C^\infty(B_{\alpha,\beta})$. Then we have expansion (14.53) with $\beta_{\lambda,\eta,k,j} = 0$ and the series converging in $\mathcal{E}(B_{\alpha,\beta})$ (see Theorems 14.24(ii) and 14.11(ii)). Now Proposition 9.18(ii) shows that the series in (14.50) converges in $\mathcal{E}(B_R)$, whence $f \in C^\infty(B_R)$.

Next, if $f \in \mathrm{QA}(B_{\alpha,\beta})$, then the coefficients of the series in (14.50) satisfy (14.51), and (14.42) is fulfilled (see Theorem 14.24(iv)). Using Proposition 9.19, we obtain $f \in \mathrm{QA}(B_R)$. Thus, Theorem 14.26 is proved. □

14.6 Problem on Admissible Rate of Decreasing. Reduction to the Helmholtz Equation

Let $T \in \mathcal{E}'(\mathbb{R}^n)$, $T \neq 0$, and let $f \in L^{1,\mathrm{loc}}(\mathbb{R}^n)$ be a nonzero function in the class $\mathcal{D}'_T(\mathbb{R}^n)$. Then f cannot decrease rapidly at infinity. For instance, if $f \in (L^1 \cap \mathcal{D}'_T)(\mathbb{R}^n)$, one has $\widehat{f}\widehat{T} = 0$. However, the set $\{x \in \mathbb{R}^n : \widehat{T}(x) = 0\}$ is dense nowhere in $\mathbb{R}^n$ because $\widehat{T}$ is an entire function. Since $\widehat{f} \in C(\mathbb{R}^n)$, we conclude that f must vanish. Thus, it is natural to ask: In general, what decay conditions on $f \in (\mathcal{D}'_T \cap L^{1,\mathrm{loc}})(\mathbb{R}^n)$ will force $f = 0$?

In this section we shall study precise growth restrictions for $f \in (\mathcal{D}'_T \cap L^{1,\mathrm{loc}})(\mathbb{R}^n)$ under which it follows that $f = 0$.

Throughout the section we suppose that $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$ and $T \neq 0$. First, we shall specify the form of (14.37) when $f \in (\mathcal{D}'_T \cap L^{1,\mathrm{loc}})(\mathbb{R}^n)$ satisfies some growth restrictions at infinity.

For $R > \xi > 0$, we denote

$$U(R,\xi) = \big\{x \in \mathbb{R}^n : R - \xi < |x| < R + \xi\big\}.$$

Theorem 14.27. *Let* $f \in (\mathcal{D}'_T \cap L^{1,\mathrm{loc}})(\mathbb{R}^n)$*, and let*

$$\liminf_{R\to+\infty} R^{-\alpha}\mathrm{e}^{-\beta R}\int_{U(R,\xi)} |f(x)|\,\mathrm{d}x = 0 \tag{14.54}$$

for some $\alpha, \beta \geqslant 0$ *and* $\xi > r(T)$*. Then* $a_{\lambda,\eta,k,j}(T, f) = 0$*, provided that* $|\mathrm{Im}\,\lambda| > \beta$*. The same is true if* $|\mathrm{Im}\,\lambda| = \beta$*,* $\lambda \neq 0$*,* $\eta \geqslant \alpha - (n-1)/2$*. In addition, if* $\alpha \leqslant n - 1 + k + 2\eta$ *and* $\beta = 0$*, then* $a_{0,\eta,k,j}(T, f) = 0$*.*

Proof. In view of (9.6), the functions $f^{k,j}$ satisfy (14.54) for all k, j. Let $\varepsilon \in (0, (\xi - r(T))/2)$, $u \in \mathcal{D}_\natural(B_\varepsilon)$, $\lambda \in \mathcal{Z}_T$, and $\nu \in \{0, \dots, n(\lambda, T)\}$. Then we have

$$\int_{U(R,\xi-\varepsilon-r(T))} \left|\left(f^{k,j} * u * T_{\lambda,\nu}\right)(x)\right| \mathrm{d}x \leqslant \gamma \int_{U(R,\xi)} \left|f^{k,j}(x)\right| \mathrm{d}x,$$

where the constant γ is independent of R. Bearing (14.31) in mind, we infer that there is an increasing infinite sequence $\{R_m\}_{m=1}^{\infty}$ of positive numbers such that $R_1 > \varepsilon$ and

$$\lim_{m\to\infty} R_m^{-\alpha}\mathrm{e}^{-\beta R_m}\int_{U(R_m,\varepsilon)} |g_{\lambda,\nu,k,j}(x)|\,\mathrm{d}x = 0,$$

where

$$g_{\lambda,\nu,k,j} = \sum_{\mu=0}^{n(\lambda,T)-\nu} a_{\lambda,\nu+\mu,k,j}(T, f*u)\binom{\nu+\mu}{\nu}_{\lambda} \Phi_{\lambda,\mu,k,j}.$$

This, together with Proposition 9.4, gives us the desired statements for $f * u$ instead of f. Now the theorem follows from the arbitrariness of u and Proposition 14.10(iii). □

Remark 14.4. The previous theorem fails in general with $\xi \in (0, r(T))$ (see Remark 3.2 in V.V. Volchkov [225], Part III). Next, condition (14.54) in Theorem 15.27 cannot be replaced by

$$\int_{U(R,\xi)} |f(x)|\,\mathrm{d}x = O\left(R^{\alpha}\mathrm{e}^{\beta R}\right) \quad \text{as } R \to +\infty. \tag{14.55}$$

To show this it is enough to consider the function $f = \Phi_{\lambda,\eta,k,j}$ (see Proposition 9.4). We note also that this function is in $L^p(\mathbb{R}^n)$, provided that $\lambda \in \mathbb{R}^1\backslash\{0\}$, $\eta = 0$, and $p \in (2n/(n-1), +\infty]$.

We set

$$\theta_T = \inf_{\lambda\in\mathcal{Z}_T} |\mathrm{Im}\,\lambda| \quad \text{and} \quad \Theta_T = \sup_{\lambda\in\mathcal{Z}_T} |\mathrm{Im}\,\lambda|.$$

Two corollaries of Theorem 14.27 are worth recording.

Corollary 14.6. *Let* $\xi > r(T)$*,* $f \in (\mathcal{D}'_T \cap L^{1,\mathrm{loc}})(\mathbb{R}^n)$*, and assume that one of the following assumptions holds:*

(1) $\theta_T < |\mathrm{Im}\,\lambda|$ *for each* $\lambda \in \mathcal{Z}_T$, *and*

$$\liminf_{R\to+\infty} \mathrm{e}^{-\gamma R} \int_{U(R,\xi)} |f(x)|\,\mathrm{d}x = 0 \tag{14.56}$$

for any $\gamma > \theta_T$;

(2) $\theta_T = |\mathrm{Im}\,\lambda|$ *for some* $\lambda \in \mathcal{Z}_T$, *and* (14.54) *is satisfied with* $\alpha = (n-1)/2$ *and* $\beta = \theta_T$.

Then $f = 0$.

The proof follows at once from Theorem 14.27 and Proposition 14.11(ii).

Several remarks are in order here. If $\theta_T < |\mathrm{Im}\,\lambda|$ for each $\lambda \in \mathcal{Z}_T$, then (14.56) cannot be replaced by the same equality with fixed $\gamma > \theta_T$. If $\theta_T = |\mathrm{Im}\,\lambda|$ for some $\lambda \in \mathcal{Z}_T$, then in condition (2) we cannot replace (14.54) with $\alpha = (n-1)/2$, $\beta = \theta_T$ by relation (14.55) with the same α, β (see Remark 14.4). If $0 \in \mathcal{Z}_T$ and $\mathrm{Im}\,\lambda \neq 0$ for each $\lambda \in \mathcal{Z}_T \backslash \{0\}$, then in condition (2), (14.54) is sufficient with $\alpha = n-1$ and $\beta = 0$ (see Theorem 14.27).

Corollary 14.7. *Let* $\xi > r(T)$, $p \in [1, +\infty)$, *and let* $f \in (\mathcal{D}'_T \cap L^{p,\mathrm{loc}})(\mathbb{R}^n)$. *Suppose that for some* $\alpha, \beta \geqslant 0$,

$$\liminf_{R\to+\infty} R^{(n-1)(p-1)-\alpha p} \mathrm{e}^{-\beta p R} \int_{U(R,\xi)} |f(x)|^p\,\mathrm{d}x = 0. \tag{14.57}$$

Then all the statements of Theorem 14.27 are valid. In particular, the following assertions hold.

(i) *If either* $\mathcal{Z}_T \cap \mathbb{R}^1 = \varnothing$ *or* $\mathcal{Z}_T \cap \mathbb{R}^1 = \{0\}$, *then* $(\mathcal{D}'_T \cap L^p)(\mathbb{R}^n) = \{0\}$.

(ii) *If* $\mathcal{Z}_T \cap \mathbb{R}^1 \neq \varnothing$, $\mathcal{Z}_T \cap \mathbb{R}^1 \neq \{0\}$, *and* $1 \leqslant p \leqslant 2n/(n-1)$, *then* $(\mathcal{D}'_T \cap L^p)(\mathbb{R}^n) = \{0\}$. *This assertion is no longer valid with* $p > 2n/(n-1)$.

Proof. Using the Hölder inequality, we see that (14.57) implies (14.54). The desired results now follow from Theorem 14.27 and Proposition 14.11(ii) (see also Remark 14.4). □

We now consider analogues of these results for $f \in (\mathcal{D}'_T \cap L^{1,\mathrm{loc}})(B_{r,+\infty})$, $r \geqslant 0$.

Theorem 14.28. *Let* $r \geqslant 0$, $f \in (\mathcal{D}'_T \cap L^{1,\mathrm{loc}})(B_{r,+\infty})$, *and assume that relation* (14.54) *holds for some* $\alpha \in \mathbb{R}^1$, $\beta \leqslant 0$, *and* $\xi > r(T)$. *Then*

$$a_{\lambda,\eta,k,j}(T, f) = b_{\lambda,\eta,k,j}(T, f) = 0,$$

provided that $|\mathrm{Im}\,\lambda| < -\beta$. *The same is true if* $|\mathrm{Im}\,\lambda| = -\beta$, $\lambda \neq 0$, *and* $\eta \geqslant \alpha-(n-1)/2$. *In addition, if* $\beta = 0$, *then* $a_{0,\eta,k,j}(T, f) = 0$ *for* $\eta \geqslant (\alpha+1-n-k)/2$ *and* $b_{0,\eta,k,j}(T, f) = 0$ *for* $\eta \geqslant (\alpha-1+k)/2$. *Next, if the condition*

$$\liminf_{R\to+\infty} \frac{1}{R^\alpha \log R} \int_{U(R,\xi)} |f(x)|\,\mathrm{d}x = 0 \tag{14.58}$$

holds instead of (14.54)*, then* $b_{0,\eta,k,j}(T, f) = 0$*, provided that* n *is even and* $2\eta \geqslant \max\{n-2+2k, k-1-\alpha\}$.

Proof. No change is required in the proof of Theorem 14.27. □

Remark 14.5. Assumption (14.54) in Theorem 14.28 cannot be replaced by (14.55). Similarly, assumption (14.58) cannot be replaced by

$$\int_{U(R,\xi)} |f(x)|\,\mathrm{d}x = O\big(R^{\alpha}\log R\big) \quad \text{as } R \to +\infty.$$

We can make sure of this by considering the function of the form

$$f = c_1\Phi_{\lambda,\eta,k,j} + c_2\Psi_{\lambda,\eta,k,j} \tag{14.59}$$

for proper $c_1, c_2 \in \mathbb{C}$. Thus, all the assertions in Theorem 14.28 are precise.

Remark 14.6. The proof of Corollary 14.7 shows that all the assumptions of Theorem 14.28 remain valid for $f \in (\mathcal{D}'_T \cap L^{p,\mathrm{loc}})(B_{r,+\infty})$, $p \in [1, +\infty)$, provided that (14.54) and (14.58) are replaced by (14.57) and the condition

$$\liminf_{R\to+\infty} R^{(n-1)(p-1)-\alpha p}(\log R)^{-p}\int_{U(R,\xi)} |f(x)|^p\,\mathrm{d}x = 0,$$

respectively.

For the rest of the section, we assume that $\mathcal{O}$ is a ζ domain in $\mathbb{R}^n$ with $\zeta = r(T)$ such that $\mathbb{R}^n\backslash\mathcal{O}$ is a nonempty bounded set. Theorem 14.28 enables us to obtain some precise uniqueness results for the class $(\mathcal{D}'_T \cap L^{1,\mathrm{loc}})(\mathcal{O})$.

Theorem 14.29. *Let* $\xi > r(T)$ *and* $\Theta_T = +\infty$*. Then the following assertions hold.*

(i) *If* $f \in (\mathcal{D}'_T \cap L^{1,\mathrm{loc}})(\mathcal{O})$ *and* (14.56) *is satisfied for all* $\gamma < 0$*, then* $f = 0$.
(ii) *For each* $\gamma < 0$*, there exists a nonzero function* $f_\gamma \in C^\infty_T(\mathcal{O})$ *such that*

$$f_\gamma(x) = O\big(\mathrm{e}^{\gamma|x|}\big) \quad \textit{as } x \to \infty. \tag{14.60}$$

Proof. Concerning part (i), first observe that $f = 0$ in $B_{r,+\infty}$ for each $r > 0$ such that $\mathbb{R}^n\backslash\mathcal{O} \subset B_r$ (see Theorem 14.28 and Proposition 14.11(ii)). Applying now Theorem 14.2(i), we arrive at (i).

To prove (ii) it is enough to consider the function $f_\gamma(x) = f(x - x_0)$, where $x_0 \in \mathbb{R}^n\backslash\mathcal{O}$, and f is defined by (14.59) with $c_1 = 1$, $c_2 = \mathrm{i}$, $\lambda \in \mathcal{Z}_T$, $\operatorname{Im}\lambda > -\gamma$ (see Proposition 9.4). □

Theorem 14.30. *Let* $\xi > r(T)$*,* $\Theta_T = 0$*, and* $0 \in \mathcal{Z}_T$*. There holds*

(i) *If* $f \in (\mathcal{D}'_T \cap L^{1,\mathrm{loc}})(\mathcal{O})$ *and* (14.54) *is fulfilled for all* $\alpha < 0$ *and* $\beta = 0$*, then* $f = 0$.
(ii) *For any* $\alpha < 0$*, there is nonzero* $f_\alpha \in C^\infty_T(\mathcal{O})$ *such that* $f_\alpha(x) = O(|x|^\alpha)$ *as* $x \to \infty$.

Proof. As in proving Theorem 14.29(i), the first item follows by Theorems 14.28 and 14.2(i) and Proposition 14.11(ii). To show (ii) it is enough to regard the function $f_\alpha(x) = \Psi_{0,0,k,j}(x - x_0)$, where $x_0 \in \mathbb{R}^n \backslash \mathcal{O}$ and $n + k - 2 + \alpha > 0$ (see Proposition 9.4). □

Theorem 14.31. *Let $\xi > r(T)$ and $\Theta_T = |\mathrm{Im}\,\lambda|$ for some $\lambda \in \mathcal{Z}_T$. Then we have the following assertions.*

(i) *If $f \in (\mathcal{D}'_T \cap L^{1,\mathrm{loc}})(\mathcal{O})$ and (14.54) holds with $\alpha = (n-1)/2$, $\beta = -\Theta_T$, then $f = 0$, provided that $\Theta_T > 0$. The same is true if $\Theta_T = 0$ and $0 \notin \mathcal{Z}_T$.*
(ii) *There exists a nonzero function $f \in C_T^\infty(\mathcal{O})$ satisfying (14.55) with $\alpha = (n-1)/2$ and $\beta = -\Theta_T$.*

We note that the first assertion of this theorem fails in general with $\xi \in (0, r(T))$ (see V.V. Volchkov [225], Part III, Remark 3.2).

Proof of Theorem 14.31. Part (i) is a direct consequence of Theorems 14.28 and 14.2(i) and Proposition 14.11(ii). Next, let f be defined by (14.59) with $c_1 = 1$, $c_2 = \mathrm{i}$, $\eta = k = 0$, $\lambda \in \mathcal{Z}_T$, and $|\mathrm{Im}\,\lambda| = \Theta_T$. Then f satisfies all the requirements of (ii) (see Proposition 9.4). Hence the theorem. □

Theorem 14.32. *Let $\xi > r(T)$ and $|\mathrm{Im}\,\lambda| < \Theta_T < +\infty$ for all $\lambda \in \mathcal{Z}_T$. There holds*

(i) *If $f \in (\mathcal{D}'_T \cap L^{1,\mathrm{loc}})(\mathcal{O})$ and (14.56) holds for all $\gamma > -\Theta_T$, then $f = 0$.*
(ii) *For each $\gamma > -\Theta_T$, there exists nonzero $f_\gamma \in C_T^\infty(\mathcal{O})$ satisfying (14.60).*

Proof. This is just a repetition of the proof of Theorem 14.29. □

As another consequence of Theorem 14.28, we shall now obtain an analog of Corollary 14.7(i), (ii).

Corollary 14.8.

(i) *If $\mathcal{Z}_T \subset \mathbb{R}^1$ and $0 \notin \mathcal{Z}_T$, then $(\mathcal{D}'_T \cap L^p)(\mathcal{O}) = \{0\}$ provided $1 \leqslant p \leqslant 2n/(n-1)$. The property fails for $p > 2n/(n-1)$.*
(ii) *If either $0 \in \mathcal{Z}_T$ or $\mathcal{Z}_T \backslash \mathbb{R}^1 \neq \varnothing$ then $(\mathcal{D}'_T \cap L^p)(\mathcal{O}) \neq \{0\}$ for each $p \in [1, +\infty]$.*

Proof. First, assume that $\mathcal{Z}_T \subset \mathbb{R}^1$, $0 \notin \mathcal{Z}_T$, and $p \in [1, 2n/(n-1)]$. Then Remark 14.6 and Proposition 14.11(ii) ensure us that $(\mathcal{D}'_T \cap L^p)(B_{r,+\infty}) = \{0\}$ for each $r \geqslant 0$. By Theorem 14.2(i) we conclude that $(\mathcal{D}'_T \cap L^p)(\mathcal{O}) = \{0\}$. In addition, Remark 14.4 shows that $(\mathcal{D}'_T \cap L^p)(\mathbb{R}^n) \neq \{0\}$ for $p > 2n/(n-1)$. This completes the proof of (i).

To prove (ii) it is enough to regard the function $f(x-x_0)$, where $x_0 \in \mathbb{R}^n \backslash \mathcal{O}$, and f is defined by (14.59) for proper $\lambda \in \mathcal{Z}_T$ and $c_1, c_2 \in \mathbb{C}$ (see Proposition 9.4). □

We now establish the following analog of Theorem 13.23.

Theorem 14.33. *Let $\{T_\nu\}_{\nu=1}^m$ be a family of nonzero distributions in $\mathcal{E}'_\natural(\mathbb{R}^n)$, and let $T = T_1 * \cdots * T_m$, $\xi > r(T)$. Assume that the set $\{1, \ldots, m\}$ is represented as a union of disjoint sets $A_1, \ldots, A_l$ such that the sets $\underset{\nu \in A_s}{\cup} \mathcal{Z}_{T_\nu}$, $s = 1, \ldots, l$, are also disjoint. Then the following assertions hold.*

(i) *Let $f_\nu \in \mathcal{D}'_{T_\nu}(\mathbb{R}^n)$, $f = \sum_{\nu=1}^m f_\nu \in L^{1,\mathrm{loc}}(\mathbb{R}^n)$, and suppose that one of assumptions (1) and (2) in Corollary 14.6 is fulfilled for given f and T. Then $\sum_{\nu \in A_s} f_\nu = 0$ for all $s \in \{1, \ldots, l\}$.*

(ii) *Let $r_\nu \geqslant 0$, $r = \max_{1 \leqslant \nu \leqslant m} r_\nu$, $\mathcal{O}_s = \bigcap_{\nu \in A_s} B_{r_\nu, +\infty}$, and let $f_\nu \in \mathcal{D}'_{T_\nu}(B_{r_\nu, +\infty})$ and $f = \sum_{\nu=1}^m f_\nu \in L^{1,\mathrm{loc}}(B_{r,+\infty})$. Assume that one of the following assumptions hold:*

  (1) *$\Theta_T = +\infty$ and (14.56) is satisfied for all $\gamma < 0$;*
  (2) *$\Theta_T = 0$, $0 \in \mathcal{Z}_T$, and (14.54) is fulfilled for all $\alpha < 0$ and $\beta = 0$;*
  (3) *$\Theta_T = |\mathrm{Im}\,\lambda| > 0$ for some $\lambda \in \mathcal{Z}_T$, and (14.54) is valid with $\alpha = (n-1)/2$ and $\beta = -\Theta_T$.*
  (4) *$\Theta_T = \mathrm{Im}\,\lambda = 0$ for some $\lambda \in \mathcal{Z}_T$, $0 \notin \mathcal{Z}_T$, and (14.54) is true with $\alpha = (n-1)/2$ and $\beta = -\Theta_T$.*

  Then $\sum_{\nu \in A_s} f_\nu = 0$ in $\mathcal{O}_s$ for each $s \in \{1, \ldots, l\}$.

Proof. (i) By the definitions of f and T we infer that $f \in \mathcal{D}'_T(\mathbb{R}^n)$. Hence, $f = 0$ because of Corollary 14.6. For $s \in \{1, \ldots, l\}$, we define $F_s \in \mathcal{D}'(\mathbb{R}^n)$ and $\Phi_s, \Psi_s \in \mathcal{E}'_\natural(\mathbb{R}^n)$ by the formulae

$$F_s = \sum_{\nu \in A_s} f_\nu, \quad \widetilde{\Phi_s} = \prod_{\nu \in A_s} \widetilde{T}_\nu, \quad \widetilde{\Psi_s} = \prod_{\substack{\nu=1 \\ \nu \notin A_s}}^m \widetilde{T}_\nu.$$

Since $f = 0$, one has $F_s \in (\mathcal{D}'_{\Phi_s} \cap \mathcal{D}'_{\Psi_s})(\mathbb{R}^n)$. Bearing in mind that $\mathcal{Z}(\widetilde{\Phi_s}) \cap \mathcal{Z}(\widetilde{\Psi_s}) = \varnothing$ and using Proposition 14.13, we see that $a_{\lambda,\eta,k,j}(\Phi_s, F_s) = 0$ for all $\lambda \in \mathcal{Z}_{\Phi_s}$, $\eta \in \{1, \ldots, n(\lambda, \Phi_s)\}$, and $k \in \mathbb{Z}_+$, $j \in \{1, \ldots, d(n,k)\}$. Because of Proposition 14.11(ii), this proves (i). Next, there is no difficulty in modifying the proof of (i) using Theorems 14.29–14.32 and 14.2(i) to obtain (ii). □

We end our considerations with the proof of the following result.

Theorem 14.34. *Let $\mathcal{U}$ be a domain in $\mathbb{R}^n$, let $\xi > r(T)$, and let $\mathcal{U}_\xi = \bigcup_{x \in \mathcal{U}} B_\xi(x)$. Assume that $f \in L^{1,\mathrm{loc}}(\mathcal{U}_\xi)$ is a nonzero function in the class $\mathcal{D}'_T(\mathcal{U}_\xi)$. Then there exists nonzero $w \in C^\infty(\mathcal{U})$ with the following properties:*

(1) *$\Delta w + \lambda^2 w = 0$ in $\mathcal{U}$ for some $\lambda \in \mathcal{Z}_T$;*
(2) *for each $r \in (r(T), \xi)$,*

$$|w(x)| \leqslant c \int_{B_r(x)} |f(y)|\,\mathrm{d}y, \quad x \in \mathcal{U},$$

where the constant $c > 0$ is independent of x.

Proof. Let $\varepsilon \in (0, \xi - r(T))$. According to Theorem 9.9, there exist $\lambda \in \mathcal{Z}_T$ and $\eta \in \{0, \dots, n(\lambda, T)\}$ such that the convolution $F = f * T^{\lambda,\eta}$ is nonzero in the set $\bigcup_{x \in \mathcal{U}} B_\varepsilon(x)$. Then $F * \varphi$ is nonzero in $\mathcal{U}$ for some $\varphi \in \mathcal{D}(B_\varepsilon)$, and $(\Delta + \lambda^2)^{\eta+1} F * \varphi = 0$ in $\mathcal{U}$ (see Proposition 9.13(i)). Hence, there exists $m \in \{0, \dots, \eta\}$ such that the function $w = (\Delta + \lambda^2)^m F * \varphi$ is nonzero in $\mathcal{U}$ and (1) is satisfied. Taking into account that $w = f * (T^{\lambda,\eta} * (\Delta + \lambda^2)^m \varphi)$ in $\mathcal{U}$, we see that the property (2) is also valid. This concludes the proof. □

This theorem often allows one to reduce the study of decay conditions for the class $\mathcal{D}'_T \cap L^{1,\mathrm{loc}}$ to the investigation of the same problem for solutions of the Helmholtz equation.

14.7 Hörmander-Type Approximation Theorems on Domains Without the Convexity Assumption

The object of this section is to consider the problem of approximation of solutions of a homogeneous convolution equation by elementary solutions.

Throughout the section $\mathcal{O}$ is a nonempty open set in $\mathbb{R}^n$. Suppose that $T \in \mathcal{E}'(\mathbb{R}^n)$, $T \neq 0$, and let

$$\mathcal{O}^T = \left\{x \in \mathbb{R}^n : x - y \in \mathcal{O} \text{ when } y \in \operatorname{supp} T\right\}.$$

Notice that $\mathcal{O}^T \supset \mathcal{O}_T$ for $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$ (see (14.1)). If $f \in \mathcal{D}'(\mathcal{O})$ and $\mathcal{O}^T \neq \varnothing$, the convolution $f * T$ is a well-defined distribution in $\mathcal{D}'(\mathcal{O}^T)$. We set

$$N_T(\mathcal{O}) = \left\{f \in C^\infty(\mathcal{O}) : f * T = 0 \text{ in } \mathcal{O}^T\right\} \quad \text{if } \mathcal{O}^T \neq \varnothing$$

and $N_T(\mathcal{O}) = C^\infty(\mathcal{O})$ if $\mathcal{O}^T = \varnothing$. For the case where $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$ and $\mathcal{O}_T \neq \varnothing$, one has $N_T(\mathcal{O}) \subset C^\infty_T(\mathcal{O})$.

Let $\mathcal{O}_1, \mathcal{O}_2$ be two nonempty subsets of $\mathbb{R}^n$ such that $\mathcal{O}_1 \subset \mathcal{O}_2$. For each $W \subset \mathcal{D}'(\mathcal{O}_2)$, we shall write $W|_{\mathcal{O}_1}$ for the set of the restrictions to $\mathcal{O}_1$ of all elements in W. If $W \subset \mathcal{D}'(\mathcal{O}_2)$ (respectively $W \subset C^\infty(\mathcal{O}_2)$), denote by $\operatorname{span}_{\mathcal{D}'(\mathcal{O}_1)} W$ (respectively $\operatorname{span}_{C^\infty(\mathcal{O}_1)} W$) the closure in $\mathcal{D}'(\mathcal{O}_1)$ (respectively $C^\infty(\mathcal{O}_1)$) of the set of all finite linear combinations of elements in $W|_{\mathcal{O}_1}$. Let E_T denote the set of all exponential solutions of the equation

$$f * T = 0 \quad \text{in } \mathbb{R}^n, \tag{14.61}$$

that is, solutions of the form

$$f(x) = p(x) e^{i\langle \zeta, x\rangle_{\mathbb{C}}}, \tag{14.62}$$

where $\zeta \in \mathbb{C}^n$, and p is a polynomial. It is not hard to check that if (14.62) holds, then (14.61) is equivalent to

$$\left(\left(\frac{\partial}{\partial x}\right)^{\alpha} p\right)\left(-\mathrm{i}\frac{\partial}{\partial \zeta_1}, \ldots, -\mathrm{i}\frac{\partial}{\partial \zeta_n}\right)\widehat{T}(\zeta) = 0 \quad \text{for all } \alpha \in \mathbb{Z}_+^n$$

(see the proof of Proposition 14.1).

The following Hörmander's result shows that for convex $\mathcal{O}$, the set $N_T(\mathcal{O})$ coincides with the closed linear span of E_T in $C^\infty(\mathcal{O})$.

Theorem 14.35. *If $\mathcal{O}$ is convex, then*

$$\operatorname{span}_{C^\infty(\mathcal{O})} E_T = N_T(\mathcal{O}). \tag{14.63}$$

For the proof, we refer the reader to Hörmander [126, Theorem 16.4.1] (see also Napalkov [159, Theorem 20.1]).

The conclusion of Theorem 14.35 may fail if the convexity requirement for $\mathcal{O}$ is dropped. Indeed, let $T \in \mathcal{E}'(\mathbb{R}^n)$ be defined by the formula

$$\langle T, f\rangle = f(\mathbf{e}_1) - f(0), \quad f \in \mathcal{E}\big(\mathbb{R}^n\big),$$

where $\mathbf{e}_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$. Assume that $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3$ where

$$\mathcal{O}_1 = \big\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 > 1\big\}, \qquad \mathcal{O}_2 = \big\{x \in \mathbb{R}^n : (-x) \in \mathcal{O}_1\big\},$$

$$\mathcal{O}_3 = \big\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : |x|^2 < 1 + x_1^2\big\}.$$

Then (14.63) implies easily that for every $f \in \operatorname{span}_{C^\infty(\mathcal{O})} E_T$, there exists $F \in C^\infty(\mathbb{R}^n)$ such that

$$F|_{\mathcal{O}} = f \quad \text{and} \quad F(x + \mathbf{e}_1) = F(x) \quad \text{for all } x \in \mathbb{R}^n. \tag{14.64}$$

Now define $f \in C^\infty(\mathcal{O})$ by letting $f = 0$ in $\mathcal{O}_2 \cup \mathcal{O}_3$ and

$$f(x) = \sin(2\pi x_1) g(x_2, \ldots, x_n) \quad \text{for } x \in \mathcal{O}_1,$$

where $g \in C^\infty(\mathbb{R}^{n-1})$ is nontrivial, and $g = 0$ in $\{x = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1} : x_2^2 + \cdots + x_n^2 \leqslant 1\}$. Obviously, $f \in N_T(\mathcal{O})$, but $f \notin \operatorname{span}_{C^\infty(\mathcal{O})}(E_T)$ since (14.64) does not hold. Thus, the convexity assumption in Theorem 14.35 is necessary in the general case.

However, for $T \in (\mathcal{E}'_\natural \cap \operatorname{Inv})(\mathbb{R}^n)$, there exist analogues of Theorem 14.35 that hold for a broad class of nonconvex $\mathcal{O}$. We shall now consider these results.

For the rest of the section, we assume that $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$, $T \neq 0$, and do not suppose that $\mathcal{O}_T \neq \varnothing$. We put $C_T^\infty(\mathcal{O}) = C^\infty(\mathcal{O})$ if $\mathcal{O}_T = \varnothing$. If A_1, A_2 are two nonempty subsets of $\mathbb{R}^n$, we set

$$\operatorname{dist}(A_1, A_2) = \inf\{|a_1 - a_2| : a_1 \in A_1, a_2 \in A_2\}.$$

Denote by $\mathcal{A}_{\mathcal{O}}$ the set of all bounded components of connection of $\mathbb{R}^n\backslash\mathcal{O}$. For $W \subset \mathcal{D}'(\mathcal{O})$ and $a \in \mathbb{R}^n$, we set

$$\tau_a W = \{f \in \mathcal{D}'(\mathcal{O}+a) : f(\cdot + a) \in W\},$$

where $\mathcal{O}+a = \{x \in \mathbb{R}^n : x - a \in \mathcal{O}\}$. In particular, $\tau_a f = f(\cdot - a)$ for each $f \in \mathcal{D}'(\mathcal{O})$. Also, define the families $\Phi_T \subset C_T^\infty(\mathbb{R}^n)$ and $\Psi_T \subset C_T^\infty(\mathbb{R}^n\backslash\{0\})$ by

$$\Phi_T = \left\{\Phi_{\lambda,\eta,k,j} : \lambda \in \mathcal{Z}_T, \eta \in \{0,\dots,n(\lambda,T)\}, k \in \mathbb{Z}_+, j \in \{1,\dots,d(n,k)\}\right\}$$

and

$$\Psi_T = \left\{\Psi_{\lambda,\eta,k,j} : \lambda \in \mathcal{Z}_T, \eta \in \{0,\dots,n(\lambda,T)\}, k \in \mathbb{Z}_+, j \in \{1,\dots,d(n,k)\}\right\}.$$

Theorem 14.36. *Let $T \in (\mathcal{E}'_\natural \cap \mathrm{Inv})(\mathbb{R}^n)$, $\mathcal{A}_{\mathcal{O}} = \varnothing$, $\Phi \subset C^\infty(\mathbb{R}^n)$, and*

$$\operatorname{span}_{C^\infty(B_R)}\Phi = C_T^\infty(B_R) \quad \textit{for each } R > 0. \tag{14.65}$$

Then

$$\operatorname{span}_{C^\infty(\mathcal{O})}\Phi = C_T^\infty(\mathcal{O}). \tag{14.66}$$

Proof. By assumption on T and Proposition 9.1(iv) there is $E \in \mathcal{D}'_\natural(\mathbb{R}^n)$ such that

$$E * T = \delta_0. \tag{14.67}$$

Take $w \in \mathcal{E}'(\mathcal{O})$ which is orthogonal to Φ and define $v = w * E$. It follows by (14.67) that

$$w = v * T. \tag{14.68}$$

We shall now need the following auxiliary fact. □

Lemma 14.4. *Let $R > r_0(w)$. Then $v = 0$ on $B_{R,+\infty}$.*

Proof. Let $\varepsilon \in (0, R - r_0(w))$ and $\psi \in \mathcal{D}_\natural\big(B_{r(T)+\varepsilon}\big)$. For each $x \in B_{R,+\infty}$, one has

$$\tau_x E * \psi * T = \tau_x \psi \tag{14.69}$$

because of (14.67). Since $\operatorname{supp} \tau_x\psi \subset B_{r(T)+\varepsilon}(x) \subset B_{R-r(T)-\varepsilon,+\infty}$, this yields $\tau_x E * \psi \in C_T^\infty(B_{R-\varepsilon})$. Appealing to (14.65), we find that

$$(v * \psi)(x) = \langle w, \tau_x E * \psi\rangle = 0.$$

As ψ is arbitrary, the desired conclusion follows. □

Passing to the proof of Theorem 14.36, consider the set

$$M = \left\{y \in \mathbb{R}^n\backslash\mathcal{O} : v = 0 \text{ on } B_{r(T)+\varepsilon}(y)\right\},$$

where

$$0 < \varepsilon < \operatorname{dist}\big(\operatorname{supp} w, \mathbb{R}^n\backslash\mathcal{O}\big). \tag{14.70}$$

Observe that M is closed and $B_{R+r(T)+\varepsilon} \cap (\mathbb{R}^n \backslash \mathcal{O}) \subset M$ due to Lemma 14.4. We claim that M is open in $\mathbb{R}^n \backslash \mathcal{O}$. Let $x \in M$, $y \in \mathbb{R}^n \backslash \mathcal{O}$, and $|x - y| < \varepsilon/2$. Then $v = 0$ on $B_{r(T)+\varepsilon/2}(y)$. In addition, (14.68) and (14.70) allow us to write $v * T = 0$ on $B_\varepsilon(y)$. By Theorem 14.2(i), $v = 0$ in $B_{r(T)+\varepsilon}(y)$. Hence, $B_{\varepsilon/2}(x) \cap (\mathbb{R}^n \backslash \mathcal{O}) \subset M$, proving the claim. According to what has been said above, this gives, by the assumption about $\mathcal{O}$, that $M = \mathbb{R}^n \backslash \mathcal{O}$. Therefore,

$$\text{supp } v \subset \left\{x \in \mathbb{R}^n : \text{dist}\left(x, \mathbb{R}^n \backslash \mathcal{O}\right) \geqslant r(T) + \varepsilon\right\} \subset \mathcal{O}_T.$$

Combining this with Lemma 14.4, we infer that $v \in \mathcal{E}'(\mathcal{O}_T)$. We then find, for each $f \in C_T^\infty(\mathcal{O})$,

$$\langle w, f\rangle = \langle v * T, f\rangle = \langle v, f * T\rangle = 0$$

(see (14.68)).Thus, every $w \in \mathcal{E}'(\mathcal{O})$ which is orthogonal to Φ is also orthogonal to $C_T^\infty(\mathcal{O})$. The Hahn–Banach theorem leads to $C_T^\infty(\mathcal{O}) \subset \text{span}_{C^\infty(\mathcal{O})}\Phi$. On the other hand, if $f \in \Phi$, then $f \in C_T^\infty(B_R)$ for each $R > 0$ (see (14.65)). Hence, $\Phi \subset C_T^\infty(\mathbb{R}^n)$, and (14.66) must be true. □

Corollary 14.9. *Let T and $\mathcal{O}$ be as assumed in Theorem 14.36. We then have the following.*

(i) $\text{span}_{C^\infty(\mathcal{O})} E_T = C_T^\infty(\mathcal{O})$.
(ii) $\text{span}_{C^\infty(\mathcal{O})} \Phi_T = C_T^\infty(\mathcal{O})$.

Proof. Part (i) is obvious from Theorems 14.36 and 14.35. Let us prove (ii). If $f \in E_T$ and (14.62) is fulfilled, then for each $R > 0$, the function f can be approximated in $C^\infty(B_R)$ by linear combinations of elements in Φ_T (see Propositions 14.1 and 14.8). Utilizing Theorem 14.35, we see that Φ_T satisfies (14.65). Hence, Theorem 14.36 is applicable, and (ii) holds. □

It can be shown that the assumption $\mathcal{A}_\mathcal{O} \neq \varnothing$ in Theorem 14.36 cannot be omitted (see Theorems 14.38–14.40 below). For the case where $\mathcal{A}_\mathcal{O} \neq \varnothing$, the analog of Theorem 14.36 goes as follows.

Theorem 14.37. *Assume that $T \in (\mathcal{E}'_\natural \cap \text{Inv})(\mathbb{R}^n)$, $\mathcal{A}_\mathcal{O} \neq \varnothing$, $A \subset \mathbb{R}^n \backslash \mathcal{O}$, and $A \cap S \neq \varnothing$ for each $S \subset \mathcal{A}_\mathcal{O}$. Let $\Phi \subset C^\infty(\mathbb{R}^n)$ satisfy (14.65). Suppose that $\Psi \subset C^\infty(\mathbb{R}^n \backslash \{0\})$ and*

$$\text{span}_{C^\infty(B_{\varepsilon,+\infty})}(\Phi \cup \Psi) = C_T^\infty(B_{\varepsilon,+\infty}) \quad \textit{for all } \varepsilon > 0. \tag{14.71}$$

Then

$$\text{span}_{C^\infty(\mathcal{O})}\left(\left(\bigcup_{a \in A} \tau_a \Psi\right) \cup \Phi\right) = C_T^\infty(\mathcal{O}).$$

Proof. Choose $E \in \mathcal{D}'_\natural(\mathbb{R}^n)$ such that (14.67) holds. Let $w \in \mathcal{E}'(\mathcal{O})$ and

$$\langle w, f\rangle = 0 \quad \text{for all } f \in \left(\bigcup_{a \in A} \tau_a \Psi\right) \cup \Phi. \tag{14.72}$$

Then (14.68) is true with $v = w * E$. Take $\varepsilon > 0$ satisfying (14.70). Now the following lemma is needed.

Lemma 14.5. *Let $a \in A$. Then $v = 0$ on $B_{r(T)+\varepsilon/2}(a)$.*

Proof. Assume that $\psi \in \mathcal{D}_\natural(B_{\varepsilon/2})$ and $y \in B_{r(T)+\varepsilon/2}(a)$. Then $y = a + x$ for some $x \in B_{r(T)+\varepsilon/2}$, and (14.69) is valid. As $\operatorname{supp} \tau_x\psi \subset B_{\varepsilon/2}(x) \subset B_{r(T)+\varepsilon}$, one obtains $\tau_x E * \psi \in C_T^\infty(B_{\varepsilon,+\infty})$. It follows from (14.65) and (14.71) that

$$\tau_{a+x}E * \psi \in \operatorname{span}_{C^\infty(B_{\varepsilon,+\infty}(a))}\big((\tau_a\Phi) \cup (\tau_a\Psi)\big) = \operatorname{span}_{C^\infty(B_{\varepsilon,+\infty}(a))}\big(\Phi \cup (\tau_a\Psi)\big). \tag{14.73}$$

By assumption on a and ε one has $w \in \mathcal{E}'(B_{\varepsilon,+\infty}(a))$. Having (14.72) in mind, we conclude from (14.73) that

$$(v * \psi)(y) = \langle w, \tau_{a+x}E * \psi\rangle = 0.$$

Again, Ψ being arbitrary, this shown that v must vanish on $B_{r(T)+\varepsilon/2}(a)$. □

The rest of the proof of Theorem 14.37 proceeds as that of Theorem 14.36, except that we must use Lemma 14.5 together with Lemma 14.4. □

Corollary 14.10. *Let $T \in \mathfrak{I}(\mathbb{R}^n)$ and assume that $\mathcal{O}$ and A satisfy the requirements in Theorem 14.37. Then*

$$\operatorname{span}_{C^\infty(\mathcal{O})}\left(\left(\bigcup_{a\in A}\tau_a\Psi_T\right) \cup \Phi_T\right) = C_T^\infty(\mathcal{O}).$$

The proof follows at once from Theorems 14.37 and 14.24(ii).

We shall now show that assumptions in Theorems 14.36 and 14.37 cannot be omitted.

Let $\mathcal{U}$ be an open subset of $\mathbb{R}^n$ such that $\mathcal{A}_{\mathcal{U}_T} \neq \varnothing$ and suppose that $\mathfrak{S} \subset \mathcal{A}_{\mathcal{U}_T}$, $\mathfrak{S} \neq \varnothing$. It is easy to check that the set

$$\mathcal{V} = \left(\bigcup_{S\in\mathfrak{S}} S\right) \cup \mathcal{U}$$

is open and

$$\mathcal{V}_T = \left(\bigcup_{S\in\mathfrak{S}} S\right) \cup \mathcal{U}_T. \tag{14.74}$$

Theorem 14.38. *Let $T \in (\mathcal{E}'_\natural \cap \operatorname{Inv})(\mathbb{R}^n)$, $f \in \mathcal{D}'(\mathcal{U})$, and assume that there is a sequence $u_m \in \mathcal{D}'_T(\mathcal{V})$, $m = 1, 2, \ldots$, such that $u_m \to f$ in $\mathcal{D}'(\mathcal{U})$. Then there exists a unique $F \in \mathcal{D}'_T(\mathcal{V})$ such that $F|_{\mathcal{U}} = f$.*

Proof. Let $S \in \mathfrak{S}$ and $0 < 4\varepsilon < \operatorname{dist}(S\backslash\mathcal{U}, (\mathbb{R}^n\backslash\mathcal{U})\backslash S) - 2r(T)$. For each $\alpha \geqslant 0$, we set

$$X_\alpha = \left\{x \in \mathbb{R}^n : \operatorname{dist}(x, S\backslash\mathcal{U}) \leqslant \alpha\right\}.$$

Select $f_\varepsilon \in \mathcal{D}'(\mathcal{U} \cup S)$ so that $f_\varepsilon = f$ on $\mathcal{U}\backslash X_\varepsilon$ and define $F_\varepsilon = f_\varepsilon * T = 0$. Since $f \in \mathcal{D}'_T(\mathcal{U})$, we get

$$\operatorname{supp} F_\varepsilon \subset X_{r(T)+\varepsilon} = \big\{x \in \mathbb{R}^n : \operatorname{dist}(x, S) \leqslant \varepsilon\big\}. \tag{14.75}$$

Suppose now that $h \in C^\infty(\mathbb{R}^n)$ and $h * T = 0$ in a neighborhood of X_ε. Choosing $\eta \in \mathcal{D}(\mathbb{R}^n)$ such that $\eta = 1$ on $X_{r(T)+2\varepsilon}$ and $\eta = 0$ on $\mathbb{R}^n\backslash X_{r(T)+3\varepsilon}$, we deduce that $\eta h \in \mathcal{D}((\mathcal{U} \cup S)_T)$ and $\operatorname{supp}(\eta h) * T \subset X_{2r(T)+3\varepsilon}\backslash X_\varepsilon$. Consequently, $(\eta h) * T \in \mathcal{D}(\mathcal{U})$ and

$$\langle u_m, (\eta h) * T\rangle = \langle u_m * T, \eta h\rangle = 0 \quad \text{for all } m \in \mathbb{N}. \tag{14.76}$$

Taking (14.75) into account, we infer that

$$\langle F_\varepsilon, h\rangle = \langle F_\varepsilon, \eta h\rangle = \langle f_\varepsilon, (\eta h) * T\rangle = \langle f, (\eta h) * T\rangle.$$

By assumption on f this, together with (14.76), yields

$$\langle F_\varepsilon, h\rangle = 0. \tag{14.77}$$

Next, let $E \in \mathcal{D}'_\natural(\mathbb{R}^n)$ satisfy (14.67), let $\psi \in \mathcal{D}_\natural(\mathbb{R}^n)$, and suppose that $x \in \mathbb{R}^n\backslash X_{r(\psi)+\varepsilon}$. Then $\delta_x * E * \psi * T = 0$ in a neighborhood of X_ε, where $\delta_x = \tau_x \delta_0$. So

$$(F_\varepsilon * E * \psi)(x) = \langle F_\varepsilon, \delta_x * E * \psi\rangle = 0$$

because of (14.77). Since ψ can be chosen arbitrarily, this leads to

$$\operatorname{supp}(F_\varepsilon * E) \subset X_\varepsilon. \tag{14.78}$$

Defining $g_\varepsilon = f_\varepsilon - F_\varepsilon * E \in \mathcal{D}'(\mathcal{U} \cup S)$, we see from (14.78) that $g_\varepsilon = f$ on $\mathcal{U}\backslash X_\varepsilon$. However, it can be verified that g_ε is independent of ε. In fact, for each $\varepsilon' \in (0, \varepsilon)$, one has $g_\varepsilon = g_{\varepsilon'}$ on $B_{r(T)+\varepsilon}(y)$ if $\operatorname{dist}(y, S\backslash\mathcal{U}) = r(T) + 2\varepsilon$. In addition, $(g_\varepsilon - g_{\varepsilon'}) * T = 0$ on the set

$$\bigcup_{x \in S} B_{4\varepsilon}(x) = \big\{x \in \mathbb{R}^n : \operatorname{dist}(x, S\backslash\mathcal{U}) < r(T) + 4\varepsilon\big\} \subset (\mathcal{U} \cup S)_T.$$

Therefore, g_ε and $g_{\varepsilon'}$ must coincide in view of Theorem 14.2(i). Thus, there exists a unique $F \in \mathcal{D}'_T(\mathcal{U} \cup S)$ such that $F|_{\mathcal{U}} = f$. As $S \in \mathfrak{S}$ above was arbitrary, the validity of the desired result is obvious. □

To continue, for each $\mu \in \mathcal{Z}_T$, denote by $\Delta_\mu(\mathcal{O})$ the set of all $f \in \mathcal{D}'(\mathcal{O})$ satisfying the equation

$$\big(\Delta + \mu^2\big)^{n(\mu,T)+1} f = 0 \quad \text{on } \mathcal{O}.$$

Also let

$$\mathcal{L}_\lambda(\mathcal{O}) = \bigcup_{\mu \in \mathcal{Z}_T\backslash\{\lambda\}} \Delta_\mu(\mathcal{O}), \quad \lambda \in \mathcal{Z}_T.$$

Theorem 14.39. *Let* $\lambda \in \mathcal{Z}_T$, $g \in \Delta_\lambda(\mathcal{U})$, *and* $g \notin \Delta_\lambda(\mathcal{V})|_{\mathcal{U}}$. *Then*

$$g \notin \operatorname{span}_{\mathcal{D}'(\mathcal{U})}\big(\mathcal{L}_\lambda(\mathcal{U}) \cup \mathcal{D}'_T(\mathcal{V})|_{\mathcal{U}}\big).$$

Proof. Assume the opposite and consider the mapping $\mathcal{K} : \mathcal{D}'(\mathcal{U}) \to \mathcal{D}'(\mathcal{U}_T)$ defined by the formula

$$\mathcal{K}(f) = f * T_{\lambda,0}, \quad f \in \mathcal{D}'(\mathcal{U}).$$

Corollary 14.3 tells us that $\mathcal{K}(f) = 0$ if $f \in \Delta_\mu(\mathcal{U})$ with $\mu \neq \lambda$ and $\mathcal{K}(f) = f$ for $f \in \Delta_\lambda(\mathcal{U})$. As $\mathcal{K}$ is continuous from $\mathcal{D}'(\mathcal{U})$ to $\mathcal{D}'(\mathcal{U}_T)$, one concludes that

$$g|_{\mathcal{U}_T} \in \operatorname{span}_{\mathcal{D}'(\mathcal{U}_T)} \Delta_\lambda(\mathcal{U}_T)$$

(see (14.74) and Theorem 14.35 with $T = (\Delta+\lambda^2)^{n(\lambda,T)+1}\delta_0$). Now Theorem 14.38 yields $g \in \Delta_\lambda(\mathcal{V}_T)|_{\mathcal{U}}$, but this is impossible since $\mathcal{V}_T \cup \mathcal{U} = \mathcal{V}$. □

Our final theorem is a generalization of Proposition 9.3(i).

Theorem 14.40. *Suppose that* $0 \leqslant a < b - 2r(T)$ *and let* $H_T = (\Phi_T \cup \Psi_T)|_{B_{a,b}}$. *Then*

$$f \notin \operatorname{span}_{\mathcal{D}'(B_{a,b})}(H_T \backslash \{f\}) \quad \textit{for each } f \in H_T.$$

Proof. The map $f \to f^{k,j} * T_{\lambda,0}$ is continuous from $\mathcal{D}'(B_{a,b})$ to $\mathcal{D}'(B_{a+r(T),b-r(T)})$ for all $\lambda \in \mathcal{Z}_T$, $k \in \mathbb{Z}_+$, and $j \in \{1,\dots,d(n,k)\}$. If $\mu \in \mathcal{Z}_T$ and $\nu \in \{0,\dots,n(\mu,T)\}$, then

$$\Phi_{\mu,\nu,k,j} * T_{\lambda,0} = \delta_{\mu,\lambda}\Phi_{\mu,\nu,k,j}$$

and

$$\Psi_{\mu,\nu,k,j} * T_{\lambda,0} = \delta_{\mu,\lambda}\Psi_{\mu,\nu,k,j}$$

in view of Proposition 14.11(i). Combining this with Proposition 9.3(i), we arrive at the desired statement. □

Chapter 15
Mean Periodic Functions on G/K

The purpose of this chapter is to present extensions of the results established for Euclidean space in Chap. 14 to noncompact symmetric spaces $X = G/K$. These include uniqueness theorems for mean periodic functions and related questions (Sects. 15.2–15.4), structure theorems and their applications (Sects. 15.5 and 15.6), and the finding of sharp growth conditions for mean periodic functions (Sect. 15.7).

If rank $X = 1$, an exact analog of Theorem 14.2 is valid for the convolution equation $f \times T = 0$ with $T \in \mathcal{E}'_\natural(X)$. The passage to higher rank of X involves new features. It turned out that the above result fails in general if $T \in \mathcal{E}'_\natural(X)$ and rank $X \geqslant 2$. The last circumstance rises the question of describing those $T \in \mathcal{E}'_\natural(X)$ for which the "correct" generalization of the uniqueness theorem does hold. We show that the class $\mathcal{E}'_{\natural\natural}(X)$ introduced in Part II provides one collection of such distributions T.

The study of the structure of mean periodic functions on X and finding sharp growth conditions for them depends heavily on many properties of generalized spherical functions. The results for symmetric spaces are different from that for Euclidean spaces (see, for example, Theorems 15.23(i), 14.24(i), and 15.30 and Corollary 14.7). The main reason for this is the difference in the asymptotic behavior of the corresponding elementary spherical functions.

The principal tools in this chapter come from Chap. 10. In addition, we use the theory developed in Chap. 13.

15.1 Preliminary Results

Throughout Chap. 15 we shall preserve the notation from Chap. 10. In particular, we assume that G is a noncompact connected semisimple Lie group with finite center, $K \subset G$ is a maximal compact subgroup, and $X = G/K$ is the associated symmetric space of noncompact type. Furthermore, we suppose that the Riemannian structure on X is induced by the Killing form in $\mathfrak{g}$.

V.V. Volchkov, V.V. Volchkov, *Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group,* Springer Monographs in Mathematics,

Let $\mathcal{O}$ be a nonempty open subset of X, and let $T \in \mathcal{E}'_{\natural}(X)$. We set

$$\mathcal{O}_T = \{x = go \in \mathcal{O} \colon g \in G_{\mathcal{O},T}\}, \tag{15.1}$$

where

$$G_{\mathcal{O},T} = \left\{g \in G : g\mathring{B}_{r(T)} \subset \mathcal{O}\right\}.$$

Throughout we suppose that $T \neq 0$ and $\mathcal{O}_T \neq \varnothing$. If $f \in \mathcal{D}'(\mathcal{O})$, the convolution $f \times T$ is well defined as a distribution in $\mathcal{D}'(\mathcal{O}_T)$. The object of this chapter is to investigate main classes of solutions of the convolution equation

$$f \times T = 0 \quad \text{in } \mathcal{O}_T. \tag{15.2}$$

Following Sect. 14.1, denote by $\mathcal{D}'_T(\mathcal{O})$ the set of all $f \in \mathcal{D}'(\mathcal{O})$ satisfying (15.2). Also we put

$$C^m_T(\mathcal{O}) = \left(\mathcal{D}'_T \cap C^m\right)(\mathcal{O}) \quad \text{for } m \in \mathbb{Z}_+ \text{ or } m = \infty,$$
$$C_T(\mathcal{O}) = C^0_T(\mathcal{O}), \quad \mathrm{QA}_T(\mathcal{O}) = \left(\mathcal{D}'_T \cap \mathrm{QA}\right)(\mathcal{O}), \quad \mathrm{RA}_T(\mathcal{O}) = \left(\mathcal{D}'_T \cap \mathrm{RA}\right)(\mathcal{O}),$$
$$G^\alpha_T(\mathcal{O}) = \left(\mathcal{D}'_T \cap G^\alpha\right)(\mathcal{O}) \quad \text{for } \alpha > 0.$$

If the set $\mathcal{O}$ is K-invariant, we define

$$\mathcal{D}'_{T,\natural}(\mathcal{O}) = \left(\mathcal{D}'_T \cap \mathcal{D}'_{\natural}\right)(\mathcal{O}).$$

Next, for $\psi \in \mathcal{E}'(\mathfrak{a})$ with $r(\psi) < R \leqslant +\infty$, we write $\mathcal{D}'_{\psi,W}(\mathcal{B}_R)$ for the set of all W-invariant $f \in \mathcal{D}'(\mathcal{B}_R)$ such that $f * \psi = 0$. As usual, let $C^m_{\psi,W}(\mathcal{B}_R) = (\mathcal{D}'_{\psi,W} \cap C^m)(\mathcal{B}_R)$ for $m \in \mathbb{Z}_+$ or $m = \infty$.

In this section we assemble certain preliminary information needed later.

Proposition 15.1.

(i) *If $f \in \mathcal{D}'_T(\mathcal{O})$, then $Df \in \mathcal{D}'_T(\mathcal{O})$ for each $D \in \mathbf{D}(X)$.*

(ii) *Let $f \in \mathcal{D}'_T(\mathcal{O})$, $\psi \in \mathcal{E}'_{\natural}(X)$, and assume that $\mathcal{O}_{T\times\psi} \neq \varnothing$. Then $f \times \psi \in \mathcal{D}'_T(\mathcal{O}_\psi)$.*

(iii) *Suppose that $\mathcal{O}$ is K-invariant and let $f \in \mathcal{D}'(\mathcal{O})$. Then $f \in \mathcal{D}'_T(\mathcal{O})$ if and only if $f_\delta \in \mathcal{D}'_T(\mathcal{O})$ for each $\delta \in \widehat{K}_M$. Next, if* rank $X = 1$*, then in order that $f \in \mathcal{D}'_T(\mathcal{O})$, it is necessary and sufficient that $f^{\delta,j} \in \mathcal{D}'_T(\mathcal{O})$ for all $\delta \in \widehat{K}_M$ and $j \in \{1, \ldots, d(\delta)\}$.*

(iv) *Let $R > r(T)$, $\delta \in \widehat{K}_M$, and $f \in \mathcal{D}'_{\check{\delta}}(B_R)$. Then $f \in \mathcal{D}'_T(B_R)$ if and only if all the entries of the matrix $\mathfrak{A}_\delta(f)$ are in the class $\mathcal{D}'_{\Lambda_+(T),W}(\mathcal{B}_R)$.*

Proof. Parts (i) and (ii) are consequences of the definition of $\mathcal{D}'_T(\mathcal{O})$. As for (iii), it is enough to apply Proposition 10.2(ii), (iii), (10.30), and (10.28). Finally, part (iv) follows at once from Theorem 10.12(ii), (iii). □

We now present analogues of Propositions 14.2 and 14.4.

Proposition 15.2.

(i) *Let* $T \in \mathcal{E}'_{\natural\natural}(X)$, *and let* p *be a polynomial such that the function* $\overset{\circ}{T}(z)/p(-z^2-|\rho|^2)$ *is entire. Suppose that* $f \in \mathcal{D}'(\mathcal{O})$ *and* $p(L)f = 0$ *in* $\mathcal{O}$. *Then* $f \in \mathrm{RA}_T(\mathcal{O})$.

(ii) *Assume that* $\operatorname{rank} X = 1$ *and let* $\mu \in \mathbb{C}$, $\nu \in \mathbb{Z}_+$, $\delta \in \widehat{K}_M$, $j \in \{1, \dots, d(\delta)\}$. *Then*

$$\Phi_{\mu,\nu,\delta,j} \times T = \sum_{\eta=0}^{\nu} \binom{\nu}{\eta}_{\mu} \overset{\circ}{T}^{\langle \nu-\eta\rangle}(\mu)\Phi_{\mu,\eta,\delta,j} \quad \text{in } X$$

and

$$\Psi_{\mu,\nu,\delta,j} \times T = \sum_{\eta=0}^{\nu} \binom{\nu}{\eta}_{\mu} \overset{\circ}{T}^{\langle \nu-\eta\rangle}(\mu)\Psi_{\mu,\eta,\delta,j} \quad \text{in } X\backslash \mathring{B}_{r(T)}.$$

In particular, if $\mu \in \mathcal{Z}_T$ *and* $\nu \in \{0, \dots, n(\lambda, T)\}$, *then* $\Phi_{\mu,\nu,\delta,j} \in \mathrm{RA}_T(X)$ *and* $\Psi_{\mu,\nu,\delta,j} \in \mathrm{RA}_T(X\backslash\{o\})$.

The proof can be given along the lines of the proof of Proposition 14.2 by using Theorem 10.7, Corollary 10.2, and Proposition 1.3.

Proposition 15.3. *Let* $T \in \mathcal{E}'_{\natural\natural}(X)$, $f \in \mathcal{D}'_T(\mathcal{O})$, *and let* $\lambda \in \mathcal{Z}_T$. *Then the following statements are valid.*

(i) $(L + \lambda^2 + |\rho|^2)f \times T_{\lambda,n(\lambda,T)} = 0$ *in* $\mathcal{O}_T$.

(ii) *If* $\lambda \neq 0$ *and* $n(\lambda, T) \geqslant 1$, *then*

$$\big(L + \lambda^2 + |\rho|^2\big)f \times T_{\lambda,n(\lambda,T)-1} + 2\lambda n(\lambda, T)f \times T_{\lambda,n(\lambda,T)} = 0 \quad \text{in } \mathcal{O}_T.$$

(iii) *If* $\lambda \neq 0$ *and* $n(\lambda, T) \geqslant 2$, *then*

$$\big(L + \lambda^2 + |\rho|^2\big)f \times T_{\lambda,\eta} + 2\lambda(\eta + 1)f \times T_{\lambda,\eta+1} + (\eta + 2)(\eta + 1)f \times T_{\lambda,\eta+2} = 0 \quad \text{in } \mathcal{O}_T$$

for each $\eta \in \{0, \dots, n(\lambda, T) - 2\}$.

(iv) *If* $0 \in \mathcal{Z}_T$ *and* $n(0, T) \geqslant 1$, *then*

$$\big(L + |\rho|^2\big)f \times T_{0,\eta} + (2\eta + 2)(2\eta + 1)f \times T_{0,\eta+1} = 0 \quad \text{in } \mathcal{O}_T$$

for all $\eta \in \{0, \dots, n(0, T) - 1\}$.

Proof. Because of (10.174), the desired results readily follow from Proposition 10.18. □

For the rest of the section, we suppose $\operatorname{rank} X = 1$.

Proposition 15.4. *Let* $R > r(T)$, $\delta \in \widehat{K}_M$, $j \in \{1, \dots, d(\delta)\}$, *and let* $f \in \mathcal{D}'_{\delta,j}(B_R)$. *Then the following items are equivalent.*

(i) $f \in \mathcal{D}'_T(B_R)$.

(ii) $\mathcal{A}_j^\delta(f) \in \mathcal{D}'_{T,\natural}(B_R)$.
(iii) $\mathfrak{A}_{\delta,j}(f) \in \mathcal{D}'_{A(T),\natural}(-R,R)$.

Proof. This is immediate from Theorems 10.25(i), (ii) and 10.21(i), (ii). □

Next, for $0 < R \leqslant +\infty$, we put

$$NB_R = \{x = ny \in X : n \in N,\ y \in B_R\}.$$

Denote by $\mathcal{D}'_{T,N}(B_R)$ the set of all $f \in \mathcal{D}'_T(B_R)$ such that $f = F|_{B_R}$ for some N-invariant $F \in \mathcal{D}'(NB_R)$. Also we set

$$C^m_{T,N}(B_R) = \big(\mathcal{D}'_{T,N} \cap C^m\big)(B_R)$$

for $m \in \mathbb{Z}_+$ or $m = \infty$.

Let $\delta \in \widehat{K}_M$, $j \in \{1,\dots,d(\delta)\}$, and let $f \in C^m_{\delta,j}(B_R)$ for some integer $m \geqslant s(\delta) + 2\alpha_X + 3$. As we already know from Theorem 10.20, there exists a function $u \in C^l(-R,R)$, $l = m - 2\alpha_X - 3$, such that

$$\int_K u\big(h\big(\tau^{-1}x\big)\big)\mathrm{e}^{\rho_X h(\tau^{-1}x)} Y_j^\delta(\tau M)\,\mathrm{d}\tau = f(x), \quad x \in B_R, \tag{15.3}$$

and

$$\big(\mathbf{D}^\delta u\big)(t) = \big(\mathbf{D}^\delta u\big)(-t), \quad t \in (-R,R).$$

Proposition 15.5. *Let $R > r(T)$ and assume that $f \in C^m_{\delta,j}(B_R)$ for some integer $m \geqslant s(\delta) + 2\alpha_X + 3$. Also let $u \in C^l(-R,R)$, $l = m - 2\alpha_X - 3$, be a solution of* (15.3) *such that the function $\mathbf{D}^\delta u$ is even. Then the function*

$$v(x) = \big(\mathbf{D}^\delta u\big)\big(h(x)\big)\mathrm{e}^{\rho_X h(x)}$$

is in the class $C^{l-s(\delta)}_{T,N}(B_R)$.

Proof. Let $\varepsilon \in (0, R - r(T))$ and $\eta \in \mathcal{D}_\natural(B_\varepsilon)$. Putting $Q = T \times \eta$ and $w = u \times Q$, we have $w \in C^\infty(B_{R-r(T)-\varepsilon})$ and $w(nx) = w(x)$ for all $x \in B_{R-r(T)-\varepsilon}$, $n \in N$ (see Proposition 10.1(ii)). In other words,

$$w(x) = w_1\big(h(x)\big)\mathrm{e}^{\rho_X h(x)}, \quad x \in B_{R-r(T)-\varepsilon}, \tag{15.4}$$

for some $w_1 \in C^\infty(-R + r(T) + \varepsilon, R - r(T) - \varepsilon)$. Relation (15.3) yields

$$\int_K w\big(\tau^{-1}x\big)Y_j^\delta(\tau M)\mathrm{d}\tau = (f \times Q)(x) = 0 \tag{15.5}$$

for each $x \in B_{R-r(T)-\varepsilon}$. Owing to Theorems 6.3, 10.7, and 10.1, there exists an even function $w_2 \in \mathcal{D}(\mathbb{R}^1)$ such that $\operatorname{supp} w_2 \subset [-\varepsilon - r(T), \varepsilon + r(T)]$ and

$$\big\langle w_2, \mathrm{e}^{\mathrm{i}zt}\big\rangle = \overset{\circ}{Q}(z) = \big\langle Q, \mathrm{e}^{(\mathrm{i}z+\rho_X)h(x)}\big\rangle, \quad z \in \mathbb{C}.$$

One has, since $z \in \mathbb{C}$ is arbitrary,

$$\langle w_2, u(t+\xi)\rangle = \big\langle Q, u\big(h(x)+\xi\big)e^{\rho_X h(x)}\big\rangle$$

for each $\xi \in (-R + r(T) + \varepsilon, R - r(T) - \varepsilon)$. Setting $\xi = h(ao)$, $a \in A$, by the definition of w_1 and Proposition 10.1(iv) we obtain

$$w_1\big(h(ao)\big) = \big\langle w_2, u\big(t + h(ao)\big)\big\rangle = \big\langle Q, u\big(h(ax)\big)e^{\rho_X h(x)}\big\rangle. \tag{15.6}$$

In view of (15.6), the function $\mathbf{D}^\delta w_1$ is even. Now it follows from (15.4), (15.5), and Corollary 10.5 that $\mathbf{D}^\delta w_1 = 0$ on $(-R + r(T) + \varepsilon, R - r(T) - \varepsilon)$. Then (15.6) implies that $v \in C_Q^{l-s(\delta)}(B_{R-\varepsilon})$. As $\eta \in \mathcal{D}_\natural(B_\varepsilon)$ is arbitrary, this, together with Proposition 10.1(ii), proves the required statement. □

Corollary 15.1. *Assume that $R > r(T)$ and $u \in \mathcal{D}'_{\Lambda(T),\natural}(-R, R)$. Then the distribution $u(h(x))e^{\rho_X h(x)}$ is in the class $\mathcal{D}'_{T,N}(B_R)$.*

Proof. Owing to Theorem 6.1, it is enough to consider the case $u \in C^\infty(-R, R)$. By Proposition 15.4 and Corollary 10.6, $\mathfrak{B}_{\delta,j}(u) \in C_T^\infty(B_R)$ for all $\delta \in \widehat{K}_M$ and $j \in \{1, \dots, d(\delta)\}$. Using now Theorem 10.23, Lemma 10.3, and Proposition 15.5, we arrive at the desired result. □

The following result describes the class $\mathcal{D}'_T(\mathcal{O})$ for the case where $r(T) = 0$.

Proposition 15.6. *Let $r(T) = 0$ and suppose that $\mathcal{O}$ is connected and K-invariant. Let $f \in \mathcal{D}'(\mathcal{O})$. Then $f \in \mathcal{D}'_T(\mathcal{O})$ if and only if for all $\delta \in \widehat{K}_M$ and $j \in \{1, \dots, d(\delta)\}$, the following relation holds:*

$$f^{\delta,j} = \sum_{\lambda \in \mathcal{Z}_T} \sum_{\eta=0}^{n(\lambda,T)} a_{\lambda,\eta,\delta,j}\Phi_{\lambda,\eta,\delta,j} + b_{\lambda,\eta,\delta,j}\Psi_{\lambda,\eta,\delta,j},$$

where $a_{\lambda,\eta,\delta,j}, b_{\lambda,\eta,\delta,j} \in \mathbb{C}$, and $b_{\lambda,\eta,\delta,j} = 0$, provided that $o \in \mathcal{O}$. In addition, if $f \in \mathcal{D}'_T(\mathcal{O})$, then the coefficients $a_{\lambda,\eta,\delta,j}$ and $b_{\lambda,\eta,\delta,j}$ are determined uniquely by f.

Proof. Thanks to Proposition 10.15, equation (15.2) can be rewritten as

$$\prod_{\lambda \in \mathcal{Z}_T} \big(L + \lambda^2 + \rho_X^2\big)^{n(\lambda,T)+1} f = 0 \quad \text{in } \mathcal{O}.$$

This, together with Corollary 10.2 and Proposition 10.4, brings us to the desired result (see Helgason [123], Chap. 3, the proof of Theorem 11.2). □

Analogues of the previous proposition for $r(T) > 0$ can be found in Sect. 15.6. We would like to end this section with the following analog of Proposition 14.9.

Proposition 15.7. *Let $\delta \in \widehat{K}_M$, $j \in \{1, \dots, d(\delta)\}$, and let E be an infinite subset of $\mathbb{C}$ such that $E \cap \mathcal{Z}(\overset{\circ}{T}) = \varnothing$ and $(-z) \in E$ for each $z \in E$. Assume that $\mathcal{O}$*

is bounded, connected, and K-invariant. Suppose that $\overset{\bullet}{B}_{r(T)+\varepsilon}(x) \subset \mathcal{O}$ for some $\varepsilon > 0$, $x \in X$. Also let $f \in (\mathcal{D}'_T \cap \mathcal{D}'_{\delta,j})(\mathcal{O})$, and let

$$\mathcal{U} = \left\{x \in X : \overset{\bullet}{B}_{\varepsilon}(x) \subset \mathcal{O}\right\}.$$

Then for each $m \in \mathbb{Z}_+$ there exists $g \in (C^m_T \cap \mathcal{D}'_{\delta,j})(\mathcal{U})$ such that $p(L)g = f$ in $\mathcal{U}$ for some polynomial p. Moreover, all the zeroes of the polynomial $q(z) = p(-z^2 - \rho_X^2)$ are simple, and $\mathcal{Z}(q) \subset E$.

Proof. It is not difficult to adapt the argument in the proof of Proposition 14.9 to show that there exists $F \in C^m_{\delta,j}(\mathcal{U})$ such that $p(L)F = f$ in $\mathcal{U}$ for some polynomial p satisfying $\mathcal{Z}(q) \cap \mathcal{Z}(q') = \varnothing$, $\mathcal{Z}(q) \subset E$, where $q(z) = p(-z^2 - \rho_X^2)$ (see Theorem 10.21). Once we have Propositions 15.6 and 15.2(ii), we can finish the proof arguing in the same way as in the proof of Proposition 14.9. □

15.2 Uniqueness Problem. Features for Higher Ranks

Let $T \in \mathcal{E}'_{\natural}(X)$, $T \neq 0$, and assume that $\mathcal{O}$ is a domain in X such that $\mathcal{O}_T \neq \varnothing$. In this section we shall study the uniqueness problem for distributions of the class $\mathcal{D}'_T(\mathcal{O})$ vanishing on some open subsets of $\mathcal{O}$. It is natural to assume that $\mathcal{O}$ is a ζ domain with $\zeta = r(T)$ (see Definition 1.1).

Theorem 15.1. *Let $T \in \mathcal{E}'_{\natural\natural}(X)$, $r(T) > 0$, and let $\mathcal{O}$ be a ζ domain in X with $\zeta = r(T)$ such that $\overset{\bullet}{B}_{r(T)} \subset \mathcal{O}$. Assume that $f \in \mathcal{D}'_T(\mathcal{O})$ and $f = 0$ in $B_{r(T)}$. Then the following assertions hold.*

(i) *If $f = 0$ in $B_{r(T)+\varepsilon}$ for some $\varepsilon > 0$, then $f = 0$ in $\mathcal{O}$. The same is true with $r(T) = 0$.*
(ii) *If $f \in C^\infty_T(\mathcal{O})$, then $f = 0$ in $\mathcal{O}$.*
(iii) *If $T = T_1 + T_2$ where $T_1 \in (\mathcal{E}'_{\natural\natural} \cap \mathcal{D})(X)$, $T_2 \in \mathcal{E}'_{\natural\natural}(X)$, and $r(T_2) < r(T)$, then $f = 0$ in $\mathcal{O}$.*
(iv) *If $\mathcal{O} = X$ and $T \notin \mathrm{Inv}_+(X)$, then $f = 0$ in $\mathcal{O}$.*
(v) *Assume that $\mathcal{O} = X$ and $T = T_1 \times T_2$ where $T_1, T_2 \in \mathcal{E}'_{\natural\natural}(X)$, $r(T_2) > 0$ and*

$$\inf_{\lambda \in \mathcal{Z}_{T_2}} \frac{\mathrm{Im}\,\lambda}{\log(2 + |\lambda|)} > 0.$$

Also let f be a distribution of finite order. Then $f = 0$ in $\mathcal{O}$.

Proof. It is enough to prove statements (i)–(iii) of Theorem 15.1 for the case where $\mathcal{O} = B_R$, $R \in (r(T), +\infty]$ (see Definition 1.1). In addition, it is easy to deduce the first assertion of Theorem 15.1 from its second assertion with the help of the standard smoothing method.

To prove (ii), suppose that $\delta \in \widehat{K}_M$. By (10.18) and Proposition 15.1(iii) we obtain

$$f_\delta = 0 \quad \text{in } B_{r(T)}$$

and $f_\delta \in C_T^\infty(\mathcal{O})$. Using now Theorem 10.12(iii) and Proposition 15.1(iv), we see that every matrix entry of $\mathfrak{A}_\delta(f_\delta)$ is in the class $\mathcal{D}'_{\Lambda_+(T),W}(\mathcal{B}_R)$ and

$$\mathfrak{A}_\delta(f_\delta) = 0 \quad \text{in } \mathcal{B}_{r(T)}.$$

Taking into account that $\mathfrak{A}_\delta(f_\delta) \in C^\infty(\mathcal{B}_R, \mathrm{Hom}(V_\delta, V_\delta^M))$ (see Theorem 10.12(iv)), we deduce from Theorem 14.2(ii) that $\mathfrak{A}_\delta(f_\delta) = 0$ in $\mathcal{B}_R$. Owing to Theorem 10.12(iii), this yields

$$f_\delta = 0 \quad \text{in } B_R.$$

Combining this with Proposition 10.2(iii), we have $f = 0$ in B_R, as contended.

The proof of (iii)–(v) is similar to that of (ii), the only change being that instead of Theorem 14.2(ii) we now use Theorems 14.2(iii), 14.4, and 10.12(v). This completes the proof of Theorem 15.1. □

The following results show that the assumptions in Theorem 15.1 cannot be considerably weakened.

Theorem 15.2.

(i) *Let $T \in \mathrm{Inv}_+(X)$ with $r(T) > 0$. Then there exists nonzero $f \in \mathcal{D}'_T(X)$ such that $f = 0$ in $B_{r(T)}$. In addition, for each $\varepsilon \in (0, r(T))$, there exists nonzero $f_\varepsilon \in C_T^\infty(X)$ such that*

$$f_\varepsilon = 0 \quad \text{in } B_{r(T)-\varepsilon}.$$

(ii) *Let $T \in \mathfrak{M}(X)$, $r(T) > 0$ and assume that $R \in (r(T), +\infty)$. Then for each $m \in \mathbb{Z}_+$, there exists nonzero $f \in C_T^m(B_R)$ such that $f = 0$ in $B_{r(T)}$. Moreover, if*

$$\frac{n_\lambda(\overset{\circ}{T}) + \mathrm{Im}\,\lambda}{\log(2 + |\lambda|)} \to 0 \quad \text{as } \lambda \to \infty,\ \lambda \in \mathcal{Z}_T, \tag{15.7}$$

then the same is valid with $R = +\infty$.

(iii) *If $r > 0$ and $\varepsilon \in (0, r)$, then there exists nonzero $T \in \mathcal{E}'_{\natural\natural}(X)$ such that $\mathrm{supp}\, T \subset \dot{B}_r$, $T \notin \mathrm{Inv}_+(X)$, and*

$$\left\{f \in C_T^\infty(X)\colon f = 0 \text{ in } B_{r-\varepsilon}\right\} \neq 0. \tag{15.8}$$

(iv) *For each $m \in \mathbb{Z}_+$, there exist nonzero functions $T \in (\mathfrak{N} \cap C^m)(X)$ and $f \in C_T^m(X)$ such that $f = 0$ in $B_{r(T)}$.*

Proof. To prove (i) and (ii), first assume that rank $X \geqslant 2$. Let $T \in \mathrm{Inv}_+(X)$, $R \in (r(T), +\infty)$, and $m \in \mathbb{Z}_+$. Theorems 14.7(i), (ii) and 14.8(iii) show that there exists nonzero $\psi \in \mathcal{D}'_{\Lambda_+(T),W}(\mathfrak{a})$ such that

$$\psi = 0 \quad \text{in } \mathcal{B}_{r(T)}.$$

Moreover, if $T \in \mathfrak{M}(X)$, then $\psi \in C^m(\mathcal{B}_R)$ and the same is true with $R = +\infty$, provided that (15.7) holds. Now define

$$f = \mathfrak{A}^{-1}(\psi).$$

By Proposition 15.1(iv) and Theorem 10.14(iii), (iv) we conclude that f is a nonzero distribution in the class $\mathcal{D}'_{T,\natural}(X)$ such that $f = 0$ in $B_{r(T)}$ and $f \in C^m(B_R)$. In addition, if (15.7) holds, then $f \in C^m(X)$. Next, let $\varepsilon \in (0, r(T))$. Choose $\eta_\varepsilon \in \mathcal{D}_\natural(B_\varepsilon)$ so that the function

$$f_\varepsilon = f \times \eta_\varepsilon$$

is nonzero. Then $f_\varepsilon \in C_T^\infty(X)$ and $f_\varepsilon = 0$ in $B_{r(T)-\varepsilon}$. Consequently, assertions (i) and (ii) are established provided that rank $X \geqslant 2$. The proofs of (i) and (ii) for the rank one case follow from the results in Sect. 15.4 (see Theorem 15.6).

Turning to (iii), let $T_1 \in \mathfrak{M}(X)$, supp $T_1 \subset \dot{B}_{r-\varepsilon/2}$, $\mathcal{Z}(\mathring{T}_1) \subset \mathbb{R}^1$, and $n_\lambda(\mathring{T}) = 1$ for all $\lambda \in \mathcal{Z}_T$. By (ii), there exists nonzero $f \in C_{T_1}(X)$ such that $f = 0$ in $B_{r(T)}$. Assume now that $w_1, w_2 \in (\mathcal{E}'_{\natural\natural} \cap \mathcal{D})(X)$, supp $w_1 \subset B_{\varepsilon/2}$, supp $w_2 \subset B_{\varepsilon/2}$, and the function $f \times w_2$ is nonzero. Setting

$$T = T_1 \times w_1,$$

we see that

$$T \in \big(\mathcal{E}'_{\natural\natural} \cap \mathcal{D}\big)(X), \quad \operatorname{supp} T \subset \dot{B}_r, \quad f \times w_2 \in C_T^\infty(X),$$

and

$$f \times w_2 = 0 \quad \text{in } B_{r-\varepsilon/2}.$$

Thus, $T \notin \mathrm{Inv}_+(X)$, and (15.8) holds (see Theorem 10.7).

To prove (iv) it is enough to consider $T \in (\mathfrak{N} \cap C^m)(X)$ with $\mathcal{Z}_T \subset \mathbb{R}^1$ and apply (ii). This concludes the proof of Theorem 15.2. □

Remark 15.1. Examining the above proof, we see that if rank $X \geqslant 2$, then for each $m \in \mathbb{Z}_+$, the function f in assertion (ii) can be chosen K-invariant. This statement is no longer valid in the rank one case (see Theorem 15.4(ii) below). The assumptions on T in (i) and (ii) cannot be omitted (see assertion (iii) of Theorem 15.1).

We shall now show that if rank $X \geqslant 2$, then the statements of Theorem 15.1 fail in general with $T \in \mathcal{D}_\natural(X)$, $T \neq 0$.

Theorem 15.3. *Assume that* rank $X \geqslant 2$. *Then there exists a nonzero function* $T \in \mathcal{D}_\natural(X)$ *with the following property: for each* $R > 0$, *there exists nonzero* K*-invariant* $f \in C_T^\infty(X)$ *such that* $f = 0$ *in* B_R.

Proof. Applying Theorem 14.6 with $n = \operatorname{rank} X$, we conclude that there exists a nonzero W-invariant function $\psi \in \mathcal{D}(\mathfrak{a})$ with the following property: for any

$R > 0$, there exists nonzero $g_R \in C^{\infty}_{\psi,W}(\mathfrak{a})$ such that $g_R = 0$ in $\mathcal{B}_R$. We define $T \in \mathcal{D}_{\natural}(X)$ by

$$\Lambda_{+}(T) = \psi$$

(see (10.103)). Then $T \neq 0$, and for every $R > 0$, the nonzero K-invariant function $f = \mathfrak{A}^{-1}(g_R)$ is in the class $C^{\infty}_{T}(X)$ (see Proposition 15.1(iv) and Theorem 10.14(iv)). In order to complete the proof, one need only remark that $f = 0$ in B_R by Theorem 10.14(iii). □

15.3 Refinements for the Rank One Case

For the rest of Chap. 15, unless otherwise stated, we assume that rank $X = 1$. For $\nu \in \mathbb{Z}$, we set $\mathcal{M}^{\nu}_{\natural}(X) = (\mathcal{M}^{\nu} \cap \mathcal{E}'_{\natural})(X)$ (see Sect. 1.2).

We now state and prove the following refinement of Theorem 15.1 for the rank one case.

Theorem 15.4. *Let $\nu, m \in \mathbb{Z}$, $m \geqslant \max\{0, 2[(1-\nu)/2]\}$, $T \in \mathcal{M}^{\nu}_{\natural}(X)$, and let $R > r(T) > 0$. Also let $f \in \mathcal{D}'_{T}(B_R)$ and assume that $f = 0$ in $B_{r(T)}$. Then the following assertions hold.*

(i) *If $f \in L^{1,\mathrm{loc}}_{m}(B_R)$, then $f^{\delta,j} = 0$ in B_R for all $\delta \in \widehat{K}_M$ and $j \in \{1, \dots, d(\delta)\}$ such that $s(\delta) \leqslant m + \nu + 1$.*
(ii) *If $f \in C^{m}(B_R)$, then $f^{\delta,j} = 0$ in B_R for all $\delta \in \widehat{K}_M$ and $j \in \{1, \dots, d(\delta)\}$ such that $s(\delta) \leqslant m + \nu + 2$.*

We point out that assumptions of this theorem in the general case cannot be weakened (see Theorem 15.7 below). In addition, by the definition of $\mathcal{M}^{\nu}(X)$, Theorem 10.7, Remark 10.1, and Proposition 10.15 we see that for each $T \in \mathcal{E}'_{\natural}(X)$, there exists $\nu \in \mathbb{Z}$ such that $T \in \mathcal{M}^{\nu}_{\natural}(X)$.

To prove Theorem 15.4 we require three further lemmas.

Lemma 15.1. *Let $T \in \mathcal{M}^{0}_{\natural}(X)$, $R > r(T) > 0$ and suppose that $\overset{\circ}{T} \in L^2(\mathbb{R}^1)$. Assume that $v \in C_{T,N}(B_R)$ and $v = 0$ in $B_{r(T)}$. Then $v = 0$ in B_R.*

Proof. By assumption on v, there exists $u \in C(-R, R)$ such that

$$v(x)\mathrm{e}^{-\rho_X h(x)} = u\big(h(x)\big) \quad \text{in } B_R$$

(see Proposition 10.1(ii)). Next,

$$\overset{\circ}{T}(z) = \big\langle T, \mathrm{e}^{(\mathrm{i}z+\rho_X)h(x)}\big\rangle = \big\langle \psi, \mathrm{e}^{\mathrm{i}zt}\big\rangle, \quad z \in \mathbb{C},$$

where $\psi = \Lambda(T) \in (L^2 \cap \mathcal{E}'_{\natural})(\mathbb{R}^1)$ and $r(\psi) = r(T)$. We have, since $z \in \mathbb{C}$ is arbitrary,

$$\int_{-r(T)}^{r(T)} \psi(t)u(t+\xi)\,\mathrm{d}t = \big\langle T, u\big(h(x)+\xi\big)\mathrm{e}^{\rho_X h(x)}\big\rangle \tag{15.9}$$

for each $\xi \in (-R + r(T), R - r(T))$. Bearing in mind that $v \in C_T(B_R)$ and using (15.9) with $\xi = h(ao)$, $a \in A$, by Proposition 10.1(iv) we find

$$\int_0^{\zeta} u\big(t + r(T)\big)\psi\big(\zeta - r(T) - t\big)\,\mathrm{d}t = 0$$

for $0 \leqslant \zeta < \min\{2r(T), R - r(T)\}$. Because $r(\psi) = r(T)$, this, together with Titchmarsh's theorem (see Corollary 6.1), implies that

$$u = 0 \quad \text{on } \big(r(T), \min\{3r(T), R\}\big).$$

The desired result is now obvious from Theorem 15.1(i) and the definition of u. □

Lemma 15.2. *Let $T \in \mathcal{M}^0_{\natural}(X)$, $R > r(T) > 0$, let $f \in (\mathcal{D}'_T \cap L^{1,\mathrm{loc}}_{\natural})(B_R)$, and assume that $f = 0$ in $B_{r(T)}$. Then $f = 0$ in B_R.*

Proof. We define a sequence of functions $f_q \in L^{1,\mathrm{loc}}_{\natural}(B_R)$, $q = 1, 2, \dots$ as follows. For $q \geqslant 2$, let

$$g_q(t) = \int_0^t \frac{1}{A_X(\xi)} \int_0^{\xi} g_{q-1}(\eta) A_X(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi, \quad t \in [0, R), \tag{15.10}$$

where the function $g_1\colon [0, R) \to \mathbb{C}$ is defined by $g_1(d(o, x)) = f(x)$. We set

$$f_q(x) = g_q\big(d(o, x)\big), \quad q = 1, 2, \dots.$$

By (15.10), for each q,

$$f_q = 0 \quad \text{in } B_{r(T)}. \tag{15.11}$$

In addition, if $q \geqslant 2$, then $f_q \in C^{2q-3}(B_R)$ and

$$Lf_q = f_{q-1} \quad \text{in } B_R \tag{15.12}$$

(see (10.16)). Using induction on q, we see from (15.12) and Proposition 15.6 that $f_q \in \mathcal{D}'_T(B_R)$ for all q (see the proof of Lemma 14.1).

Next, there exists $T_1 \in (C \cap \mathcal{E}'_{\natural})(X)$ such that

$$r(T_1) = r(T), \quad \overset{\circ}{T}_1 \in L^2\big(\mathbb{R}^1\big), \quad \text{and} \quad p(L)T_1 = T$$

for some polynomial p. Choose $q \in \mathbb{N}$ so that the function $F = p(L)f_q$ is in the class $C^{2\alpha_X+3}_{T_1}(B_R)$. According to Theorem 10.20, there is an even function $u \in C(-R, R)$ such that

$$\int_K u\big(h\big(\tau^{-1}x\big)\big)\mathrm{e}^{\rho_X h(\tau^{-1}x)}\,\mathrm{d}\tau = F(x), \quad x \in B_R. \tag{15.13}$$

Letting

$$v(x) = u\big(h(x)\big)\mathrm{e}^{\rho_X h(x)},$$

we see from (15.11), (15.13), and Corollary 10.5 that $v = 0$ in $B_{r(T)}$. Furthermore, thanks to Proposition 15.5, $v \in C_{T_1,N}(B_R)$. Applying Lemma 15.1, we deduce from (15.13) that $v = F = 0$ in B_R. In view of the ellipticity of the operator $p(L)$, relations (15.11) and (15.12) now give $f = 0$ in B_R, as we wished to prove. □

Lemma 15.3. *Let $T \in \mathcal{M}_\natural^\nu(X)$ for some $\nu \in \{0, 1\}$, and let $R > r(T) > 0$. Assume that $f \in \mathcal{D}'_T(B_R)$ and $f = 0$ in $B_{r(T)}$. Then assertions* (i) *and* (ii) *of Theorem 15.4 hold.*

Proof. First, assume that $f \in L_m^{1,\mathrm{loc}}(B_R)$ for some $m \in \mathbb{Z}_+$. Let $\varepsilon \in (0, R - r(T))$, $\eta \in \mathcal{D}_\natural(B_R)$, and let $\eta = 1$ in $B_{R-\varepsilon}$. Suppose that $\delta \in \widehat{K}_M$ and $j \in \{1, \dots, d(\delta)\}$. By (10.27) and Proposition 15.1(iii) we conclude that

$$(f\eta)^{\delta,j} \in \mathcal{D}'_T(B_{R-\varepsilon}) \quad \text{and} \quad (f\eta)^{\delta,j} = 0 \quad \text{in } B_{r(T)}.$$

Assume now that $s(\delta) \leqslant m + \nu + 1$. For convenience, we put

$$(f\eta)^{\delta,j} = u.$$

By Proposition 15.4 we have $\mathcal{A}_j^\delta(u) \in \mathcal{D}'_T(B_{R-\varepsilon})$ and $\mathcal{A}_j^\delta(u) = 0$ in $B_{r(T)}$ (see Lemma 10.6(i)). According to Proposition 10.16(iv), this yields

$$p(L)\big(\mathcal{B}_j^\delta(u) \times T\big) = 0 \quad \text{in } B_{R-\varepsilon-r(T)}$$

and

$$p(L)\mathcal{B}_j^\delta(u) = 0 \quad \text{in } B_{r(T)}$$

for some polynomial p of degree $s(\delta)$. Then by Proposition 15.6 there exists $\varphi \in \mathrm{RA}_\natural(X)$ such that $p(L)\varphi = 0$ in X and $\varphi = \mathcal{B}_j^\delta(u)$ in $B_{r(T)}$. Putting

$$v = \mathcal{B}_j^\delta(u) - \varphi, \quad w = v \times T,$$

we obtain

$$v = 0 \quad \text{in } B_{r(T)} \tag{15.14}$$

and $p(L)w = 0$ in $B_{R-\varepsilon-r(T)}$. By Theorem 10.26(i) we infer that $v \in L^{1,\mathrm{loc}}_{m+s(\delta)}(B_{R-\varepsilon})$. If $s(\delta) = 0$, then by (15.14) and Lemma 15.2 we have $v = 0$ in $B_{R-\varepsilon}$.

Now suppose that $s(\delta) \geqslant 1$. Since $s(\delta) \leqslant m + \nu + 1$, we see that $v \in L^{1,\mathrm{loc}}_{2s(\delta)-1}(B_{R-\varepsilon})$. Then the definition of $\mathcal{M}^\nu(X)$ and relation (15.14) yield

$$\big(L^q w\big)(0) = 0 \quad \text{for all } q = 0, \dots, s(\delta) - 1.$$

This, together with Proposition 15.6, implies that $w = 0$ in $B_{R-\varepsilon-r(T)}$. As before, by Lemma 15.2 we find that $v = 0$ in $B_{R-\varepsilon}$. In view of Proposition 10.16(iv), we have

$$\mathcal{A}_j^\delta(u) = p(L)\mathcal{B}_j^\delta(u) = p(L)v = 0 \quad \text{in } B_{R-\varepsilon}.$$

Lemma 10.6(i) now gives $u = 0$ in $B_{R-\varepsilon}$. Taking into account that $u = f^{\delta,j}$ in $B_{R-\varepsilon}$, we obtain, since $\varepsilon \in (0, R - r(T))$ was arbitrary, $f^{\delta,j} = 0$ in B_R. Thus, assertion (i) of Theorem 15.4 holds for $\nu \in \{0, 1\}$. The proof of assertion (ii) is similar to that of (i), the only change being that instead of Theorem 10.26(i) we now use Theorem 10.26(ii). Lemma 15.3 is thereby established. □

Proof of Theorem 15.4. Let $\delta \in \widehat{K}_M$, $j \in \{1, \ldots, d(\delta)\}$, $\varepsilon \in (0, R - r(T))$, and let u be defined as Lemma 15.3. Then by assumption on f and Proposition 15.1(iii) we deduce $u = 0$ in $B_{r(T)}$ and $u \in \mathcal{D}'_T(B_{R-\varepsilon})$. In order to conclude the proof of the theorem, it is sufficient to repeat the arguments in the proof of Theorem 14.1 using Lemmas 15.3 and 10.7, Corollary 10.4, and the definition of $\mathcal{M}^\nu(X)$. □

We recall from Theorem 10.7 that for each $T \in \mathcal{E}'_\natural(X)$ the following estimate holds:

$$\big|\overset{\circ}{T}(z)\big| \leqslant \gamma_1(1 + |z|)^{\gamma_2} \mathrm{e}^{r(T)|\mathrm{Im}\, z|}, \quad z \in \mathbb{C}, \tag{15.15}$$

where $\gamma_1 > 0$ and $\gamma_2 \in \mathbb{R}^1$ are independent of z. The value of γ_2 in (15.15) and Remark 10.1 allow us come to a conclusion on a character of smoothness of T.

As a consequence of Theorem 15.4, we obtain the following statement.

Corollary 15.2. *Let $T \in \mathcal{E}'_\natural(X)$, $R > r(T) > 0$, and let*

$$f \in C^m_T(B_R) \quad \textit{for some integer } m \geqslant \max\big\{0, 2\big[(2\alpha_X + 5 - [-\gamma_2])/2\big]\big\},$$

where $\gamma_2 \in \mathbb{R}^1$ is the constant from estimate (15.15)*. Assume that $f = 0$ in $B_{r(T)}$. Then $f^{\delta,j} = 0$ in B_R for all $\delta \in \widehat{K}_M$ and $j \in \{1, \ldots, d(\delta)\}$ with $s(\delta) \leqslant m - 2\alpha_X - 2 + [-\gamma_2]$.*

Proof. We set $\nu = -2\alpha_X - 4 + [-\gamma_2]$. Taking (15.15) and Remark 10.1 into account, by (10.73) we have $T \in \mathcal{M}^\nu(X)$. Using now Theorem 15.4(ii), we get the required result. □

Our goal in the rest of this section is to present another refinement of Theorem 15.1(ii). For $r > 0$, let

$$S_r^+ = \left\{x \in S_r : \frac{\mathrm{d}}{\mathrm{d}t}\big(d(a_t x, o)\big)\bigg|_{t=0} \geqslant 0\right\}.$$

We note that $a_r o \in S_r^+$ and $a_{-r} o \notin S_r^+$. In addition,

$$mx \in S_r^+ \quad \text{for all } m \in M,\ x \in S_r^+.$$

The set S_r^+ is an analog of a hemisphere for the space X (see (2.11), (2.30), (2.43), and (2.62)).

Theorem 15.5. *Let $T \in \mathcal{E}'_\natural(X)$, $r(T) > 0$, let $\mathcal{O}$ be a ζ domain in X with $\zeta = r(T)$ containing the ball $\mathring{B}_{r(T)}$, and let $f \in \mathcal{D}'_T(\mathcal{O})$. Assume that $f = 0$ in $B_{r(T)}$ and $f \in C^\infty(\mathcal{O}_1)$ for some open subset $\mathcal{O}_1$ of $\mathcal{O}$ such that $S^+_{r(T)} \subset \mathcal{O}_1$. Then $f = 0$ in $\mathcal{O}$.*

We point out that the set $S^+_{r(T)}$ in this theorem cannot be decreased in general (see Remark 15.2 below).

Proof of Theorem 15.5. It is enough to consider the case where $\mathcal{O} = B_R$ for some $R > r(T)$. First, assume that $T \in (\mathcal{E}'_\natural \cap C^m)(X)$, where $m = 2\alpha_X + 1$. For $\lambda \in \mathcal{Z}_T$ and $\eta \in \{0, \dots, n(\lambda, T)\}$, we set

$$F = f \times T^{\lambda,\eta},$$

where $T^{\lambda,\eta}$ is defined by (10.175). Owing to Remark 10.1 and (10.75), $T^{\lambda,\eta} \in (\mathcal{E}'_\natural \cap C)(X)$, $r(T^{\lambda,\eta}) = r(T)$, and

$$\left(L + \lambda^2 + \rho_X^2\right)^{\eta+1} F = 0 \quad \text{in } B_{R-r(T)}. \tag{15.16}$$

Let $\mathcal{O}_2 \subset X$ be an open set such that $S^+_{r(T)} \subset \mathcal{O}_2$ and the closure of $\mathcal{O}_2$ is contained in $\mathcal{O}_1$. By the definition of $S^+_{r(T)}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(d(a_t x, o)\right)\bigg|_{t=0} < 0 \quad \text{for all } x \notin S^+_{r(T)}.$$

Hence, there exists $\varepsilon_0 > 0$ such that $d(a_t x, o) < r(T)$ for all $x \in S_{r(T)} \setminus \mathcal{O}_2$, $t \in (0, \varepsilon_0)$.

Let $\mathcal{U} \subset K$ be an open neighborhood of the unity in K such that $kS^+_{r(T)} \subset \mathcal{O}_2$ for each $k \in \mathcal{U}$. According to what has been said above, for some $\varepsilon_1 \in (0, \varepsilon_0)$, we can write

$$F(ka_t o) = \int_{B_{r(T)}} T^{\lambda,\eta}(x) f(ka_t x)\,\mathrm{d}x, \quad k \in \mathcal{U},\ t \in (0, \varepsilon_1).$$

By assumption on f this yields

$$\lim_{t\to+0}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^\nu F(ka_t o) = 0, \quad k \in \mathcal{U},\ \nu \in \mathbb{Z}_+. \tag{15.17}$$

In view of the ellipticity of the operator $(L + \lambda^2 + \rho_X^2)^{\eta+1}$ and (15.16), we have $F \in \mathrm{RA}(B_{R-r(T)})$. Thus, (15.17) implies that

$$F(ka_t o) = 0 \quad \text{for all } k \in \mathcal{U},\ t \in (0, \varepsilon_1).$$

Therefore, $F = 0$ in $B_{R-r(T)}$. This, together with Theorem 10.28, gives $f = 0$ in B_R.

Let us now consider the general case. By Theorem 10.7 and (10.73) there exists $T_1 \in (\mathcal{E}'_\natural \cap C^{2\alpha_X+1})(X)$ such that $r(T_1) = r(T)$ and $p(L)T_1 = T$ for some polynomial p. Then

$$p(L)f \in \mathcal{D}'_{T_1}\left(B_R\right), \quad p(L)f = 0 \quad \text{in } B_{r(T_1)},$$

and $p(L)f \in C^\infty(\mathcal{O}_1)$. The above arguments show that $p(L)f = 0$ in B_R. Since $f = 0$ in $B_{r(T)}$, the desired result is now obvious from the ellipticity of the operator $p(L)$. □

Corollary 15.3. *Let $T \in \mathcal{E}'_\natural(X)$, $r(T) > 0$, and let $p(L)T = 0$ in $B_{r(T)}$ for some nonzero polynomial p. Assume that $R > r(T)$ and $\varepsilon \in (0, R - r(T))$. Then the following statements are valid.*

(i) *If $f \in \mathcal{D}'_T(B_R)$, $f = 0$ in $B_{r(T)-\varepsilon,r(T)}$, and $f \in C^\infty(\mathcal{O})$, where $\mathcal{O}$ is an open subset of B_R such that $S^+_{r(T)} \subset \mathcal{O}$, then $f = 0$ in $B_{r(T)-\varepsilon,r(T)+\varepsilon}$.*
(ii) *If $f \in C^\infty_T(B_R)$ and $f = 0$ in $B_{r(T),r(T)+\varepsilon}$, then $f = 0$ in $B_{r(T)-\varepsilon,r(T)+\varepsilon}$.*

Proof. There is no difficulty in modifying the proof of Corollary 14.2 using Theorems 15.5, 15.1(ii), 10.25(i), (vii), and Proposition 15.1(iii) to obtain the desired result. □

Remark 15.2. Let $T \in \mathrm{Inv}_+(X)$ with $r(T) > 0$. The example

$$f(x) = \zeta'_{\Lambda(T)}\big(h(x)\big)e^{\rho_X h(x)}, \quad x \in X,$$

shows that

$$\big\{f \in \mathcal{D}'_{T,N}(X)\colon\ f = 0 \text{ in } NB_{r(T)}\big\} \neq \{0\} \tag{15.18}$$

(see Corollary 15.1 and Proposition 8.20(i), (ii), (v)). However, we cannot replace $\mathcal{D}'_{T,N}(X)$ in (15.18) by $C_{T,N}(X)$ (see Lemma 15.1).

15.4 Counterexamples to the Uniqueness Problem

In this section we show that the assertions of Theorem 15.4 cannot be refined in the general case.

Let $T \in \mathrm{Inv}_+(X)$. Following Sect. 14.3, for all $\delta \in \widehat{K}_M$ and $j \in \{1, \dots, d(\delta)\}$, we define the distribution $\zeta_{T,\delta,j}$ by letting

$$\zeta_{T,\delta,j} = -\mathfrak{A}^{-1}_{\delta,j}\big(\zeta'_{\Lambda(T)}\big). \tag{15.19}$$

In the case where δ is an identity representation, we have $j = 1$ and write $\zeta_{T,\mathrm{triv}}$ instead of $\zeta_{T,\delta,j}$. Now we state a result similar to Theorem 14.7.

Theorem 15.6. *If $T \in \mathrm{Inv}_+(X)$, then the following assertions hold.*

(i) *$\zeta_{T,\delta,j} \in (\mathcal{D}'_T \cap \mathcal{D}'_{\delta,j})(X)$ and $\mathcal{A}^\delta_j(\zeta_{T,\delta,j}) = \zeta_{T,\mathrm{triv}}$.*
(ii) *If $r(T) > 0$, then $\zeta_{T,\delta,j} = 0$ in $\mathring{B}_{r(T)}$ and $S_{r(T)} \subset \operatorname{supp} \zeta_{T,\delta,j}$.*
(iii) *If $R > r(T)$, $x \in X$, $u \in \mathcal{E}'_\natural(X)$, and $\zeta_{T,\delta,j} \times u = 0$ in $B_R(x)$, then*

$$u = T \times v$$

for some $v \in \mathcal{E}'_\natural(X)$.

(iv) *If* $T \in \mathfrak{M}(X)$, *then*

$$\zeta_{T,\delta,j} = \sum_{\eta=0}^{n(0,T)} \frac{a^{0,0}_{2(n(0,T)-\eta)}(\overset{\circ}{T})}{(2\eta)!} \Phi_{0,\eta,\delta,j} + 2 \sum_{\lambda \in \mathcal{Z}_T \setminus \{0\}} \sum_{\eta=0}^{n(\lambda,T)} a^{\lambda,\eta}_{n(\lambda,T)}\big(\overset{\circ}{T}\big)(\eta\Phi_{\lambda,\eta-1,\delta,j} + \lambda\Phi_{\lambda,\eta,\delta,j}), \tag{15.20}$$

where the series converges in $\mathcal{D}'(X)$, *and the first sum is set to be equal to zero if* $0 \notin \mathcal{Z}_T$.

(v) *If* $T \in \mathfrak{M}(X)$, $m \in \mathbb{Z}_+$, *and* $R \in (0, +\infty)$, *then there exists* $\sigma = \sigma(m, R, T) > 0$ *such that* $\zeta_{T,\delta,j} \in C^m(B_R)$, *provided that* $s(\delta) > \sigma$. *In addition, if*

$$\frac{n(\lambda, T) + \operatorname{Im}\lambda}{\log(2 + |\lambda|)} \to 0 \quad \text{as } \lambda \to \infty,\ \lambda \in \mathcal{Z}_T,$$

then the same is true with $R = +\infty$.

Proof. The arguments are quite parallel to the proof of Theorem 14.7 except that we now use Theorems 10.21, 10.25, 15.1, and Proposition 10.7(i). □

To go further, for $r > 0$, we define the distribution $\delta_{S_r} \in \mathcal{E}'_\natural(X)$ by the formula

$$\langle \delta_{S_r}, u\rangle = \int_K u(ka_t o)\,\mathrm{d}k, \qquad t = \kappa r,\ u \in \mathcal{E}(X)$$

(see Sect. 10.1). Also we set

$$\chi_{B_r}(x) = \begin{cases} 1 & \text{if } x \in B_r, \\ 0 & \text{if } x \in X \setminus B_r. \end{cases}$$

We shall now investigate $\zeta_{T,\delta,j}$ in greater detail in the cases where $T = \delta_{S_r}$ and $T = \chi_{B_r}$.

Theorem 15.7. *Let* $r > 0$, *and let* $\delta \in \widehat{K}_M$ *and* $j \in \{1, \dots, d(\delta)\}$. *Then the following results are true.*

(i) *If* $T = \delta_{S_r}$, *then* $\zeta_{T,\delta,j} \in L^{p,\mathrm{loc}}_{s(\delta)-2}(X)$, *provided that* $s(\delta) \geqslant 2$, $p \in [1, +\infty)$, *and* $\zeta_{T,\delta,j} \in C^{s(\delta)-3}(X)$ *for* $s(\delta) \geqslant 3$.

(ii) *If* $T = \chi_{B_r}$ *then* $\zeta_{T,\delta,j} \in L^{p,\mathrm{loc}}_{s(\delta)-3}(X)$, *provided that* $s(\delta) \geqslant 3$, $p \in [1, +\infty)$, *and* $\zeta_{T,\delta,j} \in C^{s(\delta)-4}(X)$ *for* $s(\delta) \geqslant 4$.

Proof. First, we assume that $T = \delta_{S_r}$. Then by (10.137), (10.71), and Theorem 10.3 we find

$$\overset{\circ}{T}(z) = \varphi^{(\xi,\eta)}_\lambda(\kappa r), \qquad z \in \mathbb{C}, \tag{15.21}$$

where $\xi = \alpha_X$, $\eta = \beta_X$, $\lambda = z/\kappa$. Because of Proposition 7.6(i), all the zeros of $\overset{\circ}{T}$ are real, simple, and $0 \notin \mathcal{Z}(\overset{\circ}{T})$. So (15.20) can be written

$$\zeta_{T,\delta,j} = \sum_{m=1}^{\infty} \frac{2\lambda_m}{\overset{\circ}{T}{}'(\lambda_m)} \Phi_{\lambda_m,0,\delta,j}, \tag{15.22}$$

where $\{\lambda_1, \lambda_2, \dots\}$ is the sequence of all positive zeros of $\overset{\circ}{T}$ numbered in the ascending order. Using now (15.21), Proposition 7.4, and Proposition 7.6(ii), we see that

$$\frac{\overset{\circ}{T}{}'(\lambda_m)}{\lambda_m} = c(-1)^m m^{-\alpha_X - 3/2} + O\big(m^{-\alpha_X - 5/2}\big) \quad \text{as } m \to \infty, \tag{15.23}$$

where the constant $c \neq 0$ is independent of m. Combining this with Theorem 10.3 and Propositions 7.2(ii), and 7.4, we infer that for $R > r$ and $s(\delta) \geqslant 3$, series (15.22) converges in $C^{s(\delta)-3}(\dot{B}_R \backslash B_{r/2})$. In particular, $\zeta_{T,\delta,j} \in C^{s(\delta)-3}(X)$ for $s(\delta) \geqslant 3$ (see Theorem 15.6(ii)).

To continue, let us assume that $s(\delta) \geqslant 2$ and $x = ka_t o \in X \backslash B_{r/2}$, where $k \in K$ and $a_t \in A$ (see (10.22)). Then

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{s(\delta)-2} \zeta_{T,\delta,j}(x) = \Bigg(u_1(t) \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \cos\left(\frac{\pi m t}{r}\right) + u_2(t) \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sin\left(\frac{\pi m t}{r}\right) + u_3(t)\Bigg) Y_j^{\delta}(kM),$$

where $u_1, u_2, u_3 \in C(r\kappa/2, +\infty)$ (see (15.23) and Proposition 10.5). Bearing in mind that for $t \in (-\pi, \pi)$,

$$\sum_{m=1}^{\infty} \frac{(-1)^m}{m} \cos mt = \log(2 + 2\cos t)$$

and

$$\sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sin mt = -t,$$

we see that $\zeta_{T,\delta,j} \in L^{p,\mathrm{loc}}_{s(\delta)-2}(X \backslash B_{r/2})$ for $p \in [1, +\infty)$. According to Theorem 15.6(ii), this proves part (i).

Let us prove (ii). Since $T = \chi_{B_r}$, we conclude from (10.15), (10.57), and Theorem 10.3 that

$$\overset{\circ}{T}(z) = \int_0^r \varphi_{z/\kappa}^{(\alpha_X, \beta_X)}(\kappa t) A_X(t)\, \mathrm{d}t, \quad z \in \mathbb{C}.$$

This, together with (10.14) and Proposition 7.2(iii), yields

$$\overset{\circ}{T}(z) = c\varphi_{z/\kappa}^{(\alpha_X+1,\beta_X+1)}(\kappa r), \quad z \in \mathbb{C}, \tag{15.24}$$

where the constant $c > 0$ is independent of z. The rest of the proof of (ii) is quite similar to that of (i), the only change being that instead of (15.21) we now use (15.24). □

It can be shown that $\delta_{s_r} \in \mathcal{M}_{\natural}^0(X)$ and $\chi_{B_r} \in \mathcal{M}_{\natural}^1(X)$ for any $r > 0$. Thus, Theorem 15.7 shows that the assertions of Theorem 15.4 are precise.

To conclude we give an analog of Theorem 14.10.

Theorem 15.8. *Let* $T \in \mathrm{Inv}_+(X)$, $r(T) > 0$, $\delta \in \widehat{K}_M$, *and* $j \in \{1, \ldots, d(\delta)\}$. *Then the following statements are valid.*

(i) *If* $0 < r \leqslant r(T) < R \leqslant +\infty$ *and* $f \in (\mathcal{D}'_T \cap \mathcal{D}'_{\delta,j})(B_R)$, *then in order that* $f = 0$ *in* B_r, *it is necessary and sufficient that*

$$f = \zeta_{T,\delta,j} \times U \quad \textit{in } B_R \tag{15.25}$$

for some $U \in \mathcal{E}'_{\natural}(X)$ *with* $\operatorname{supp} U \subset \dot{B}_{r(T)-r}$.

(ii) *If* $R \in (r(T), +\infty]$ *and* $f \in (C_T^\infty \cap \mathcal{D}'_{\delta,j})(B_R)$, *then* $(Df)(o) = 0$ *for each differential operator* D *if and only if relation* (15.25) *is fulfilled for some* $U \in \mathcal{D}_{\natural}(X)$ *with* $\operatorname{supp} U \subset \dot{B}_{r(T)}$.

This theorem can be proved in the same way as Theorem 14.10 with attention to Theorems 10.21 and 15.6.

15.5 Generalized Spherical Functions Series

The aim of this section is to obtain symmetric space analogues of results from Sect. 14.4. Throughout the section we assume that $T \in \mathcal{E}'_{\natural}(X)$, $T \neq 0$, and that $\mathcal{O}$ is a K-invariant domain in X such that the set $\mathcal{O}_T$ is nonempty (see (15.1)).

Let

$$\lambda \in \mathcal{Z}_T, \quad \eta \in \{0, \ldots, n(\lambda, T)\}, \quad \delta \in \widehat{K}_M, \quad j \in \{1, \ldots, d(\delta)\}.$$

For each $f \in \mathcal{D}'_T(\mathcal{O})$, we define the coefficients $a_{\lambda,\eta,\delta,j}(T, f)$ and $b_{\lambda,\eta,\delta,j}(T, f)$ as follows. Using Proposition 15.1(iii) and Proposition 10.18, we conclude that $f^{\delta,j} \in (\mathcal{D}'_T \cap \mathcal{D}'_{\delta,j})(\mathcal{O})$ and

$$\left(L + \lambda^2 + \rho_X^2\right)^{n(\lambda,T)+1}\left(f^{\delta,j} \times T_{\lambda,0}\right) = 0 \quad \text{in } \mathcal{O}_T.$$

According to Propositions 15.6 and 10.15, there exist constants $a_{\lambda,\eta,\delta,j}(T, f)$, $b_{\lambda,\eta,\delta,j}(T, f) \in \mathbb{C}$ such that

$$f^{\delta,j} \times T_{\lambda,0} = \sum_{\eta=0}^{n(\lambda,T)} a_{\lambda,\eta,\delta,j}(T, f)\Phi_{\lambda,\eta,\delta,j} + b_{\lambda,\eta,\delta,j}(T, f)\Psi_{\lambda,\eta,\delta,j} \quad \text{in } \mathcal{O}_T \tag{15.26}$$

and $b_{\lambda,\eta,\delta,j}(T, f) = 0$, provided that $o \in \mathcal{O}$.

Let us now investigate main properties of the coefficients $a_{\lambda,\eta,\delta,j}(T, f)$ and $b_{\lambda,\eta,\delta,j}(T, f)$. First of all, we note that for these coefficients, the analogues of properties (14.25)–(14.30) hold (see Proposition 10.4).

The following results are analogues of Propositions 14.10–14.12.

Proposition 15.8.

(i) *If* $\nu \in \{0, \dots, n(\lambda, T)\}$ *and* $f \in \mathcal{D}'_T(\mathcal{O})$, *then*

$$f^{\delta,j} \times T_{\lambda,\nu} = \sum_{\mu=0}^{n(\lambda,T)-\nu} \binom{\nu+\mu}{\nu}_{\lambda} \big(a_{\lambda,\nu+\mu,\delta,j}(T, f)\Phi_{\lambda,\mu,\delta,j} + b_{\lambda,\nu+\mu,\delta,j}(T, f)\Psi_{\lambda,\mu,\delta,j}\big) \quad \text{in } \mathcal{O}_T.$$

(ii) *Let* $f_m \in \mathcal{D}'_T(\mathcal{O})$, $m = 1, 2, \dots$, *and assume that* $f_m \to f$ *in* $\mathcal{D}'(\mathcal{O})$ *as* $m \to \infty$. *Then* $a_{\lambda,\eta,\delta,j}(T, f_m) \to a_{\lambda,\eta,\delta,j}(T, f)$ *and* $b_{\lambda,\eta,\delta,j}(T, f_m) \to b_{\lambda,\eta,\delta,j}(T, f)$ *as* $m \to \infty$.

(iii) *Let* $f \in \mathcal{D}'_T(\mathcal{O})$, $u \in \mathcal{E}'_{\natural}(X)$, *and let* $\mathcal{O}_{T\times u} \neq \varnothing$. *Then*

$$a_{\lambda,\eta,\delta,j}(T, f \times u) = \sum_{\nu=\eta}^{n(\lambda,T)} a_{\lambda,\nu,\delta,j}(T, f)\binom{\nu}{\eta}_{\lambda} \overset{\circ}{u}{}^{\langle\nu-\eta\rangle}(\lambda)$$

and

$$b_{\lambda,\eta,\delta,j}(T, f \times u) = \sum_{\nu=\eta}^{n(\lambda,T)} b_{\lambda,\nu,\delta,j}(T, f)\binom{\nu}{\eta}_{\lambda} \overset{\circ}{u}{}^{\langle\nu-\eta\rangle}(\lambda).$$

In particular, for each polynomial p,

$$a_{\lambda,\eta,\delta,j}\big(T, p(L)f\big) = \sum_{\nu=\eta}^{n(\lambda,T)} a_{\lambda,\nu,\delta,j}(T, f)\binom{\nu}{\eta}_{\lambda} q^{\langle\nu-\eta\rangle}(\lambda)$$

and

$$b_{\lambda,\eta,\delta,j}\big(T, p(L)f\big) = \sum_{\nu=\eta}^{n(\lambda,T)} b_{\lambda,\nu,\delta,j}(T, f)\binom{\nu}{\eta}_{\lambda} q^{\langle\nu-\eta\rangle}(\lambda),$$

where $q(z) = p(-z^2 - \rho_X^2)$.

Proof. For the case where $n(\lambda, T) = 0$, part (i) is obvious from (15.26). If $n(\lambda, T) > 0$, we can prove (i) by induction on ν using Theorem 10.4 and Proposition 15.3. Next, applying (i) repeatedly for $\nu = n(\lambda, T), \dots, 0$ and taking (10.27) into account, we obtain part (ii). Finally, (iii) follows by (15.26), (10.30), and Proposition 15.2(ii). □

Proposition 15.9.

(i) *If* $\mu \in \mathcal{Z}_T$ *and* $\nu \in \{0, \dots, n(\mu, T)\}$, *then*

$$a_{\lambda,\eta,\delta,j}(T, \Phi_{\mu,\nu,\delta,j}) = \delta_{\lambda,\mu}\delta_{\eta,\nu}$$

and

$$b_{\lambda,\eta,\delta,j}(T, \Phi_{\mu,\nu,\delta,j}) = 0.$$

If, in addition, $o \notin \mathcal{O}$, *then*

$$a_{\lambda,\eta,\delta,j}(T, \Psi_{\mu,\nu,\delta,j}) = 0$$

and

$$b_{\lambda,\eta,\delta,j}(T, \Psi_{\mu,\nu,\delta,j}) = \delta_{\lambda,\mu}\delta_{\eta,\nu}.$$

(ii) *Assume that* $\delta \in \widehat{K}_M$ *and* $j \in \{1, \dots, d(\delta)\}$ *are fixed, and let* $a_{\lambda,\eta,\delta,j}(T, f) = b_{\lambda,\eta,\delta,j}(T, f) = 0$ *for all* $\lambda \in \mathcal{Z}_T$ *and* $\eta \in \{0, \dots, n(\lambda, T)\}$. *Then* $f^{\delta,j} = 0$ *in* $\mathcal{O}$.

(iii) *If* $o \in \mathcal{O}$, $f \in C_T^m(\mathcal{O})$, $m = \operatorname{ord} T + 4\alpha_X + 6 + s(\delta)$, *then*

$$a_{\lambda,\eta,\delta,j}(T, f) = \langle T_{\lambda,\eta,\delta,j}, f\rangle.$$

Proof. Relation (15.26), together with Proposition 15.2(ii) and (10.179), easily implies (i). Assertion (ii) is evident from Proposition 15.8(i) and Theorem 10.28. To prove (iii) it is sufficient to repeat the arguments in the proof of Proposition 14.11 using Theorem 10.25(i), (vi) and (10.181). □

Proposition 15.10. *Let* $f \in \mathcal{D}'_T(\mathcal{O})$, *and let*

$$T = \left(L + \lambda_1^2 + \rho_X^2\right)^{s_1} \cdots \left(L + \lambda_l^2 + \rho_X^2\right)^{s_l} Q,$$

where $Q \in \mathcal{E}'_\natural(X)$, $s_1, \dots, s_l \in \mathbb{N}$, *and* $\{\lambda_1, \dots, \lambda_l\}$ *is a set of distinct complex numbers in* $\mathcal{Z}_T$. *Then* $r(Q) = r(T)$, $s_m \leqslant n(\lambda_m, T) + 1$ *for all* $m \in \{1, \dots, l\}$, *and the distribution*

$$g = f - \sum_{m=1}^{l} \sum_{\eta=n(\lambda_m,T)+1-s_m}^{n(\lambda_m,T)} a_{\lambda_m,\eta,\delta,j}(T, f)\Phi_{\lambda_m,\eta,\delta,j} + b_{\lambda_m,\eta,\delta,j}(T, f)\Psi_{\lambda_m,\eta,\delta,j}$$

is in $\mathcal{D}'_Q(\mathcal{O})$. *In addition, if* $\lambda \in \mathcal{Z}_Q$, *then* $n(\lambda, Q) \leqslant n(\lambda, T)$ *and*

$$a_{\lambda,\eta,\delta,j}(Q, g) = a_{\lambda,\eta,\delta,j}(T, f),$$
$$b_{\lambda,\eta,\delta,j}(Q, g) = b_{\lambda,\eta,\delta,j}(T, f)$$

for all $\eta \in \{0, \dots, n(\lambda, Q)\}$.

Proof. There is no difficulty in modifying the proof of Proposition 14.12 using Theorem 10.7, (10.179), (10.177), Proposition 15.2, and Proposition 10.17(iv) to obtain the desired result. □

To continue, let $f \in \mathcal{D}'_T(\mathcal{O})$. We recall from Sect. 10.2 that the Fourier series

$$f = \sum_{\delta\in\widehat{K}_M} \sum_{j=1}^{d(\delta)} f^{\delta,j} \tag{15.27}$$

unconditionally converges to f in $\mathcal{D}'(\mathcal{O})$. We now associate with each distribution $f^{\delta,j}$ the series

$$f^{\delta,j} \sim \sum_{\lambda\in\mathcal{Z}_T} \sum_{\eta=0}^{n(\lambda,T)} a_{\lambda,\eta,\delta,j}(T,f)\Phi_{\lambda,\eta,\delta,j} + b_{\lambda,\eta,\delta,j}(T,f)\Psi_{\lambda,\eta,\delta,j}. \tag{15.28}$$

Theorem 15.9.

(i) *If $f \in \mathcal{D}'_T(\mathcal{O})$ and the series in (15.28) converges in $\mathcal{D}'(\mathcal{O})$, then its sum coincides with $f^{\delta,j}$.*
(ii) *Let $f \in \mathcal{D}'(\mathcal{O})$ and assume that for all $\delta \in \widehat{K}_M$ and $j \in \{1,\dots,d(\delta)\}$,*

$$f^{\delta,j} = \sum_{\lambda\in\mathcal{Z}_T} \sum_{\eta=0}^{n(\lambda,T)} \alpha_{\lambda,\eta,\delta,j}\Phi_{\lambda,\eta,\delta,j} + \beta_{\lambda,\eta,\delta,j}\Psi_{\lambda,\eta,\delta,j}, \quad \alpha_{\lambda,\eta,\delta,j}, \beta_{\lambda,\eta,\delta,j} \in \mathbb{C},$$

where $\beta_{\lambda,\eta,\delta,j}$ are set to be equal to zero if $o \in \mathcal{O}$, and the series converges in $\mathcal{D}'(\mathcal{O})$. Then $f \in \mathcal{D}'_T(\mathcal{O})$ and

$$\alpha_{\lambda,\eta,\delta,j} = a_{\lambda,\eta,\delta,j}(T,f), \quad \beta_{\lambda,\eta,\delta,j} = b_{\lambda,\eta,\delta,j}(T,f).$$

The proof follows from Proposition 15.8(ii) and Proposition 15.9(i), (ii) (see the proof of Theorem 13.9).

We now establish some convenient estimates of the coefficients $a_{\lambda,\eta,\delta,j}(T,f)$ and $b_{\lambda,\eta,\delta,j}(T,f)$.

Theorem 15.10. *Assume that $r(T) > 0$. Then the following assertions hold.*

(i) *Let $f \in \mathcal{D}'_T(\mathcal{O})$, and let p be a polynomial. Then there exists $c_1 > 0$ independent of f such that for all $\lambda \in \mathcal{Z}_T$, $|\lambda| > c_1$, the following estimates hold:*

$$\max_{0\leqslant\eta\leqslant n(\lambda,T)} |a_{\lambda,\eta,\delta,j}(T,f)| \leqslant \frac{c_2}{|p(-\lambda^2-\rho_X^2)|} \max_{0\leqslant\eta\leqslant n(\lambda,T)} \big|a_{\lambda,\eta,\delta,j}\big(T,p(L)f\big)\big|,$$
$$\max_{0\leqslant\eta\leqslant n(\lambda,T)} |b_{\lambda,\eta,\delta,j}(T,f)| \leqslant \frac{c_2}{|p(-\lambda^2-\rho_X^2)|} \max_{0\leqslant\eta\leqslant n(\lambda,T)} \big|b_{\lambda,\eta,\delta,j}\big(T,p(L)f\big)\big|$$

with the constant $c_2 > 0$ independent of λ, f.

(ii) *Suppose that* $R \in (r(T), +\infty]$, $m \in \mathbb{Z}_+$, *and let* $f \in C_T^{2m}(B_R)$. *Then there exist* $c_3, c_4, c_5, c_6 > 0$ *independent of* f, m *such that for all* $\lambda \in \mathcal{Z}_T$, $|\lambda| > c_3$,

$$\max_{0 \leqslant \eta \leqslant n(\lambda,T)} |a_{\lambda,\eta,\delta,j}(T, f)| \leqslant \sigma_\lambda\big(\mathring{T}\big) c_4^{m+1} |\lambda|^{c_5 - 2m} \bigg(\int_{B_{r(T)}} \big| L^m \big(f^{\delta,j}\big)(x) \big| \, \mathrm{d}x + c_6^m c_7 \bigg),$$

where $c_7 > 0$ *is independent of* m, λ.

(iii) *Let* $f \in \mathcal{D}'_T(X)$ *and assume that* $\operatorname{ord} f < +\infty$. *Then for each* $\alpha > 0$,

$$|a_{\lambda,\eta,\delta,j}(T, f)| \leqslant \sigma_\lambda\big(\mathring{T}\big)(2 + |\lambda|)^{c_8} c_9^{n(\lambda,T)} n(\lambda, T)! \mathrm{e}^{-\alpha \operatorname{Im} \lambda},$$

where $c_8, c_9 > 0$ *are independent of* λ, η.

The proof of this theorem is similar to that of Theorem 14.12, the only change being that we now use Propositions 15.8, 15.10, 10.5 and (10.29), (10.180).

Corollary 15.4. *Let* $R \in (r(T), +\infty]$. *Then the following statements are valid.*

(i) *Let* $f \in \mathcal{D}'_T(B_R)$. *Then*

$$|a_{\lambda,\eta,\delta,j}(T, f)| \leqslant (2 + |\lambda|)^{c_1} \sigma_\lambda\big(\mathring{T}\big).$$

where $c_1 > 0$ *is independent of* λ, η. *In particular, if* $T \in \mathfrak{M}(X)$, *then*

$$|a_{\lambda,\eta,\delta,j}(T, f)| \leqslant (2 + |\lambda|)^{c_2},$$

where $c_2 > 0$ *is independent of* λ, η.

(ii) *If* $T \in \mathfrak{M}(X)$ *and* $f \in C_T^\infty(B_R)$, *then for each* $\alpha > 0$,

$$|a_{\lambda,\eta,\delta,j}(T, f)| \leqslant c_3 (2 + |\lambda|)^{-\alpha},$$

where $c_3 > 0$ *is independent of* λ, η.

(iii) *If* $\alpha > 0$, $T \in \mathfrak{G}_\alpha(X)$, $r(T) > 0$, *and* $f \in C_T^\infty(B_R) \cap G^\alpha(\dot{B}_{r(T)})$, *then*

$$|a_{\lambda,\eta,\delta,j}(T, f)| \leqslant c_4 \exp\big(-c_5 |\lambda|^{1/\alpha}\big),$$

where $c_4, c_5 > 0$ *are independent of* λ, η.

(iv) *If* $T \in \mathfrak{M}(X)$, $r(T) > 0$, *and* $f \in C_T^\infty(B_R) \cap \mathrm{QA}(\dot{B}_{r(T)})$, *then*

$$\max_{0 \leqslant \eta \leqslant n(\lambda,T)} |a_{\lambda,\eta,\delta,j}(T, f)| \leqslant M_{q,\delta,j} (1 + |\lambda|)^{-2q}$$

for all $\lambda \in \mathcal{Z}_T$ *and* $q \in \mathbb{N}$, *where the constants* $M_{q,\delta,j} > 0$ *are independent of* λ, *and*

$$\sum_{\nu=1}^{\infty} \frac{1}{\inf_{q \geqslant \nu} M_{q,\delta,j}^{1/2q}} = +\infty. \tag{15.29}$$

(v) *Let* $T \in \mathfrak{E}(X)$*,* $f \in \mathcal{D}'_T(X)$*, and suppose that* f *is of finite order in* X*. Then for each* $\alpha > 0$*,*

$$|a_{\lambda,\eta,\delta,j}(T, f)| \leqslant c_6 \mathrm{e}^{-\alpha \operatorname{Im} \lambda},$$

where $c_6 > 0$ *is independent of* λ, η*.*

Proof. No change is required in the proof of Corollary 14.4 except that we now apply Proposition 15.7, (10.26), and Theorem 15.10. □

Theorem 15.11. *Let* $r(T) > 0$ *and assume that*

$$0 \leqslant r < r' < R' < R \leqslant +\infty, \quad R' - r' = 2r(T). \tag{15.30}$$

Let $m \in \mathbb{Z}_+$ *and suppose that* $f \in C_T^{2m}(B_{r,R})$*. Then there exist* $c_1, c_2, c_3, c_4, c_5 > 0$ *independent of* f, m *such that for all* $\lambda \in \mathcal{Z}_T$*,* $|\lambda| > c_1$*,*

$$\begin{aligned}
&\max_{0 \leqslant \eta \leqslant n(\lambda,T)} \big(|a_{\lambda,\eta,\delta,j}(T, f)| + |b_{\lambda,\eta,\delta,j}(T, f)|\big) \\
&\quad \leqslant \sigma_\lambda\big(\mathring{T}\big) c_2^{m+1} |\lambda|^{c_3(n(\lambda,T)+1)-2m} \mathrm{e}^{c_4(\operatorname{Im}\lambda+1)(n(\lambda,T)+1)} \\
&\quad \times \left(c_5^m c_6 + \int_{B_{r',R'}} \big|L^m\big(f^{\delta,j}\big)(x)\big| \,\mathrm{d}x\right),
\end{aligned}$$

where $c_6 > 0$ *is independent of* m, λ*.*

Proof. Take $k \in K$ such that $Y_j^\delta(kM) \neq 0$ and select $a_{t_0} \in A$ so that $t_0 > 0$ and $a_{t_0} o \in S_{(r'+R')/2}$. Suppose that $t \in \mathbb{R}^1$ and $a_{t+t_0} o \in B_{r,R}$, and define

$$u_{\lambda,\eta}(t) = \Phi_{\lambda,\eta,\delta,j}(k a_{t+t_0} o), \quad v_{\lambda,\eta}(t) = \Psi_{\lambda,\eta,\delta,j}(k a_{t+t_0} o),$$

where $\lambda \in \mathcal{Z}_T$ and $\eta \in \{0, \dots, n(\lambda, T)\}$. For the rest of the proof, now it is possible to adapt the arguments in the proof of Theorem 14.13 (see (10.65), (10.68), (7.26), and (6.1)). □

Corollary 15.5. *Assume that* (15.30) *is satisfied. Then the following results are true.*

(i) *Let* $f \in \mathcal{D}'_T(B_{r,R})$*. Then*

$$|a_{\lambda,\eta,\delta,j}(T, f)| + |b_{\lambda,\eta,\delta,j}(T, f)| \leqslant \mathrm{e}^{c_1(1+\operatorname{Im}\lambda+\log(1+|\lambda|))(n(\lambda,T)+1)} \sigma_\lambda\big(\mathring{T}\big),$$

where $c_1 > 0$ *is independent of* λ, η*. In particular, if* $T \in \mathfrak{N}(X)$*, then*

$$|a_{\lambda,\eta,\delta,j}(T, f)| + |b_{\lambda,\eta,\delta,j}(T, f)| \leqslant (2 + |\lambda|)^{c_2},$$

where $c_2 > 0$ *is independent of* λ, η*.*

(ii) *If* $T \in \mathfrak{N}(X)$ *and* $f \in C_T^\infty(B_{r,R})$*, then for each* $\alpha > 0$*,*

$$|a_{\lambda,\eta,\delta,j}(T, f)| + |b_{\lambda,\eta,\delta,j}(T, f)| \leqslant c_3(2 + |\lambda|)^{-\alpha},$$

where $c_3 > 0$ *is independent of* λ, η*.*

(iii) *If* $\alpha > 0$*,* $T \in \mathfrak{D}_\alpha(X)$*, and* $f \in C_T^\infty(B_{r,R}) \cap G^\alpha(\overset{\bullet}{B}_{r',R'})$*, then*

$$|a_{\lambda,\eta,\delta,j}(T, f)| + |b_{\lambda,\eta,\delta,j}(T, f)| \leqslant c_4 \exp\bigl(-c_5|\lambda|^{1/\alpha}\bigr),$$

where $c_4, c_5 > 0$ *are independent of* λ, η*.*

(iv) *If* $T \in \mathfrak{N}(X)$ *and* $f \in C_T^\infty(B_{r,R}) \cap \mathrm{QA}(\overset{\bullet}{B}_{r',R'})$*, then*

$$\max_{0\leqslant\eta\leqslant n(\lambda,T)} \bigl(|a_{\lambda,\eta,\delta,j}(T, f)| + |b_{\lambda,\eta,\delta,j}(T, f)|\bigr) \leqslant M_{q,\delta,j}(1 + |\lambda|)^{-2q}$$

for all $\lambda \in \mathcal{Z}_T$ *and* $q \in \mathbb{N}$*, where the constants* $M_{q,\delta,j} > 0$ *are independent of* λ *and satisfy* (15.29).

The proof depends on Theorem 15.11 and Proposition 15.6.

Analogs of Theorems 14.14 and 14.15 run as follows.

Theorem 15.12. *Let* U *be a* K*-invariant compact subset of* $\mathcal{O}$ *such that* $\overset{\bullet}{B}_{r(T)}(x) \subset U$ *for some* $x \in X$*. Let* $f \in C_T^\infty(\mathcal{O})$ *and assume that there exist* $\alpha > 0$ *and* $\beta \geqslant 0$ *such that*

$$\liminf_{q\to+\infty} \alpha^{-2q} q^{-\beta} \int_U \bigl|\bigl(L + \rho_X^2\bigr)^q f(x)\bigr| \,\mathrm{d}x = 0.$$

Then

$$a_{\lambda,\eta,\delta,j}(T, f) = b_{\lambda,\eta,\delta,j}(T, f) = 0,$$

provided that $|\lambda| > \alpha$*, and the same is valid when* $|\lambda| = \alpha$ *and* $\eta \geqslant \beta$*. In particular, if* $\alpha \leqslant \min_{\lambda\in\mathcal{Z}_T} |\lambda|$ *and* $\beta = 0$*, then* $f = 0$*.*

This is optimal as, for example,

$$f = c_1\Phi_{\lambda,\eta,\delta,j} + c_2\Psi_{\lambda,\eta,\delta,j}, \quad c_1, c_2 \in \mathbb{C}$$

shows (see Propositions 10.7(iii) and 10.8(iii)). Theorem 15.12 follows at once from Theorems 15.10(ii) and 15.11 and the argument used to prove Theorem 14.14.

Theorem 15.13. *Let* $T \in \mathcal{E}'_\natural(X)$*,* $T \neq 0$*, and let* (10.186) *hold. Assume that* $0 \leqslant r < R \leqslant +\infty$ *and*

$$f = \sum_{\lambda\in\mathcal{Z}_T} \sum_{\eta=0}^{n(\lambda,T)} a_{\lambda,\eta}\Phi_{\lambda,\eta,\delta,j} + b_{\lambda,\eta}\Psi_{\lambda,\eta,\delta,j} \quad \text{in } B_{r,R},$$

where $a_{\lambda,\eta}, b_{\lambda,\eta} \in \mathbb{C}$*, and properties* (10.190) *and* (10.191) *are valid with* $c_{\lambda,\eta} = a_{\lambda,\eta}, b_{\lambda,\eta}$*. If* $f = 0$ *in* B_{r_1,r_2} *for some* $r_1 > r$*,* $r_2 < R$*,* $r_1 < r_2$*, then* $a_{\lambda,\eta} = b_{\lambda,\eta} = 0$ *for all* λ, η*.*

Proof. The proof of this theorem is similar to that of Theorem 14.15, the change being that we now use Theorems 10.22, 10.21(ii), and 15.9(ii) and Propositions 10.7(ii) and 10.8(ii). □

Finally, we mention the following analog of Propositions 13.8 and 14.13 which will be used in the sequel.

Proposition 15.11. *Let $H \in \mathcal{E}'_\natural(X)$, $H \neq 0$, and $R = r(T) + r(H)$. Assume that $\lambda \in \mathcal{Z}_T$, $n(\lambda, T) > n(\lambda, H)$, and $\eta \in \{n(\lambda, H) + 1, \ldots, n(\lambda, T)\}$, where the number $n(\lambda, H)$ is set to be equal to -1 if $\lambda \notin \mathcal{Z}_H$. Then the following assertions hold.*

(i) *If $\mathring{B}_R(x) \subset \mathcal{O}$ for some $x \in X$ and $f \in (\mathcal{D}'_T \cap \mathcal{D}'_H)(\mathcal{O})$, then*

$$a_{\lambda,\eta,\delta,j}(T, f) = b_{\lambda,\eta,\delta,j}(T, f) = 0. \tag{15.31}$$

(ii) *If $B_R(x) \subset \mathcal{O}$ for some $x \in X$ and $f \in C^\infty_T(\mathcal{O} \cup B_{0,+\infty}) \cap C^\infty_H(\mathcal{O})$, then (15.31) holds.*

(iii) *If $B_R(x) \subset \mathcal{O}$ for some $x \in X$, $H \in \mathcal{D}_\natural(X)$, and $f \in \mathcal{D}'_T(\mathcal{O} \cup B_{0,+\infty}) \cap \mathcal{D}'_H(\mathcal{O})$, then (15.31) is satisfied.*

This result is proved similarly to Propositions 13.8 and 14.13 by using Proposition 15.8(iii) and Theorem 15.1(i).

15.6 Structure Theorems and Their Applications

In this section we shall obtain a very utilitarian description for some classes of solutions of convolution equations.

The following is an analog of Theorem 14.16.

Theorem 15.14. *Let $T \in \mathrm{Inv}_+(X)$, $R \in (r(T), +\infty]$, and $f \in \mathcal{D}'(B_R)$. Then $f \in \mathcal{D}'_T(B_R)$ if and only if for all $\delta \in \widehat{K}_M$, $j \in \{1, \ldots, d(\delta)\}$,*

$$f^{\delta,j} = \zeta_{T,\delta,j} \times u_{\delta,j} \quad \text{in } B_R$$

for some $u_{\delta,j} \in \mathcal{E}'_\natural(X)$ with $\operatorname{supp} u_{\delta,j} \subset \mathring{B}_{r(T)}$.

The proof of this theorem copies the proof of Theorem 14.16, but instead of the mapping $\mathfrak{A}_{k,j}$ we now use $\mathfrak{A}_{\delta,j}$.

Theorem 15.15. *Let $T \in \mathfrak{M}(X)$ and $R \in (r(T), +\infty]$.*

(i) *If $f \in \mathcal{D}'(B_R)$, then for f to belong to $\mathcal{D}'_T(B_R)$, it is necessary and sufficient that for all $\delta \in \widehat{K}_M$, $j \in \{1, \ldots, d(\delta)\}$, the following relation holds:*

$$f^{\delta,j} = \sum_{\lambda \in \mathcal{Z}_T} \sum_{\eta=0}^{n(\lambda,T)} \alpha_{\lambda,\eta,\delta,j} \Phi_{\lambda,\eta,\delta,j}, \tag{15.32}$$

where $\alpha_{\lambda,\eta,\delta,j} \in \mathbb{C}$, and the series converges in $\mathcal{D}'(B_R)$.

(ii) *Let* $f \in C^{\infty}(B_R)$. *Then* $f \in C_T^{\infty}(B_R)$ *if and only if for all* $\delta \in \widehat{K}_M$ *and* $j \in \{1, \dots, d(\delta)\}$, *decomposition* (15.32) *holds with the series converging in* $\mathcal{E}(B_R)$.

(iii) *Let* $f \in \mathcal{D}'_{\delta,j}(B_R)$ *for some* $\delta \in \widehat{K}_M$ *and* $j \in \{1, \dots, d(\delta)\}$. *Then* $f \in \mathrm{QA}_T(B_R)$ *if and only if relation* (15.32) *is satisfied with the series converging in* $\mathcal{E}(B_R)$,

$$\max_{0 \leqslant \eta \leqslant n(\lambda,T)} |\alpha_{\lambda,\eta,\delta,j}| \leqslant M_{q,\delta,j}(1+|\lambda|)^{-2q}$$

for all $\lambda \in \mathcal{Z}_T$ *and* $q \in \mathbb{N}$, *and the constants* $M_{q,\delta,j} > 0$ *are independent of* λ *and satisfy* (15.29).

The proof follows from Proposition 10.23, Corollary 15.4, and Theorem 15.9.

In combination with Proposition 10.23, Corollary 15.4, and Theorem 15.9, this theorem shows that for $T \in \mathfrak{M}(X)$, the class $(\mathcal{D}'_T \cap \mathcal{D}'_{\delta,j})(B_R)$ coincides with the set of series of the form (15.32), where the coefficients $\alpha_{\lambda,\eta,\delta,j}$ increase no faster than a positive power of $|\lambda|$. Similarly, $(C_T^{\infty} \cap \mathcal{D}'_{\delta,j})(B_R)$ is just the set of series of the form (15.32) where

$$\max_{0 \leqslant \eta \leqslant n(\lambda,T)} |\alpha_{\lambda,\eta,\delta,j}| = O\big(|\lambda|^{-\sigma}\big) \quad \text{as } \lambda \to \infty \tag{15.33}$$

for each fixed $\sigma > 0$.

We shall now show that the assumption on T in Theorem 15.15(i), (ii) cannot be omitted.

Theorem 15.16. *For* $\delta \in \widehat{K}_M$ *and* $j \in \{1, \dots, d(\delta)\}$, *the following assertions hold.*

(i) *Let* $T \in \mathcal{E}'_\natural(X)$, $T \neq 0$, *and assume that there is a sequence* $\{\lambda_m\}_{m=1}^{\infty} \subset \mathcal{Z}_T$ *such that*

$$\lim_{m\to\infty} \frac{\operatorname{Im}\lambda_m}{\log(2+|\lambda_m|)} = +\infty. \tag{15.34}$$

Then there exist constants $\alpha_{\lambda,\eta,\delta,j} \in \mathbb{C}$ $(\lambda \in \mathcal{Z}_T,\ \eta \in \{0, \dots, n(\lambda, T)\})$ *satisfying* (15.33) *and such that the series in* (15.32) *is not convergent in* $\mathcal{D}'(B_R)$ *for each* $R > 0$.

(ii) *For any function* $\beta\colon \mathbb{C} \to \mathbb{R}^1_+$ *satisfying the condition* $\lim_{z\to\infty}\beta(z) = +\infty$, *there exists* $T \in \mathcal{E}'_\natural(X)$ *with the following properties:*

(1) $\mathcal{Z}(\overset{\circ}{T}) \subset \mathbb{R}^1$, *and*

$$\big|\overset{\circ}{T}{}'(\lambda)\big| \geqslant (2+|\lambda|)^{-\beta(\lambda)} \tag{15.35}$$

for all $\lambda \in \mathcal{Z}(\overset{\circ}{T})$;

(2) *for every* $R \in (r(T), +\infty]$, *there is* $f \in C_T^{\infty}(B_R)$ *for which* $f^{\delta,j}$ *cannot be presented as series* (15.32) *converging in* $\mathcal{D}'(B_R)$.

Proof. To prove (i) we define $\alpha_{\lambda,\eta,\delta,j}$ for $\lambda \in \mathcal{Z}_T$ and $\eta \in \{0,\dots,n(\lambda,T)\}$ as follows. Let

$$\alpha_{\lambda,\eta,\delta,j} = \exp\bigl(-\sqrt{\operatorname{Im}\lambda \log(2+|\lambda|)}\bigr)$$

if $\lambda = \lambda_m$ for some $m \in \mathbb{N}$ and $\eta = 0$. Otherwise we set $\alpha_{\lambda,\eta,\delta,j} = 0$. Relation (15.34) shows that $\alpha_{\lambda,\eta,\delta,j}$ satisfy (15.33) for each fixed $\sigma > 0$. Assume that the series in (15.32) with $\alpha_{\lambda,\eta,\delta,j}$ as above converges in $\mathcal{D}'(B_R)$ for some $R > 0$. Then the series

$$\sum_{\lambda\in\mathcal{Z}_T} \alpha_{\lambda,0,\delta,j} \cos\lambda t \tag{15.36}$$

converges in $\mathcal{D}'(-R,R)$ (see Theorem 10.21(iv), (vii)). Using now [225, Part III, the proof of Theorem 1.5(i)], we have a contradiction. Hence, the desired conditions hold for given $\alpha_{\lambda,\eta,\delta,j}$.

Turning to (ii), let us consider a function $\beta\colon \mathbb{C} \to \mathbb{R}^1_+$ such that $\lim_{z\to\infty}\beta(z) = +\infty$. For $m \in \mathbb{N}$ and $z \in \mathbb{C}$, we set

$$\varepsilon_m = \frac{1}{3}\exp\bigl(-\sqrt{\beta(m)}\log m\bigr),$$
$$\mu(z) = \prod_{m=1}^{\infty}\left(1 - \frac{z^2}{(m+\varepsilon_m)^2}\right).$$

If $c > 0$ is large enough, then there is $T \in \mathcal{E}'_\natural(X)$ such that $\overset{\circ}{T}(z) = c\mu(z)(\sin\pi z)/z$ and (15.35) is satisfied (see [225, Part III, the proof of Theorem 1.5] and Theorem 10.7). We note that $\mathcal{Z}(\overset{\circ}{T}) \subset \mathbb{R}^1$ and define

$$f = \sum_{m=1}^{\infty} \varepsilon_m^{-1/2}(\Phi_{m,0,\delta,j} - \Phi_{m+\varepsilon_m,0,\delta,j}). \tag{15.37}$$

Using (10.66), it is easy to verify that series (15.37) converges in $\mathcal{E}(X)$. Thus, $f = f^{\delta,j}$, and by virtue of Proposition 15.2(ii), we have $f \in C^\infty_T(X)$. Assume now that for some $R \in (r(T),+\infty]$ and $\alpha_{\lambda,\eta,\delta,j} \in \mathbb{C}$, equality (15.32) holds, where the series converges in $\mathcal{D}'(B_R)$. Then series (15.36) converges in $\mathcal{D}'(-R,R)$ in view of Theorem 10.21(iv), (vii). This, however, contradicts [225, Part III, the proof of Theorem 1.5(ii)]. Thus, T possesses the properties (1) and (2). Hence the theorem. □

Theorem 15.17.

(i) *Let $\alpha > 0$, $T \in \mathfrak{G}_\alpha(X)$, $R \in (r(T),+\infty]$, and $f \in C^\infty(B_R) \cap G^\alpha(\mathring{B}_{r(T)})$. Then $f \in C^\infty_T(B_R)$ if and only if for all $\delta \in \widehat{K}_M$ and $j \in \{1,\dots,d(\delta)\}$, equality (15.32) is satisfied with the series converging in $\mathcal{E}(B_R)$.*

(ii) *Let $T \in \mathfrak{E}(X)$, $f \in \mathcal{D}'(X)$, and assume that f is of finite order in X. Then $f \in \mathcal{D}'_T(X)$ if and only if for all $\delta \in \widehat{K}_M$ and $j \in \{1,\dots,d(\delta)\}$, relation (15.32) holds with the series converging in $\mathcal{E}(X)$. In particular, if $f \in \mathcal{D}'_T(X)$, then $f^{\delta,j} \in C^\infty_T(X)$ for all $\delta \in \widehat{K}_M$ and $j \in \{1,\dots,d(\delta)\}$.*

The proof immediately follows from Theorem 15.9, Corollary 15.4, and Proposition 10.23.

We have, just as in Sect. 14.5, the following consequence.

Theorem 15.18. *Let $T \in \mathrm{Inv}_+(X)$, $R > r(T)$, and let $\delta \in \widehat{K}_M$ and $j \in \{1, \dots, d(\delta)\}$. Then for each $f \in (\mathcal{D}'_T \cap \mathcal{D}'_{\delta,j})(B_R)$, there exists a unique $F \in (\mathcal{D}'_T \cap \mathcal{D}'_{\delta,j})(X)$ such that $F = f$ in B_R. Moreover, if $f \in \mathfrak{M}(X)$, then the following results are true.*

(i) *If $f \in C^\infty(B_R)$, then $F \in C^\infty(X)$.*
(ii) *If $f \in \mathrm{QA}(B_R)$, then $F \in \mathrm{QA}(X)$.*

The proof proceeds as that of Theorem 14.19 except that we now apply Theorem 15.15, Proposition 10.23, Theorem 15.14, and Theorem 15.1(i).

For the case where $T \in \mathfrak{G}_\alpha(X)$ and $\alpha > 0$, the first part of Theorem 15.18, with corresponding changes, is also valid (see Remark 14.2, Theorem 15.17(i), Proposition 10.23, and Corollary 15.4(iii)).

Next, suppose that $T \in \mathcal{E}'_\natural(X)$, $R > r(T) > 0$, and $f \in (C^\infty_T \cap C^\infty_{\delta,j})(B_R)$ for some $\delta \in \widehat{K}_M$ and $j \in \{1, \dots, d(\delta)\}$. Then it follows by Proposition 15.4(ii) and Theorem 10.25(v) that

$$\left\langle T, \mathcal{A}^\delta_j\left(L^\nu f\right)\right\rangle = 0 \quad \text{for all } \nu \in \mathbb{Z}_+. \tag{15.38}$$

Therefore, for $r(T) > 0$, Theorem 15.18 can be refined by the following way.

Theorem 15.19. *Let $T \in \mathcal{E}'_\natural(X)$, $r(T) > 0$, $\delta \in \widehat{K}_M$, and $j \in \{1, \dots, d(\delta)\}$. Assume that $f \in C^\infty_{\delta,j}(\mathring{B}_{r(T)})$ and that (15.38) is fulfilled. Then the following assertions hold.*

(i) *If $T \in \mathfrak{M}(X)$, then there is a unique $F \in (C^\infty_T \cap C^\infty_{\delta,j})(X)$ such that $F = f$ in $\mathring{B}_{r(T)}$. Moreover, if $f \in \mathrm{QA}(\mathring{B}_{r(T)})$, then $F \in \mathrm{QA}_T(X)$.*
(ii) *Let $\alpha > 0$, $T \in \mathfrak{G}_\alpha(X)$, and $f \in G^\alpha(\mathring{B}_{r(T)})$. Then there exists a unique $F \in (G^\alpha_T \cap C^\infty_{\delta,j})(X)$ such that $F = f$ in $\mathring{B}_{r(T)}$.*

The proof follows from Corollaries 10.10, 10.11, 15.4, Proposition 10.23, and Theorems 15.9 and 10.29.

The following results show that the assumption on T in the previous theorem cannot be considerably relaxed.

Theorem 15.20. *Let $T \in \mathcal{E}'_\natural(X)$, $T \neq 0$, $R > r(T)$, and suppose that*

$$\sup_{\lambda \in \mathcal{Z}_T} \frac{n(\lambda, T) + \operatorname{Im} \lambda}{\log(2 + |\lambda|)} = +\infty.$$

Then for all $\delta \in \widehat{K}_M$ and $j \in \{1, \dots, d(\delta)\}$, there exists $f \in (C^\infty_T \cap \mathcal{D}'_{\delta,j})(B_R)$ such that if $\varepsilon > 0$ and $F \in \mathcal{D}'_{\delta,j}(B_{R+\varepsilon})$, then $F|_{B_R} \neq f$.

Proof. We can imitate the proof of Theorem 14.21 (see Proposition 15.4 and Corollary 10.6). □

Theorem 15.21. *Let $\alpha > 0$, $T \in \mathcal{E}'_{\natural}(X)$, $T \neq 0$, and*

$$\sup_{\lambda \in \mathcal{Z}_T} \frac{\operatorname{Im}\lambda}{(1+|\lambda|)^{1/\alpha}} = +\infty.$$

Then for all $\delta \in \widehat{K}_M$, $j \in \{1,\dots,d(\delta)\}$, and $R > r(T)$, there is $f \in (C_T^\infty \cap G^\alpha)(B_R)$ such that if $\varepsilon > 0$ and $F \in \mathcal{D}'_{\delta,j}(B_{R+\varepsilon})$, then $F|_{B_R} \neq f$.

The proof is similar to that of Theorem 14.21 except that we now apply Corollary 10.7.

Theorem 15.22. *There exists $T \in \mathcal{D}_{\natural}(X)$ such that the following assertions hold.*

(i) $r(T) > 0$, $\mathcal{Z}(\overset{\circ}{T}) \subset \mathbb{R}^1$, *and $n(\lambda, T) = 0$ for all $\lambda \in \mathcal{Z}_T$.*
(ii) *For all $\delta \in \widehat{K}_M$, $j \in \{1,\dots,d(\delta)\}$, and $R > r(T)$, there exists $f \in (C_T^\infty \cap \mathcal{D}'_{\delta,j})(B_R)$ such that if $\varepsilon > 0$ and $F \in \mathcal{D}'_{\delta,j}(B_{R+\varepsilon})$, then $F|_{B_R} \neq f$.*

The proof is identical to that of Theorem 14.22 (see Sect. 10.8).

Let us pass to analogs of Theorems 15.15 and 15.17(i) for a spherical annulus.

Theorem 15.23. *Let $T \in \mathcal{E}'_{\natural}(X)$, $T \neq 0$, and assume that (15.30) holds. Then the following statements are valid.*

(i) *Let $T \in \mathfrak{N}(X)$ and $f \in \mathcal{D}'(B_{r,R})$. Then $f \in \mathcal{D}'_T(B_{r,R})$ if and only if for all $\delta \in \widehat{K}_M$ and $j \in \{1,\dots,d(\delta)\}$,*

$$f^{\delta,j} = \sum_{\lambda \in \mathcal{Z}_T} \sum_{\eta=0}^{n(\lambda,T)} \alpha_{\lambda,\eta,\delta,j} \Phi_{\lambda,\eta,\delta,j} + \beta_{\lambda,\eta,\delta,j} \Psi_{\lambda,\eta,\delta,j}, \tag{15.39}$$

where $\alpha_{\lambda,\eta,\delta,j}, \beta_{\lambda,\eta,\delta,j} \in \mathbb{C}$, and the series converges in $\mathcal{D}'(B_{r,R})$.
(ii) *Let $T \in \mathfrak{N}(X)$ and $f \in C^\infty(B_{r,R})$. Then $f \in C_T^\infty(B_R)$ if and only of for all $\delta \in \widehat{K}_M$ and $j \in \{1,\dots,d(\delta)\}$, decomposition (15.39) holds with the series converging in $\mathcal{E}(B_{r,R})$.*
(iii) *Let $\alpha > 0$, $T \in \mathfrak{D}_\alpha(X)$, and $f \in C^\infty(B_{r,R}) \cap G^\alpha(\dot{B}_{r',R'})$. Then $f \in C_T^\infty(B_{r,R})$ if and only if for all $\delta \in \widehat{K}_M$ and $j \in \{1,\dots,d(\delta)\}$, equality (15.39) holds with the series converging in $\mathcal{E}(B_{r,R})$.*
(iv) *Let $T \in \mathfrak{N}(X)$ and suppose $f \in \mathcal{D}'_{\delta,j}(B_{r,R})$ for some $\delta \in \widehat{K}_M$ and $j \in \{1,\dots,d(\delta)\}$. Then $f \in \mathrm{QA}_T(B_{r,R})$ if and only if equality (15.39) is satisfied with the series converging in $\mathcal{E}(B_{r,R})$,*

$$\max_{0 \leqslant \eta \leqslant n(\lambda,T)} (|\alpha_{\lambda,\eta,\delta,j}| + |\beta_{\lambda,\eta,\delta,j}|) \leqslant M_{q,\delta,j}(1+|\lambda|)^{-2q}$$

for all $\lambda \in \mathcal{Z}_T$ and $q \in \mathbb{N}$, and the constants $M_{q,\delta,j} > 0$ are independent of λ and satisfy (15.29).

The proof follows from Theorem 15.9, Corollary 15.5, and Propositions 15.2, 10.23, and 10.24.

Theorem 15.24. *Let $T \in \mathfrak{N}(X)$, $0 \leqslant r < R \leqslant +\infty$, $R - r > 2r(T)$, and assume that $\delta \in \widehat{K}_M$ and $j \in \{1, \dots, d(\delta)\}$. Then for each $f \in (\mathcal{D}'_T \cap \mathcal{D}'_{\delta,j})(B_{r,R})$, there exists a unique $F \in (\mathcal{D}'_T \cap \mathcal{D}'_{\delta,j})(X \setminus \{o\})$ such that $F = f$ in $B_{r,R}$ and the following assertions hold.*

(i) *If $f \in C^\infty(B_{r,R})$, then $F \in C^\infty(X \setminus \{o\})$.*
(ii) *If r' and R' satisfy (15.30) and $f \in C^\infty(B_{r,R}) \cap \mathrm{QA}(\dot{B}_{r',R'})$, then $F \in \mathrm{QA}(X \setminus \{o\})$.*

The proof follows from Theorems 15.23, 15.9, and 15.1(i) and Propositions 10.23 and 10.24 (see the proof of Theorem 14.25).

Using Theorem 15.23(iii), we can also get the analog of Remark 14.3 for $T \in \mathfrak{D}_\alpha(X)$.

We conclude this section with the following theorem on a removable singularity for some classes of solutions of convolution equation.

Theorem 15.25. *Let $T \in \mathfrak{N}(X)$, $R \in (2r(T), +\infty]$, and let $f \in (\mathcal{D}'_T \cap \mathcal{D}'_{\delta,j})(B_R)$ for some $\delta \in \widehat{K}_M$ and $j \in \{1, \dots, d(\delta)\}$. Assume that $0 < \alpha < \beta < R$ and $\beta - \alpha > 2r(T)$. Then the following assertions are true.*

(i) *If $f \in C^\infty(B_{\alpha,\beta})$, then $f \in C^\infty(B_R)$.*
(ii) *If $f \in \mathrm{QA}(B_{\alpha,\beta})$, then $f \in \mathrm{QA}(B_R)$.*

Proof. We can essentially use the same arguments as in the proof of Theorem 14.26. However, we now apply Theorems 15.15(i), 15.23, and 15.9 and Propositions 10.23 and 10.24. □

15.7 Sharp Growth Estimates. Comparing with Eigenfunctions of the Laplacian

Assume that $T \in \mathcal{E}'_\natural(X)$, $T \neq 0$, and let $f \in L^{1,\mathrm{loc}}(X)$ be a nonzero function satisfying the equation

$$(f \times T)(x) = 0, \quad x \in X. \tag{15.40}$$

In this section we shall show that f cannot decrease rapidly at infinity. Moreover, we shall investigate precise assumptions on the behavior of f at infinity under which (15.40) implies that $f = 0$.

According to results in Sect. 15.5, we can associate with each function $f^{\delta,j}$ the series in the right-hand side of (15.28) where $b_{\lambda,\eta,\delta,j}(T, f) = 0$. The following result makes more precise the form of expansion (15.28) for the case where f satisfies some restrictions of growth.

For $R > \xi > 0$, we set

$$U(R,\xi) = \{x \in X\colon R-\xi < d(o,x) < R+\xi\}.$$

Theorem 15.26. *Let $T \in \mathcal{E}'_\natural(X)$, $T \neq 0$, and let $f \in (L^{1,\mathrm{loc}} \cap \mathcal{D}'_T)(X)$. Assume that for some $\alpha, \beta \geqslant 0$ and $\xi > r(T)$,*

$$\liminf_{R\to+\infty} R^{-\alpha}\mathrm{e}^{-\beta R}\int_{U(R,\xi)} |f(x)|\,\mathrm{d}x = 0. \tag{15.41}$$

Then

$$a_{\lambda,\eta,\delta,j}(T,f) = 0$$

for $\operatorname{Im}\lambda > \beta - \rho_X$ *and for* $\operatorname{Im}\lambda = \beta - \rho_X$, $\lambda \neq 0$, $\eta \geqslant \alpha$. *In addition, if* $0 \in \mathcal{Z}_T$ *and* $\beta = \rho_X$, *then*

$$a_{0,\eta,\delta,j}(T,f) = 0,$$

provided that $2\eta + 1 \geqslant \alpha$.

Proof. It follows by (10.26) that, together with f, all the functions $f^{\delta,j}$ satisfy (15.41). Let $\varepsilon \in (0, (\xi - r(T))/2)$ and suppose that $u \in \mathcal{D}_\natural(B_\varepsilon)$. It is easy to verify that for all and $\lambda \in \mathcal{Z}_T$, $\nu \in \{0,\dots,n(\lambda,T)\}$, the following estimate holds:

$$\int_{U(R,\xi-\varepsilon-r(T))} \left|\left(f^{\delta,j} \times u \times T_{\lambda,\nu}\right)(x)\right|\mathrm{d}x \leqslant c\int_{U(R,\xi)} \left|f^{\delta,j}(x)\right|\mathrm{d}x,$$

where the constant $c > 0$ is independent of R. In combination with (15.41), this ensures us that there exists an increasing infinite sequence $\{R_n\}_{n=1}^\infty$ of positive numbers such that $R_1 > \varepsilon$ and

$$\lim_{n\to\infty} R_n^{-\alpha}\mathrm{e}^{-\beta R_n}\int_{U(R_n,\varepsilon)} |g_{\lambda,\nu,\delta,j}(x)|\,\mathrm{d}x = 0,$$

where

$$g_{\lambda,\nu,\delta,j} = \sum_{\mu=0}^{n(\lambda,T)-\nu} a_{\lambda,\nu+\mu,\delta,j}(T, f\times u)\binom{\nu+\mu}{\nu}_\lambda \Phi_{\lambda,\mu,\delta,j}.$$

Using now (10.15) and Proposition 10.6, we have

$$a_{\lambda,\eta,\delta,j}(T, f\times u) = 0$$

for $\operatorname{Im}\lambda > \beta - \rho_X$ and for $\operatorname{Im}\lambda = \beta - \rho_X$, $\lambda \neq 0$, $\eta \geqslant \alpha$. In addition, if $0 \in \mathcal{Z}_{T_1}$ and $\beta = \rho_X$, then

$$a_{0,\eta,\delta,j}(T, f\times u) = 0$$

for $2\eta + 1 \geqslant \alpha$. Since $u \in \mathcal{D}_\natural(B_\varepsilon)$ was arbitrary, the same is true for the constants $a_{\lambda,\eta,\delta,j}(T,f)$ (see Proposition 15.8(iii)). Hence the theorem. □

Remark 15.3. Since the function $\Phi_{\lambda,\eta,\delta,j}$ is in the class $C_T^\infty(X)$, provided that $\lambda \in \mathcal{Z}_T$ and $\eta \in \{0, \dots, n(\lambda, T)\}$, it follows by (10.15) and Proposition 10.6 that assumption (15.41) in Theorem 15.26 cannot be replaced by the relation

$$\int_{U(R,\xi)} |f(x)|\mathrm{d}x = O\big(R^\alpha \mathrm{e}^{\beta R}\big) \quad \text{as } R \to +\infty. \tag{15.42}$$

Thus, all the assumptions in Theorem 15.26 are precise.

To continue, for $T \in \mathcal{E}'_\natural(X)$, $T \neq 0$, we denote

$$\varkappa_T = \inf\big\{|\mathrm{Im}\,\lambda| \colon \overset{\circ}{T}(\lambda) = 0\big\}.$$

One of the consequences of Theorem 15.26 is the following uniqueness result.

Corollary 15.6. *Let $T \in \mathcal{E}'_\natural(X)$, $T \neq 0$, $\xi > r(T)$, and let $f \in (L^{1,\mathrm{loc}} \cap \mathcal{D}'_T)(X)$. Assume that one of the following assumptions holds.*

(1) $\varkappa_T < \mathrm{Im}\,\lambda$ *for any* $\lambda \in \mathcal{Z}_T$, *and* (15.41) *is fulfilled with each* $\beta > \varkappa_T + \rho_X$ *and* $\alpha = 0$.
(2) $\varkappa_T = \mathrm{Im}\,\lambda$ *for some* $\lambda \in \mathcal{Z}_T$, *and* (15.41) *is satisfied with* $\alpha = 0$ *and* $\beta = \varkappa_T + \rho_X$.
(3) $0 \in \mathcal{Z}_T$, $\mathrm{Im}\,\lambda > 0$ *for each* $\lambda \in \mathcal{Z}_T \backslash \{0\}$, *and* (15.41) *is valid with* $\alpha = 1$, $\beta = \rho_X$.

Then $f = 0$.

Proof. By Theorems 15.26 and 15.9(i) we conclude that $f^{\delta,j} = 0$ for all $\delta \in \widehat{K}_M$ and $j \in \{1, \dots, d(\delta)\}$. The desired assertion is now obvious from (15.27). □

Remark 15.3 shows that assumptions (1)–(3) in Corollary 15.6 cannot be relaxed either.

We now turn to analogous results for the class $(L^{1,\mathrm{loc}} \cap \mathcal{D}'_T)(B_{r,+\infty})$, $r \geqslant 0$. First, we shall specify the form of expansion (15.28) for the case where f satisfies (15.41).

Theorem 15.27. *Let $T \in \mathcal{E}'_\natural(X)$, $T \neq 0$, and let $f \in (L^{1,\mathrm{loc}} \cap \mathcal{D}'_T)(B_{r,+\infty})$, $r \geqslant 0$. Assume that relation* (15.41) *is satisfied for some $\alpha, \beta \geqslant 0$ and $\xi > r(T)$. Then*

$$a_{\lambda,\eta,\delta,j}(T, f) = b_{\lambda,\eta,\delta,j}(T, f) = 0$$

for $\mathrm{Im}\,\lambda > \beta - \rho_X$ *and for* $\mathrm{Im}\,\lambda = \beta - \rho_X$, $\lambda \neq 0$, $\eta \geqslant \alpha$. *In addition, if* $0 \in \mathcal{Z}_T$ *and* $\beta = \rho_X$, *then*

$$a_{0,\eta,\delta,j}(T, f) = 0$$

for $2\eta + 1 \geqslant \alpha$, *and*

$$b_{0,\eta,\delta,j}(T, f) = 0$$

for $2\eta \geqslant \alpha$.

The proof of this theorem is similar to that of Theorem 15.26, the only change being that together with Proposition 10.6 we now use Proposition 10.8(iv).

Remark 15.4. Assumption (15.41) in Theorem 15.27 cannot be replaced by estimate (15.42). We may come to this conclusion having regarded the function

$$f = c_1 \Phi_{\lambda,\eta,\delta,j} + c_2 \Psi_{\lambda,\eta,\delta,j} \tag{15.43}$$

for proper $c_1, c_2 \in \mathbb{C}$ (see Propositions 10.6 and 10.8(iv)). So all the statements in Theorem 15.27 are precise.

Corollary 15.7. *Let $T \in \mathcal{E}'_{\natural}(X)$, $T \neq 0$, $\xi > r(T)$, and let $\mathcal{O}$ be a ζ domain with $\zeta = r(T)$ such that the set $X \backslash \mathcal{O}$ is bounded and nonempty. Assume that $f \in (L^{1,\mathrm{loc}} \cap \mathcal{D}'_T)(\mathcal{O})$ and that one of assumptions* (1), (2) *in Corollary 15.6 holds. Then $f = 0$. The same is no longer valid if instead of* (1) *or* (2) *assumption* (3) *of Corollary 15.6 is fulfilled.*

Proof. Thanks to Theorem 15.1(i), we can assume, without loss of generality, that $\mathcal{O} = B_{r,\infty}$ for some $r \geqslant 0$. Let $\delta \in \widehat{K}_M$ and $j \in \{1, \dots, d(\delta)\}$. By Theorem 15.27 each of assumptions (1), (2) in Corollary 15.6 implies that $f^{\delta,j} = 0$ in $\mathcal{O}$. In view of (15.27), this yields $f = 0$.

Assume now that $0 \in \mathcal{Z}_T$ and $\operatorname{Im} \lambda > 0$ for all $\lambda \in \mathcal{Z}_T \backslash \{0\}$. Choose $g \in G$ so that $go \in X \backslash \mathcal{O}$. Then the function

$$f(x) = \Psi_{0,0,\delta,j}\big(g^{-1}x\big)$$

belongs to $C_T^{\infty}(\mathcal{O})$, and (15.41) holds with $\alpha > 0$ and $\beta = \rho_X$ (see Proposition 10.8(iv)). This completes the proof. □

Corollaries 15.6 and 15.7 admit the following generalization.

Theorem 15.28. *Let $\{T_\nu\}_{\nu=1}^m$ be a family of nonzero distributions in the class $\mathcal{E}'_{\natural}(X)$, and let $T = T_1 \times \cdots \times T_m$, $\xi > r(T)$. Assume that the set $\{1, \dots, m\}$ is represented as a union of disjoint sets $A_1, \dots, A_l$ such that the sets*

$$\bigcup_{\nu \in A_k} \mathcal{Z}_{T_\nu}, \quad k = 1, \dots, l,$$

are also disjoint. Then the following assertions hold.

(i) *Let $f_\nu \in \mathcal{D}'_{T_\nu}(X)$, $f = \sum_{\nu=1}^m f_\nu \in L^{1,\mathrm{loc}}(X)$, and assume that one of assumptions* (1)–(3) *in Corollary 15.6 is fulfilled for given f and T. Then*

$$\sum_{\nu \in A_k} f_\nu = 0$$

for all $k \in \{1, \dots, l\}$.

(ii) *Let $r_\nu \geqslant 0$, $r = \max_{1 \leqslant \nu \leqslant m} r_\nu$, $\mathcal{O}_k = \bigcap_{\nu \in A_k} B_{r_\nu,+\infty}$, and let $f_\nu \in \mathcal{D}'_{T_\nu}(B_{r_\nu,+\infty})$, $f = \sum_{\nu=1}^m f_\nu \in L^{1,\mathrm{loc}}(B_{r,+\infty})$. Assume that one of assumptions* (1), (2) *in*

Corollary 15.6 holds for given f and T. Then

$$\sum_{\nu \in A_k} f_\nu = 0$$

in $\mathcal{O}_k$ for each $k \in \{1, \ldots, l\}$.

Proof. To prove (i), first note that $f \in \mathcal{D}'_T(X)$. Then Corollary 15.6 yields $f = 0$. The desired statement now follows from Proposition 15.11 (see the proof of Theorem 14.33).

Turning to (ii), we have, as above,

$$\sum_{\nu \in A_k} f_\nu = 0$$

in $B_{r,+\infty}$ for all k (see Corollary 15.7). Since

$$\sum_{\nu \in A_k} f_\nu \in \mathcal{D}'_T(\mathcal{O}_k),$$

this, together with Theorem 15.1(i), finishes the proof. □

Let us now regard $L^{p,\mathrm{loc}}$ versions of Theorems 15.26 and 15.27.

Theorem 15.29. *Let $p \in [1, +\infty)$, $\alpha, \beta, r \geqslant 0$, $R > 0$. For $\sigma = p\beta - 2(p-1)\rho_X \geqslant 0$, let*

$$h_p(\alpha, \beta, R) = \begin{cases} R^{p\alpha} \mathrm{e}^{\sigma R} & \text{if } \sigma > 0, \\ R^{1+p\alpha} & \text{if } \sigma = 0, \end{cases}$$

and assume that $T \in \mathcal{E}'_\natural(X)$, $T \neq 0$. Then the following results are true.

(i) *Let $f \in (L^{p,\mathrm{loc}} \cap \mathcal{D}'_T)(X)$ and*

$$\liminf_{R \to +\infty} \frac{1}{h_p(\alpha, \beta, R)} \int_{B_R} |f(x)|^p \, \mathrm{d}x = 0. \tag{15.44}$$

Then all the assertions of Theorem 15.26 hold.

(ii) *Let $f \in (L^{p,\mathrm{loc}} \cap \mathcal{D}'_T)(B_{r,+\infty})$ and*

$$\liminf_{R \to +\infty} \frac{1}{h_p(\alpha, \beta, R)} \int_{B_{r+1,R}} |f(x)|^p \, \mathrm{d}x = 0. \tag{15.45}$$

Then all the statements of Theorem 15.27 are valid.

Proof. It is enough to prove that for some $\xi > r(T)$, relation (15.41) is satisfied. Suppose on the contrary that there exist $c > 0$ and $\xi > r(T)$ such that

$$c < R^{-\alpha} \mathrm{e}^{-\beta R} \int_{U(R,\xi)} |f(x)| \, \mathrm{d}x \tag{15.46}$$

for all sufficiently large $R > 0$. Using (10.14), (10.15), and the Hölder inequality, we conclude from (15.46) that

$$c_1 R^{p\alpha} e^{\sigma R} < \int_{U(R,\xi)} |f(x)|^p \, dx$$

for some $c_1 > 0$ not depending on R. By the latter inequality and the definition of $h_p(\alpha, \beta, R)$, we obtain

$$h_p(\alpha, \beta, R) = O\left(\int_{B_{r+1,R}} |f(x)|^p \, dx\right) \quad \text{as } R \to +\infty.$$

This contradicts (15.44) and (15.45), proving the theorem. □

As before, the function in (15.43), together with (10.15) and Propositions 10.6 and 10.8(iv), shows that assumptions (15.44) and (15.45) in Theorem 15.29 cannot be relaxed either.

We shall now obtain the following uniqueness result important for applications.

Theorem 15.30. *Let $T \in \mathcal{E}'_\natural(X)$, $T \neq 0$, and let $\mathcal{O}$ be a ζ domain with $\zeta = r(T)$ such that the set $X \backslash \mathcal{O}$ is bounded or empty. Assume that $f \in (L^p \cap \mathcal{D}'_T)(\mathcal{O})$ for some $p \in [1, +\infty)$. Then the following assertions hold.*

(i) *If $p \in [1, 2]$, then $f = 0$.*
(ii) *If $\varkappa_T \geqslant \rho_X$, then $f = 0$.*
(iii) *If $\varkappa_T < \rho_X$ and $1 \leqslant p \leqslant 2\rho_X/(\rho_X - \varkappa_T)$, then $f = 0$. The statement fails in general for $p > 2\rho_X/(\rho_X - \varkappa_T)$.*

It is easy to see that the situations described in assertions (ii) and (iii) actually occur for suitable T.

Proof of Theorem 15.30. It is enough to consider the case where $\mathcal{O} = B_{r,+\infty}$, $r > 0$ (see Theorem 15.1(i)). To prove (i), first extend f on X by letting $f = 0$ in $\mathring{B}_r$. Now define

$$u = f \times T$$

and let $\delta \in \widehat{K}_M$ and $j \in \{1, \ldots, d(\delta)\}$. Using the Hölder inequality, we see that $f^{\delta,j} \in L^p(X)$. By (10.30) we deduce that

$$u^{\delta,j} = f^{\delta,j} \times T$$

and $u^{\delta,j} \in \mathcal{E}'_{\delta,j}(X)$. Then $\mathcal{F}^\delta_j(u^{\delta,j})$ is an entire function of exponential type (see Theorem 10.11). According to (10.72) and (10.93), we have

$$\mathcal{F}^\delta_j\big(u^{\delta,j}\big)(\lambda) = \mathcal{F}^\delta_j\big(f^{\delta,j}\big)(\lambda)\overset{\circ}{T}(\lambda) \tag{15.47}$$

for almost all $\lambda \in \mathbb{R}^1$. Owing to results in Sect. 10.5, $\mathcal{F}^\delta_j(f^{\delta,j}) \in L^{2,\text{loc}}(\mathbb{R}^1)$. Since the functions $\mathcal{F}^\delta_j(u^{\delta,j})$ and $\overset{\circ}{T}$ are entire, this, together with (15.47), implies that $\mathcal{F}^\delta_j(f^{\delta,j})$ is also entire. Moreover, a result of Malgrange [150, p. 306] shows that $\mathcal{F}_j^\delta(f^{\delta,j})$ is an entire function of exponential type. Now it follows by Theorem 10.11 that $f^{\delta,j} \in \mathcal{E}'_{\delta,j}(X)$. As $f^{\delta,j} \in \mathcal{D}'_T(B_{r,\infty})$, Theorem 15.1(i) yields $f^{\delta,j} = 0$ in $B_{r,\infty}$. Combining this with (15.27), we obtain (i).

Next, let $\alpha = 0$, $\beta = \rho_X + \varkappa_T$, and let $\sigma = p\beta - 2(p-1)\rho_X$. Then the inequality $\varkappa_T \geqslant \rho_X$ implies that $\sigma \geqslant 0$ for all $p \geqslant 1$. Assertion (ii) is now obvious from Theorem 15.29(ii). Similarly, if $\varkappa_T < \rho_X$ and $1 \leqslant p \leqslant 2\rho_X/(\rho_X - \varkappa_T)$, then $\sigma \geqslant 0$, and by Theorem 15.29(ii) $f = 0$.

Assume now that $p > 2\rho_X/(\rho_X - \varkappa_T)$. Then

$$p > \frac{2\rho_X}{\rho_X - \operatorname{Im}\lambda}$$

for some $\lambda \in \mathcal{Z}_T$. Because of (10.14), (10.15), and Propositions 10.6, the function $\Phi_{\lambda,0,\delta,1}$ is in the class $(L^p \cap C_T^\infty)(X)$ if δ is identity representation. This gives us (iii) and completes the proof of Theorem 15.30. □

To conclude we establish some related results on spaces of arbitrary rank.

Theorem 15.31. *Let* rank $X \geqslant 1$*, and let* $\{T_\nu\}_{\nu=1}^m$ *be a family of nonzero distributions in the class* $\mathcal{E}'_\natural(X)$*. Assume that* $f_\nu \in \mathcal{D}'_{T_\nu}(X)$ *and*

$$f = \sum_{\nu=1}^m f_\nu \in L^p(X)$$

for some $p \in [1, 2]$*. Then* $f = 0$*. Moreover, if* $T_\nu \in \mathcal{E}'_{\natural\natural}(X)$ *for all* ν *and the set* $\{1, \dots, m\}$ *is represented as a union of disjoint sets* $A_1, \dots, A_l$ *such that the sets*

$$\bigcup_{\nu \in A_k} \mathcal{Z}_{T_\nu}, \quad k = 1, \dots, l, \tag{15.48}$$

are also disjoint, then

$$\sum_{\nu \in A_k} f_\nu = 0 \quad \text{for all } k \in \{1, \dots, l\}. \tag{15.49}$$

These statements fail in general for $p > 2$*.*

Proof. Setting $T = T_1 \times \cdots \times T_m$, we see that $f \in (L^p \cap \mathcal{D}'_T)(X)$. Let $V \in \mathcal{D}_\natural(X)$ and $g \in G$. Now define

$$f_1 = f \times V,$$

$$f_2(x) = f_1(gx), \quad f_3(x) = \int_K f_2(kx)\, dk, \quad x \in X.$$

Using the Hölder inequality, we deduce that $f_1, f_2 \in (L^p \cap C_T^\infty)(X)$ and $f_3 \in (L^p \cap C_{T,\natural}^\infty)(X)$. Applying the spherical transform, we find

$$0 = \widetilde{f_3 \times T}(\lambda) = \widetilde{f_3}(\lambda)\widetilde{T}(\lambda)$$

for almost all $\lambda \in \mathfrak{a}^*$ (see Proposition 10.10). Since $\widetilde{T}$ is a nonzero entire function, the Lebesgue measure of the set $\{\lambda \in \mathfrak{a}^* : \widetilde{T}(\lambda) = 0\}$ is equal to zero (see Collingwood and Lohwater [53, Theorem 8.1]). Thus, $\widetilde{f_3} = 0$ almost everywhere on $\mathfrak{a}^*$, and hence $f_3 = 0$. In particular,

$$f_3(o) = f_2(o) = f_1(go) = 0.$$

As $g \in G$ and $V \in \mathcal{D}_\natural(X)$ were arbitrary, this gives $f = 0$.

Next, according to Theorem 10.12(ii), (iii), for each $\delta \in \widehat{K}_M$,

$$\sum_{\nu=1}^{m} \mathfrak{A}_\delta\big((f_\nu)_\delta\big) = 0 \quad \text{in } \mathfrak{a},$$

and $\mathfrak{A}_\delta((f_\nu)_\delta)$ is a matrix whose entries are in $\mathcal{D}'_{\Lambda_+(T_\nu),W}(\mathfrak{a})$, $\nu = 1, \dots, m$. If $T_\nu \in \mathcal{E}'_{\natural\natural}(X)$ and the sets in (15.48) are disjoint, then

$$\sum_{\nu \in A_k} \mathfrak{A}_\delta\big((f_\nu)_\delta\big) = 0 \quad \text{for each } k \in \{1, \dots, l\}$$

(see Theorem 14.33). Using now Theorem 10.12(iii) and Helgason [123, Chap. 3, Proposition 5.10], we obtain (15.49).

To continue, consider the case where $\widetilde{T_1}(\lambda) = 0$ for some $\lambda \in \mathfrak{a}^*$. We set

$$f_1 = \varphi_\lambda \quad \text{and} \quad f_\nu = 0 \quad \text{if } 2 \leqslant \nu \leqslant m.$$

In view of (10.8) and (10.34), we conclude that $f_1 \in (L^p \cap C^\infty_{T_1})(X)$ for each $p \in (2, +\infty]$. Thus, the results in Theorem 15.31 are no longer valid with $p > 2$. □

Theorem 15.32. *Let* rank $X \geqslant 1$, $T \in \mathcal{E}'_{\natural\natural}(X)$, $T \neq 0$, *and* $\xi > r(T)$. *Assume that* $\mathcal{O}$ *is a domain in* X, *and let*

$$\mathcal{O}_\xi = \bigcup_{x \in \mathcal{O}} B_\xi(x).$$

Let $f \in L^{1,\mathrm{loc}}(\mathcal{O}_\xi)$ *be a nonzero function in the class* $\mathcal{D}'_T(\mathcal{O}_\xi)$. *Then there exists nonzero* $w \in C^\infty(\mathcal{O})$ *with the following properties:*

(1) $Lw = -(\langle\lambda, \lambda\rangle + |\rho|^2)w$ *in* $\mathcal{O}$ *for some* $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ *such that* $\widetilde{T}(\lambda) = 0$;
(2) *for each* $r \in (r(T), \xi)$,

$$|w(x)| \leqslant c \int_{B_r(x)} |f(y)|\,\mathrm{d}y, \quad x \in \mathcal{O},$$

where $c > 0$ *is independent of* x.

Proof. This is just a repetition of the proof of Theorem 14.34 with the reference to Theorem 9.9 and Proposition 9.13(i) replaced by the reference to Theorem 10.28 and Proposition 10.18(i). □

Chapter 16
Mean Periodic Functions on Compact Symmetric Spaces of Rank One

A number of results in the previous chapter was obtained for the class $\mathcal{D}'_T(B_R)$, $0 < R \leqslant +\infty$. In particular, if $R = +\infty$, we cover the case of the whole space G/K. For compact symmetric spaces, the situation is more delicate. The representative features of this case are present already for mean periodic functions on the circle $\mathbb{S}^1$. Let $f \in L^1(\mathbb{S}^1)$ and suppose that the integral of f over any interval of length $2r$ vanishes. Denoting the characteristic function of the interval $(\mathrm{e}^{-\mathrm{i}r}, \mathrm{e}^{\mathrm{i}r})$ by χ_r, we have (in the sense of Fourier series)

$$\chi_r\left(\mathrm{e}^{\mathrm{i}\theta}\right) = \sum_{n=-\infty}^{\infty} \frac{2}{n} \sin(nr)\mathrm{e}^{\mathrm{i}n\theta},$$

where the constant term is understood to be $2r$. The equation

$$\int_{\theta-r}^{\theta+r} f\left(\mathrm{e}^{\mathrm{i}t}\right) \mathrm{d}t = 0$$

may be written as the convolution

$$(f * \chi_r)\left(\mathrm{e}^{\mathrm{i}\theta}\right) = \int_0^{2\pi} f\left(\mathrm{e}^{\mathrm{i}(\theta-t)}\right)\chi_r\left(\mathrm{e}^{\mathrm{i}t}\right) \mathrm{d}t = 0.$$

Taking Fourier series on both sides, we get

$$\frac{2}{n} \sin(nr)a_n = 0, \quad \text{where } a_n = \int_{-\pi}^{\pi} f\left(\mathrm{e}^{\mathrm{i}t}\right)\mathrm{e}^{-\mathrm{i}nt} \, \mathrm{d}t.$$

Thus $a_n = 0$ whenever $\sin(nr) \neq 0$ $(n \neq 0)$. It follows that f is the zero function if r is not a rational multiple of π. On the other hand, for an arbitrary $r \in (0, \pi)$, the function $\mathrm{e}^{\mathrm{i}\pi t/r}$ is mean periodic on $\mathbb{S}^1 \backslash \{1\}$ with respect to χ_r. Moreover, the space of all such functions is infinite-dimensional and can be described by means of Theorem 13.14.

V.V. Volchkov, V.V. Volchkov, *Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group,* Springer Monographs in Mathematics,

Similar phenomenons persist in the general case. In Sect. 16.3 we give different characterizations of mean periodic functions on subsets of compact symmetric spaces $\mathcal{X}$ of rank one. (The case of the whole space $\mathcal{X}$ is considered in Sect. 21.1 of Part IV in more general context.) In Sect. 16.2 we discuss support properties of mean periodic functions on $\mathcal{X}$. In contrast to the noncompact case, these questions involve additional difficulties related to finding explicit formulas for differential operators from the Lie algebra of the isometry group. The corresponding formulas are presented in Sect. 16.1 (see Propositions 16.2 and 16.4).

16.1 Group and Infinitesimal Properties

Our aim in this chapter is to get analogues of the main results from Chap. 15 for symmetric spaces $\mathcal{X}$ of compact type of rank one. Here the notational conventions accepted in Chap. 11 will be in force. In particular, we assume that the diameter of $\mathcal{X}$ is $\pi/2$ and use the realizations for $\mathcal{X}$ given in Chap. 3.

Let T be a nonzero distribution in $\mathcal{E}'_{\natural}(\mathfrak{X})$, and let $\mathcal{O}_T = \{g0\colon g \in I(\mathcal{X}), g\mathring{B}_{r(T)} \subset \mathcal{O}\}$, where $\mathcal{O}$ is an open subset of $\mathcal{X}$. Throughout we suppose that $\mathcal{O}_T \neq \varnothing$. In analogy with Sect. 15.1 we define the following classes:

$$\mathcal{D}'_T(\mathcal{O}) = \big\{f \in \mathcal{D}'(\mathcal{O})\colon f \times T = 0 \text{ in } \mathcal{O}_T\big\},$$
$$C^s_T(\mathcal{O}) = \big(\mathcal{D}'_T \cap C^s\big)(\mathcal{O}), \quad s \in \mathbb{Z}_+ \cup \{\infty\}, \qquad C_T(\mathcal{O}) = C^0_T(\mathcal{O}),$$
$$\mathrm{RA}_T(\mathcal{O}) = \big(\mathcal{D}'_T \cap \mathrm{RA}\big)(\mathcal{O}).$$

If the set $\mathcal{O}$ is $K_{\mathcal{X}}$-invariant, we set

$$\mathcal{D}'_{T,\natural}(\mathcal{O}) = \big(\mathcal{D}'_T \cap \mathcal{D}'_{\natural}\big)(\mathcal{O}), \quad C^s_{T,\natural}(\mathcal{O}) = \big(C^s_T \cap \mathcal{D}'_{\natural}\big)(\mathcal{O}).$$

The present section is devoted to a preliminary study of these classes.

We put

$$\mathcal{Z}_T = \big\{\lambda \in \mathcal{Z}\big(\widetilde{T}\big) : \mathrm{Re}\,\lambda \geqslant 0, \mathrm{i}\lambda \notin (0, +\infty)\big\},$$
$$n(\lambda, T) = \begin{cases} n_\lambda\big(\widetilde{T}\big) - 1 & \text{if } \lambda \in \mathcal{Z}_T\backslash\{0\}, \\ n_\lambda\big(\widetilde{T}\big)/2 - 1 & \text{if } \lambda = 0 \in \mathcal{Z}_T. \end{cases}$$

Proposition 16.1. *Let $\lambda \in \mathbb{C}$, $\eta \in \mathbb{Z}_+$, and let $k \in \mathbb{Z}_+$, $m \in \{0, \dots, M_{\mathcal{X}}(k)\}$, $j \in \{1, \dots, d^{k,m}_{\mathcal{X}}\}$. Then*

$$\Phi_{\lambda,\eta,k,m,j} \times T = \sum_{\nu=0}^{\eta} \binom{\eta}{\nu}_{\lambda} \widetilde{T}^{\langle \eta-\nu\rangle}(\lambda)\Phi_{\lambda,\nu,k,m,j} \quad \text{in } B_{\pi/2-r(T)}$$

and

$$\Psi_{\lambda,\eta,k,m,j} \times T = \sum_{\nu=0}^{\eta} \binom{\eta}{\nu}_{\lambda} \widetilde{T}^{\langle \eta-\nu\rangle}(\lambda)\Psi_{\lambda,\nu,k,m,j} \quad \text{in } B_{r(T),\pi/2-r(T)}$$

for $r(T) < \pi/4$. *In particular, if* $\lambda \in \mathcal{Z}_T$ *and* $\eta \in \{0, \dots, n(\lambda, T)\}$, *then* $\Phi_{\lambda,\eta,k,m,j} \in \mathrm{RA}_T(\mathfrak{X})$ *and*

$$\Psi_{\lambda,\eta,k,m,j} \in \mathrm{RA}_T(\mathfrak{X}\backslash\{0\}), \quad \textit{provided that } r(T) < \pi/4.$$

Proof. Using (11.42) and Proposition 11.10, we obtain the desired statement with $\eta = 0$. Now the general case is a consequence of (11.23)–(11.28). □

Introduce the differential operators A_j, $1 \leqslant j \leqslant a_{\mathcal{X}}$, on $\mathfrak{X}$ as follows:

- $\mathcal{X} = \overline{\mathbb{R}^n}$:

$$A_j = \left(1 - |x|^2\right)\frac{\partial}{\partial x_j} + 2x_j \sum_{l=1}^{n} x_l \frac{\partial}{\partial x_l}; \tag{16.1}$$

- $\mathcal{X} = \mathbb{P}^n_{\mathbb{R}}$:

$$A_j = \frac{\partial}{\partial x_j} + x_j \sum_{l=1}^{n} x_l \frac{\partial}{\partial x_l}; \tag{16.2}$$

- $\mathcal{X} = \mathbb{P}^n_{\mathbb{C}}$:

$$\begin{aligned} A_j &= \frac{\partial}{\partial z_j} + \overline{z}_j \sum_{l=1}^{n} \overline{z}_l \frac{\partial}{\partial \overline{z}_l}, \quad 1 \leqslant j \leqslant n, \\ A_j &= \frac{\partial}{\partial \overline{z}_{j-n}} + z_{j-n} \sum_{l=1}^{n} z_l \frac{\partial}{\partial z_l}, \quad n+1 \leqslant j \leqslant 2n; \end{aligned} \tag{16.3}$$

- $\mathcal{X} = \mathbb{P}^n_{\mathbb{Q}}$:

$$\begin{aligned} A_j &= \frac{\partial}{\partial z_j} + \overline{z}_j \sum_{l=1}^{2n} \overline{z}_l \frac{\partial}{\partial \overline{z}_l} - z_{j+n} \sum_{l=1}^{n} \left(\overline{z}_{n+l} \frac{\partial}{\partial z_l} - \overline{z}_l \frac{\partial}{\partial z_{n+l}} \right), \quad 1 \leqslant j \leqslant n, \\ A_j &= \frac{\partial}{\partial z_j} + \overline{z}_j \sum_{l=1}^{2n} \overline{z}_l \frac{\partial}{\partial \overline{z}_l} - z_{j-n} \sum_{l=1}^{n} \left(\overline{z}_l \frac{\partial}{\partial z_{n+l}} - \overline{z}_{n+l} \frac{\partial}{\partial z_l} \right), \\ &\quad n+1 \leqslant j \leqslant 2n, \\ A_j &= \frac{\partial}{\partial \overline{z}_{j-2n}} + z_{j-2n} \sum_{l=1}^{2n} z_l \frac{\partial}{\partial z_l} - \overline{z}_{j-n} \sum_{l=1}^{n} \left(z_{n+l} \frac{\partial}{\partial \overline{z}_l} - z_l \frac{\partial}{\partial \overline{z}_{n+l}} \right), \\ &\quad 2n+1 \leqslant j \leqslant 3n, \\ A_j &= \frac{\partial}{\partial \overline{z}_{j-2n}} + z_{j-2n} \sum_{l=1}^{2n} z_l \frac{\partial}{\partial z_l} - \overline{z}_{j-3n} \sum_{l=1}^{n} \left(z_l \frac{\partial}{\partial \overline{z}_{n+l}} - z_{n+l} \frac{\partial}{\partial \overline{z}_l} \right), \\ &\quad 3n+1 \leqslant j \leqslant 4n; \end{aligned} \tag{16.4}$$

- $\mathcal{X} = \mathbb{P}^2_{\mathbb{C}a}$:

$$A_j = \left(1 - \sum_{l=1}^{8} x_{2l-1}^2\right)\frac{\partial}{\partial x_j} - \varepsilon_j \sum_{l=1}^{8} P_{l,\lambda_j}(x)\frac{\partial}{\partial x_{2l}} + 2x_j \sum_{l=1}^{16} x_l \frac{\partial}{\partial x_l}, \quad j \in \{1, 3, \dots, 15\}, \tag{16.5}$$

$$A_j = \left(1 - \sum_{l=1}^{8} x_{2l}^2\right)\frac{\partial}{\partial x_j} - \varepsilon_{j-1} \sum_{l=1}^{8} Q_{l,\lambda_{j-1}}(x)\frac{\partial}{\partial x_{2l-1}} + 2x_j \sum_{l=1}^{16} x_l \frac{\partial}{\partial x_l}, \quad j \in \{2, 4, \dots, 16\}, \tag{16.6}$$

where

$$\varepsilon_1 = 1, \qquad \varepsilon_j = -1, \quad j \in \{3, 5, \dots, 15\},$$
$$\lambda_1 = 0, \quad \lambda_3 = 4, \quad \lambda_5 = 2, \quad \lambda_7 = 6, \quad \lambda_9 = 1, \quad \lambda_{11} = 5, \quad \lambda_{13} = 3, \quad \lambda_{15} = 7,$$

and the polynomials $P_{l,s}\colon \mathbb{R}^{16} \to \mathbb{R}^1$ and $Q_{l,s}\colon \mathbb{R}^{16} \to \mathbb{R}^1$, $0 \leqslant s \leqslant 7$, are determined by the relations

$$\begin{aligned}&\mathbf{i}_s\big(p_1(x) + p_5(x)\mathbf{i}_1 + p_3(x)\mathbf{i}_2 + p_7(x)\mathbf{i}_3 + p_2(x)\mathbf{i}_4 + p_6(x)\mathbf{i}_5 + p_4(x)\mathbf{i}_6 + p_8(x)\mathbf{i}_7\big)\\ &\quad = P_{1,s}(x) + P_{5,s}(x)\mathbf{i}_1 + P_{3,s}(x)\mathbf{i}_2 + P_{7,s}(x)\mathbf{i}_3 + P_{2,s}(x)\mathbf{i}_4 + P_{6,s}(x)\mathbf{i}_5\\ &\qquad + P_{4,s}(x)\mathbf{i}_6 + P_{8,s}(x)\mathbf{i}_7,\\ &\big(p_1(x) + p_5(x)\mathbf{i}_1 + p_3(x)\mathbf{i}_2 + p_7(x)\mathbf{i}_3 + p_2(x)\mathbf{i}_4 + p_6(x)\mathbf{i}_5 + p_4(x)\mathbf{i}_6 + p_8(x)\mathbf{i}_7\big)\mathbf{i}_s\\ &\quad = Q_{1,s}(x) + Q_{5,s}(x)\mathbf{i}_1 + Q_{3,s}(x)\mathbf{i}_2 + Q_{7,s}(x)\mathbf{i}_3 + Q_{2,s}(x)\mathbf{i}_4 + Q_{6,s}(x)\mathbf{i}_5\\ &\qquad + Q_{4,s}(x)\mathbf{i}_6 + Q_{8,s}(x)\mathbf{i}_7\end{aligned}$$

(see Sect. 1.1). The importance of these operators for the theory of convolution equations on $\mathcal{X}$ is given in the following:

Proposition 16.2. *Let $f \in \mathcal{D}'_T(\mathcal{O})$. Then $A_j f \in \mathcal{D}'_T(\mathcal{O})$ for all $1 \leqslant j \leqslant a_{\mathcal{X}}$.*

Proof. The standard approximation argument shows that it suffices to verify the assertion of Proposition 16.2 for $f \in C^\infty_T(\mathcal{O})$. First, suppose that $\mathcal{X} = \overline{\mathbb{R}^n}$ or $\mathcal{X} = \mathbb{P}^n_{\mathbb{K}}$ with $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{Q}$. Put $s = n$ if $\mathcal{X} = \overline{\mathbb{R}^n}, \mathbb{P}^n_{\mathbb{R}}, \mathbb{P}^n_{\mathbb{C}}$, and $s = 2n$ if $\mathcal{X} = \mathbb{P}^n_{\mathbb{Q}}$. We associate to each $\mathcal{X}$ the mappings

$$\varphi_{t,j} = \frac{1}{g_{t,j}}\big(f^1_{t,j}, \dots, f^s_{t,j}\big), \quad t \in \mathbb{C}, 1 \leqslant j \leqslant a_X,$$

where the functions $g_{t,j}$ and $f^l_{t,j}$, $1 \leqslant l \leqslant s$, are defined as follows:

- $\mathcal{X} = \overline{\mathbb{R}^n}$:

$$g_{t,j}(x) = 1 + 2tx_j + t^2|x|^2,$$
$$f^l_{t,j}(x) = \begin{cases} (1+t^2)x_l, & l \neq j, \\ (1-t^2)x_j - t(1-|x|^2), & l = j; \end{cases}$$

- $\mathcal{X} = \mathbb{P}^n_{\mathbb{R}}$:

$$g_{t,j}(x) = 1 + tx_j,$$
$$f^l_{t,j}(x) = \begin{cases} \sqrt{1+|t|^2}x_l, & l \neq j, \\ x_j - t, & l = j; \end{cases}$$

- $\mathcal{X} = \mathbb{P}^n_{\mathbb{C}}$:

$$g_{t,j}(z) = \begin{cases} 1 + tz_j, & 1 \leqslant j \leqslant n, \\ 1 - \mathrm{i}tz_{j-n}, & n+1 \leqslant j \leqslant 2n, \end{cases}$$
$$f^l_{t,j}(z) = \begin{cases} \sqrt{1+|t|^2}z_l, & l \neq j, \\ z_j - t, & l = j, \end{cases} \quad \text{if } 1 \leqslant j \leqslant n \quad \text{and}$$
$$f^l_{t,j}(z) = f^l_{\mathrm{i}t,j-n}(z) \quad \text{if } n+1 \leqslant j \leqslant 2n;$$

- $\mathcal{X} = \mathbb{P}^n_{\mathbb{Q}}$:

$$g_{t,j}(z) = \begin{cases} 1 + 2t\,\mathrm{Re}\,z_j + t^2(|z_j|^2 + |z_{j+n}|^2), & 1 \leqslant j \leqslant n, \\ 1 + 2t\,\mathrm{Re}\,z_j + t^2(|z_j|^2 + |z_{j-n}|^2), & n+1 \leqslant j \leqslant 2n, \\ 1 + 2t\,\mathrm{Re}(\mathrm{i}\overline{z}_{j-2n}) + t^2(|z_{j-n}|^2 + |z_{j-2n}|^2), & 2n+1 \leqslant j \leqslant 3n, \\ 1 + 2t\,\mathrm{Re}(\mathrm{i}\overline{z}_{j-2n}) + t^2(|z_{j-2n}|^2 + |z_{j-3n}|^2), & 3n+1 \leqslant j \leqslant 4n, \end{cases}$$
$$f^l_{t,j}(z) = \begin{cases} \sqrt{1+|t|^2}(z_l + t(z_l\overline{z}_j + z_{n+j}\overline{z}_{n+l})), & 1 \leqslant l \leqslant n,\ l \neq j, \\ z_j + t(|z_j|^2 + |z_{n+j}|^2 - 1) - t^2\overline{z}_j, & l = j, \\ \sqrt{1+|t|^2}(z_l + t(z_l\overline{z}_j - z_{n+j}\overline{z}_{l-n})), & n+1 \leqslant l \leqslant 2n,\ l \neq n+j, \\ (1+|t|^2)z_{n+j}, & l = n+j, \end{cases}$$
$$\text{if } 1 \leqslant j \leqslant n,$$
$$f^l_{t,j}(z) = \begin{cases} \sqrt{1+|t|^2}(z_l + t(z_l\overline{z}_j - z_{j-n}\overline{z}_{n+l})), & 1 \leqslant l \leqslant n,\ l \neq j-n, \\ (1+|t|^2)z_{j-n}, & l = j-n, \\ \sqrt{1+|t|^2}(z_l + t(z_{j-n}\overline{z}_{l-n} + z_l\overline{z}_j)), & n+1 \leqslant l \leqslant 2n,\ l \neq j, \\ z_j + t(|z_j|^2 + |z_{j-n}|^2 - 1) - t^2\overline{z}_j, & l = j, \end{cases}$$
$$\text{if } n+1 \leqslant j \leqslant 2n, \quad \text{and}$$
$$f^l_{t,j}(z) = f^l_{\mathrm{i}t,j-2n}(z) \quad \text{if } 1 \leqslant l \leqslant 2n,\ 2n+1 \leqslant j \leqslant 4n.$$

Fix a point $p \in \mathcal{O}_T$. Using Proposition 3.1 and its analogues for the spaces $\overline{\mathbb{R}^n}$, $\mathbb{P}^n_{\mathbb{C}}$, and $\mathbb{P}^n_{\mathbb{Q}}$ (see Chap. 3), we derive from the condition $f \in \mathcal{D}'_T(\mathcal{O})$ that

$$\big((f \circ \varphi_{t,j}) \times T\big)(p) = 0, \quad 1 \leqslant j \leqslant a_{\mathcal{X}}, \tag{16.7}$$

in some interval of $\mathbb{R}^1$ around $t = 0$. Let us differentiate equality (16.7) with respect to t and put $t = 0$. After some computation we obtain the desired statement for $\mathcal{X} = \overline{\mathbb{R}^n}$ and $\mathcal{X} = \mathbb{P}^n_{\mathbb{K}}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{Q}$). Now assume that $\mathcal{X} = \mathbb{P}^2_{\mathbb{C}a}$. Set

$$\tau(z_1, z_2) = (\overline{z}_2, \overline{z}_1), \quad z_1, z_2 \in \mathbb{C}a,$$
$$\tau_s(z_1, z_2) = (\mathbf{i}_s z_1, z_2 \mathbf{i}_s), \quad z_1, z_2 \in \mathbb{C}a, 1 \leqslant s \leqslant 7.$$

According to Example 1.1, $\tau, \tau_s \in \mathrm{O}_{\mathbb{C}a}(2)$. Denote by φ_t, $t \in \mathbb{R}^1$, the mapping $\phi_3 \circ \sigma_t \circ \phi_3^{-1}$ defined in Sect. 3.5, i.e.,

$$\varphi_t(z_1, z_2) = \Big((t - z_1)(t z_1 + 1)^{-1}, -\sqrt{1 + t^2}(t\overline{z}_1 + 1)^{-1} z_2\Big),$$
$$z_1 \in \mathbb{C}a \setminus \big\{-t^{-1}\big\}, \ z_2 \in \mathbb{C}a.$$

Also let $\psi_t = -\varphi_t$. For fixed $p \in \mathcal{O}_T$, there exists $\varepsilon > 0$ such that

$$\big((f \circ \psi_t) \times T\big)(p) = 0, \qquad \big((f \circ \tau \circ \psi_t \circ \tau) \times T\big)(p) = 0,$$
$$\big((f \circ \tau_s \circ \varphi_t \circ \tau_s) \times T\big)(p) = 0, \quad \big((f \circ \tau \circ \tau_s \circ \varphi_t \circ \tau_s \circ \tau) \times T\big)(p) = 0,$$
$$1 \leqslant s \leqslant 7,$$

provided that $|t| < \varepsilon$. As above, this implies the required result in the case $\mathcal{X} = \mathbb{P}^2_{\mathbb{C}a}$. Thus, Proposition 16.2 is proved. □

In the next statements of this section we shall assume that the set $\mathcal{O}$ is $K_{\mathcal{X}}$-invariant.

Proposition 16.3.

(i) *If $f \in \mathcal{D}'_T(\mathcal{O})$, then $f^{k,m,i,j} \in \mathcal{D}'_T(\mathcal{O})$ for all $k \in \mathbb{Z}_+$, $m \in \{0, \dots, M_{\mathcal{X}}(k)\}$ and $i, j \in \{1, \dots, d^{k,m}_{\mathcal{X}}\}$.*

(ii) *Let $f \in \mathcal{D}'(\mathcal{O})$ and suppose that $f^{k,m,j} \in \mathcal{D}'_T(\mathcal{O})$ for all k, m, j. Then $f \in \mathcal{D}'_T(\mathcal{O})$.*

Proof. Our result follows from (11.13) and (11.14). □

We shall now obtain a refinement of Proposition 16.2 in the case where $f \in C^s_{k,m,j}(\mathcal{O})$.

Proposition 16.4. *Let $Y \in \mathcal{H}^{k,m}_{\mathcal{X}} \setminus \{0\}$ and suppose that $\varphi(\varrho)Y(\sigma) \in C^s_T(\mathcal{O})$ for some $s \geqslant 1$. Then the following assertions hold.*

(i) *For all $j \in \{1, \dots, d^{k+1,m}_{\mathcal{X}}\}$,*

$$\big(D\big(-k, m + 1 - \mathcal{N}_{\mathcal{X}}(k + 1)\big)\varphi\big)(\varrho) Y^{k+1,m}_j(\sigma) \in C^{s-1}_T(\mathcal{O}).$$

(ii) *If* $m \leqslant M_{\mathcal{X}}(k+1)-1$, *then*

$$\big(D(-k, \beta_{\mathcal{X}}-m)\varphi\big)(\varrho)Y_j^{k+1,m+1}(\sigma) \in C_T^{s-1}(\mathcal{O})$$

for all $j \in \{1, \dots, d_{\mathcal{X}}^{k+1,m+1}\}$.

(iii) *If* $k \geqslant 1$ *and* $m \leqslant M_{\mathcal{X}}(k-1)$, *then*

$$\big(D\big(k+2\alpha_{\mathcal{X}}, \mathcal{N}_{\mathcal{X}}(k)+\gamma_{\mathcal{X}}-m\big)\varphi\big)(\varrho)Y_j^{k-1,m}(\sigma) \in C_T^{s-1}(\mathcal{O})$$

for all $j \in \{1, \dots, d_{\mathcal{X}}^{k-1,m}\}$.

(iv) *If* $m \geqslant 1$, *then*

$$\big(D(k+2\alpha_{\mathcal{X}}, \alpha_{\mathcal{X}}+m)\varphi\big)(\varrho)Y_j^{k-1,m-1}(\sigma) \in C_T^{s-1}(\mathcal{O})$$

for all $j \in \{1, \dots, d_{\mathcal{X}}^{k-1,m-1}\}$.

Proof. Thanks to Propositions 16.2 and 16.3 (i),

$$A_l\big(\psi(\varrho)H(\varrho\sigma)\big) \in C_T^{s-1}(\mathcal{O}) \quad (1 \leqslant l \leqslant a_{\mathcal{X}}) \tag{16.8}$$

for every $H \in \mathcal{H}_{\mathcal{X}}^{k,m}$, where

$$\psi(\varrho) = \frac{\varphi(\varrho)}{\varrho^k}.$$

We require a number of relations concerning the functions (16.8) for some l and H. We shall distinguish four cases.

(a) The case $\mathcal{X} = \overline{\mathbb{R}^n}$ or $\mathcal{X} = \mathbb{P}_{\mathbb{R}}^n$.

If $\mathcal{X} = \overline{\mathbb{R}^n}$, then (16.1) yields

$$\begin{aligned} A_1\big(\psi(\varrho)(x_1+\mathrm{i}x_2)^k\big) &= \left(\big(1+\varrho^2\big)\varphi'(\varrho) - k\frac{\varphi(\varrho)}{\varrho}\big(1-\varrho^2\big)\right)\varrho^{-1-k}H_2(x) \\ &\quad + \left(\big(1+\varrho^2\big)\varphi'(\varrho) + (n+k-2)\frac{\varphi(\varrho)}{\varrho}\big(1-\varrho^2\big)\right) \\ &\quad \times \frac{k}{n+2k-2}\varrho^{1-k}H_1(x), \end{aligned} \tag{16.9}$$

where

$$H_1(x) = (x_1+\mathrm{i}x_2)^{k-1},$$
$$H_2(x) = x_1(x_1+\mathrm{i}x_2)^k - \frac{k\varrho^2}{n+2k-2}(x_1+\mathrm{i}x_2)^{k-1}.$$

Obviously, $H_1 \in \mathcal{H}_1^{n,k-1}$ for $k \geqslant 1$. In addition, $H_2 \in \mathcal{H}_1^{n,k+1}$ since

$$(n+2k-2)x_l h - \varrho^2\frac{\partial h}{\partial x_l} \in \mathcal{H}_1^{n,k+1} \quad \text{for all } h \in \mathcal{H}_1^{n,k}, l \in \{1, \dots, n\} \tag{16.10}$$

(see [225, Part I, Proposition 5.1 (2)]). Analogously, by virtue of (16.2),

$$A_1\big(\psi(\varrho)(x_1+\mathrm{i}x_2)^k\big)=\frac{k}{n+2k-2}\bigg(\big(1+\varrho^2\big)\varphi'(\varrho)+(n+k-2)\frac{\varphi(\varrho)}{\varrho}\bigg)\times\varrho^{1-k}H_1(x)+\bigg(\big(1+\varrho^2\big)\varphi'(\varrho)-k\frac{\varphi(\varrho)}{\varrho}\bigg)\varrho^{-1-k}H_2(x)\tag{16.11}$$

for $\mathcal{X}=\mathbb{P}^n_{\mathbb{R}}$.

(b) The case $\mathcal{X}=\mathbb{P}^n_{\mathbb{C}}$.
In this situation we have from (16.3)

$$A_1\big(\psi(\varrho)z_1^{k-m}\overline{z}_2^m\big)=\frac{1}{2}\bigg(\big(1+\varrho^2\big)\varphi'(\varrho)-\frac{\varphi(\varrho)}{\varrho}\big(k+(k-2m)\varrho^2\big)\bigg)\times\varrho^{-1-k}H_4(z)+\frac{k-m}{2(n+k-1)}\bigg(\big(1+\varrho^2\big)\varphi'(\varrho)+\frac{\varphi(\varrho)}{\varrho}\big(2n+k-2-(k-2m)\varrho^2\big)\bigg)\varrho^{1-k}H_3(z),\tag{16.12}$$

$$A_1\big(\psi(\varrho)z_1^{m}\overline{z}_2^{k-m}\big)=\bigg(\big(1+\varrho^2\big)\varphi'(\varrho)-\frac{\varphi(\varrho)}{\varrho}\big(k+(2m-k)\varrho^2\big)\bigg)\times\varrho^{-1-k}H_6(z)+\frac{m}{2(n+k-1)}\bigg(\big(1+\varrho^2\big)\varphi'(\varrho)+\frac{\varphi(\varrho)}{\varrho}\big(2n+k-2+(k-2m)\varrho^2\big)\bigg)\varrho^{1-k}H_5(z)\tag{16.13}$$

with

$$H_3(z)=z_1^{k-m-1}\overline{z}_2^m,\qquad H_4(z)=\overline{z}_1z_1^{k-m}\overline{z}_2^m-\frac{k-m}{n+k-1}\varrho^2z_1^{k-m-1}\overline{z}_2^m,$$
$$H_5(z)=z_1^{m-1}\overline{z}_2^{k-m},\qquad H_6(z)=\overline{z}_1z_1^{m}\overline{z}_2^{k-m}-\frac{m}{n+k-1}\varrho^2z_1^{m-1}\overline{z}_2^{k-m}.$$

Owing to Lemma 4.11 and (16.10), $H_3\in\mathcal{H}_3^{n,k-1,m}$ and $H_4\in\mathcal{H}_3^{n,k+1,m+1}$ for $m\leqslant[(k-1)/2]$. Likewise, $H_5\in\mathcal{H}_3^{n,k-1,m-1}$ if $m\geqslant1$, and $H_6\in\mathcal{H}_3^{n,k+1,m}$.

(c) The case $\mathcal{X}=\mathbb{P}^n_{\mathbb{Q}}$.
For $\alpha,\beta\in\mathbb{Z}$, we put

$$P_{\alpha,\beta}(z)=\overline{z}_1^{\alpha-2\beta}(\overline{z}_1z_{n+2}-\overline{z}_2z_{n+1})^{\beta},$$
$$Q_{\alpha,\beta}(z)=z_{n+1}\overline{z}_1^{\alpha-2\beta-1}(\overline{z}_1z_{n+2}-\overline{z}_2z_{n+1})^{\beta}.$$

By Lemmas 4.12 and 4.31, $P_{k,m}\in\mathcal{H}_5^{n,k,m}$ and $Q_{k,m}\in\mathcal{H}_5^{n,k,m}$ if $m\leqslant[(k-1)/2]$. Using (16.4), we find

$$A_1\big(\psi(\varrho)P_{k,m}(z)\big)=\frac{1}{2}\bigg(\big(1+\varrho^2\big)\varphi'(\varrho)+\frac{\varphi(\varrho)}{\varrho}\big(-k+(k-2m)\varrho^2\big)\bigg)\varrho^{-k-1}P_{k+1,m}(z),\tag{16.14}$$

$$A_{3n+2}\big(\psi(\varrho)P_{k,m}(z)\big) - A_2\big(\psi(\varrho)Q_{k,m}(z)\big)$$
$$= \frac{1}{2}\bigg(\big(1+\varrho^2\big)\varphi'(\varrho) + \frac{\varphi(\varrho)}{\varrho}\big(-k+(2m-k-2)\varrho^2\big)\bigg)\varrho^{-k-1}P_{k+1,m+1}(z). \tag{16.15}$$

Next,

$$A_{n+2}\big(\psi(\varrho)P_{k,m}(z)\big) = \frac{1}{2}\bigg(\big(1+\varrho^2\big)\varphi'(\varrho) + \frac{\varphi(\varrho)}{\varrho}\big(-k+(k-2m)\varrho^2\big)\bigg)$$
$$\times\varrho^{-k-1}H_7(z) + \frac{m}{4n+2k-2}\bigg(\big(1+\varrho^2\big)\varphi'(\varrho)$$
$$+\frac{\varphi(\varrho)}{\varrho}\big(4n+k-2+(k-2m)\varrho^2\big)\bigg)\varrho^{1-k}P_{k-1,m-1}(z), \tag{16.16}$$

$$A_{2n+1}\big(\psi(\varrho)P_{k,m}(z)\big) + A_{n+1}\big(\psi(\varrho)Q_{k,m}(z)\big)$$
$$= \frac{1}{2}\bigg(\big(1+\varrho^2\big)\varphi'(\varrho) + \frac{\varphi(\varrho)}{\varrho}\big(-k+(2m-k-2)\varrho^2\big)\bigg)\varrho^{-k-1}\big(H_8(z)+H_9(z)\big)$$
$$+\frac{k-m+1}{4n+2k-2}\bigg(\big(1+\varrho^2\big)\varphi'(\varrho) + \frac{\varphi(\varrho)}{\varrho}\big(4n+k-2+(2m-k-2)\varrho^2\big)\bigg)$$
$$\times\varrho^{1-k}P_{k-1,m}(z), \tag{16.17}$$

where

$$H_7(z) = \overline{z}_{n+2}P_{k,m}(z) - \frac{\varrho^2}{2n+k-1}\frac{\partial P_{k,m}}{\partial z_{n+2}},$$
$$H_8(z) = z_1 P_{k,m}(z) - \frac{\varrho^2}{2n+k-1}\frac{\partial P_{k,m}}{\partial \overline{z}_1},$$
$$H_9(z) = \overline{z}_{n+1}Q_{k,m}(z) - \frac{\varrho^2}{2n+k-1}\frac{\partial Q_{k,m}}{\partial z_{n+1}}.$$

According to (16.10), the polynomials H_7, H_8, and H_9 belong to the space $\mathcal{H}_1^{4n,k+1}$.

(d) The case $\mathcal{X} = \mathbb{P}^2_{\mathbb{C}a}$.
For $\alpha, \beta \in \mathbb{Z}$, we put

$$R_{\alpha,\beta}(x) = \begin{cases} \displaystyle\sum_{l=0}^{[\alpha/2]-\beta} a_{\alpha,\beta,l}\,\Phi_{\alpha,\beta,l}(x) & \text{if } \beta \leqslant [\alpha/2], \\ 0 & \text{if } \beta > [\alpha/2], \end{cases} \tag{16.18}$$

where

$$a_{\alpha,\beta,l} = \left(-\frac{1}{4}\right)^l (l+1)(l+2)\binom{\alpha-2\beta-l+2}{l+2},$$
$$\Phi_{\alpha,\beta,l}(x) = (x_1+\mathrm{i}x_2)^{\alpha-2\beta-2l}\big(p_9(x)-p_{10}(x)+2\mathrm{i}p_1(x)\big)^{\beta+l}$$

(for the definition of $p_9(x)$ and $p_{10}(x)$, see Sect. 1.1). In view of Remark 4.4, $R_{k,m} \in \mathcal{H}_6^{k,m}$. By (16.5) and (16.6),

$$\begin{aligned}
&(-A_1 - \mathrm{i}A_2)\big(\psi(\varrho)R_{k,m}(x)\big)\\
&\quad = \varphi_1(\varrho)(x_1 + \mathrm{i}x_2)R_{k,m}(x) - \psi(\varrho)\bigg(\big(p_9(x) - 1\big)\frac{\partial R_{k,m}}{\partial x_1} + \mathrm{i}\big(p_{10}(x) - 1\big)\frac{\partial R_{k,m}}{\partial x_2}\\
&\qquad + \sum_{l=1}^{8} p_l(x)\bigg(\frac{\partial R_{k,m}}{\partial x_{2l}} + \mathrm{i}\frac{\partial R_{k,m}}{\partial x_{2l-1}}\bigg)\bigg)
\end{aligned}$$

with

$$\varphi_1(\varrho) = \bigg(\varrho + \frac{1}{\varrho}\bigg)\psi'(\varrho) + 2k\psi(\varrho).$$

It follows from (16.18) and (1.17) that

$$\begin{aligned}
\frac{\partial R_{k,m}}{\partial x_1} + \mathrm{i}\frac{\partial R_{k,m}}{\partial x_2} &= 0,\\
p_9(x)\frac{\partial R_{k,m}}{\partial x_1} + \mathrm{i}p_{10}(x)\frac{\partial R_{k,m}}{\partial x_2} &= \big(p_9(x) - p_{10}(x)\big)\Sigma_1,\\
\sum_{l=1}^{8} p_l(x)\bigg(\frac{\partial R_{k,m}}{\partial x_{2l}} + \mathrm{i}\frac{\partial R_{k,m}}{\partial x_{2l-1}}\bigg) &= 2\mathrm{i}p_1(x)\Sigma_1,
\end{aligned}$$

where

$$\Sigma_1 = \sum_{l=0}^{[k/2]-m} a_{k,m,l}\big((k - 2m - 2l)\Phi_{k-1,m,l}(x) + (2m + 2l)\Phi_{k-1,m-1,l}(x)\big).$$

Therefore,

$$\begin{aligned}
&(-A_1 - \mathrm{i}A_2)\big(\psi(\varrho)R_{k,m}(x)\big)\\
&\quad = \varphi_1(\varrho)(x_1 + \mathrm{i}x_2)R_{k,m}(x) - \psi(\varrho)\big(p_9(x) - p_{10}(x) + 2\mathrm{i}p_1(x)\big)\Sigma_1.
\end{aligned} \tag{16.19}$$

Furthermore, the definition of $R_{k,m}$ shows that

$$2\binom{k - 2m + 1}{2}\Phi_{k+1,m+1,0}(x) = R_{k+1,m+1}(x) - \Sigma_2 \tag{16.20}$$

and

$$\begin{aligned}
&2\binom{k - 2m + 3}{2}\Phi_{k+1,m,0}(x)\\
&\quad = R_{k+1,m}(x) - \Sigma_3 + \frac{k - 2m + 2}{4}\big(R_{k+1,m+1}(x) - \Sigma_2\big)
\end{aligned} \tag{16.21}$$

with

$$\Sigma_2 = \sum_{1\leqslant l\leqslant[(k-1)/2]-m} a_{k+1,m+1,l}\Phi_{k+1,m+1,l}(x),$$

$$\Sigma_3 = \sum_{2\leqslant l\leqslant[(k+1)/2]-m} a_{k+1,m,l}\Phi_{k+1,m,l}(x).$$

(Here and henceforth, sums are set to be equal to zero if the set of indices of summation is empty.) Relations (16.18)–(16.21) give

$$\begin{aligned}(-A_1 - \mathrm{i}A_2)\big(\psi(\varrho)R_{k,m}(x)\big) &= \frac{k-2m+1}{k-2m+3}\varphi_2(\varrho)\big(R_{k+1,m}(x) - \Sigma_2\big)\\ &\quad + \frac{k-2m+5}{4(k-2m+3)}\varphi_3(\varrho)\big(R_{k+1,m+1}(x) - \Sigma_3\big)\\ &\quad + \varphi_2(\varrho)\Sigma_4 - \psi(\varrho)(\Sigma_5 + \Sigma_6)\end{aligned} \tag{16.22}$$

with

$$\varphi_2(\varrho) = \varrho^{-k-1}\left(\big(1+\varrho^2\big)\varphi'(\varrho) + \frac{\varphi(\varrho)}{\varrho}\big(-k + (k-2m)\varrho^2\big)\right),$$

$$\varphi_3(\varrho) = \varrho^{-k-1}\left(\big(1+\varrho^2\big)\varphi'(\varrho) + \frac{\varphi(\varrho)}{\varrho}\big(-k + (2m-k-6)\varrho^2\big)\right),$$

$$\Sigma_4 = \sum_{2\leqslant l\leqslant[k/2]-m} a_{k,m,l}\Phi_{k+1,m,l}(x),$$

$$\Sigma_5 = \sum_{1\leqslant l\leqslant[k/2]-m} (k-2m-2l)a_{k,m,l}\Phi_{k+1,m+1,l}(x),$$

$$\Sigma_6 = \sum_{2\leqslant l\leqslant[k/2]-m} 2la_{k,m,l}\Phi_{k+1,m,l}(x).$$

In a similar way,

$$\begin{aligned}&(-A_1 + \mathrm{i}A_2)\big(\psi(\varrho)R_{k,m}(x)\big)\\ &\quad = \frac{4m(k-2m+1)}{(14+2k)(k-2m+3)}\varphi_4(\varrho)\big(R_{k-1,m-1}(x) - \Sigma_7\big)\\ &\qquad + \frac{(k-m+3)(k-2m+5)}{(14+2k)(k-2m+3)}\varphi_5(\varrho)\big(R_{k-1,m}(x) - \Sigma_8\big) + \frac{\varphi_4(\varrho)}{14+2k}(\Sigma_9 + \Sigma_{10})\\ &\qquad - \frac{\varrho^2\psi(\varrho)}{14+2k}(\Sigma_{11} + \Sigma_{12}) + \varphi_2(\varrho)H_{10}(x) + \psi(\varrho)H_{11}(x),\end{aligned} \tag{16.23}$$

where

$$\varphi_4(\varrho) = \varrho^{-k+1}\left(\big(1+\varrho^2\big)\varphi'(\varrho) + \frac{\varphi(\varrho)}{\varrho}\big(k + 14 + (k-2m)\varrho^2\big)\right),$$

$$\varphi_5(\varrho) = \varrho^{-k+1}\left(\big(1+\varrho^2\big)\varphi'(\varrho) + \frac{\varphi(\varrho)}{\varrho}\big(k + 14 + (2m-k-6)\varrho^2\big)\right),$$

$$\Sigma_7 = \sum_{2\leqslant l\leqslant[(k+1)/2]-m} a_{k-1,m-1,l}\Phi_{k-1,m-1,l}(x),$$

$$\Sigma_8 = \sum_{1\leqslant l\leqslant[(k-1)/2]-m} a_{k+1,m+1,l}\Phi_{k-1,m,l}(x),$$

$$\Sigma_9 = \sum_{1\leqslant l\leqslant[k/2]-m} 2(k-2m-2l)a_{k,m,l}\Phi_{k-1,m,l}(x),$$

$$\Sigma_{10} = \sum_{2\leqslant l\leqslant[k/2]-m} 4(m+l)a_{k,m,l}\Phi_{k-1,m-1,l}(x),$$

$$\Sigma_{11} = \sum_{1\leqslant l\leqslant[k/2]-m} (k-2m-2l)(14+2k+4l)a_{k,m,l}\Phi_{k-1,m,l}(x),$$

$$\Sigma_{12} = \sum_{2\leqslant l\leqslant[k/2]-m} 8l(m+l)a_{k,m,l}\Phi_{k-1,m-1,l}(x),$$

and

$$H_{10}(x) = (x_1 - \mathrm{i}x_2)R_{k,m}(x) - \frac{\varrho^2}{14+2k}\left(\frac{\partial R_{k,m}}{\partial x_1} - \mathrm{i}\frac{\partial R_{k,m}}{\partial x_2}\right),$$

$$H_{11}(x) = \sum_{l=0}^{[k/2]-m} 2la_{k,m,l}\left(\frac{\varrho^2}{14+2k}\left(\frac{\partial \Phi_{k,m,l}}{\partial x_1} - \mathrm{i}\frac{\partial \Phi_{k,m,l}}{\partial x_2}\right) - (x_1 - \mathrm{i}x_2)\Phi_{k,m,l}(x)\right).$$

Remark 4.4 and (16.10) imply that $H_{10} \in \mathcal{H}_1^{16,k+1}$, $H_{11} \in \mathcal{H}_1^{16,k+1}$.

Now we are in a position to complete the proof of Proposition 16.4. By (16.8) and Proposition 16.3(i), all the terms of the Fourier expansion (11.9) of the function $A_l(\psi(\varrho)H(\varrho\sigma))$ belong to $C_T^{s-1}(\mathcal{O})$. Using (16.9)–(16.17), (16.22), and (16.23) and taking Remark 4.4 into account, we obtain the required statement. □

Proposition 16.5. *Let $R \in (r(T), \pi/2]$, and let $k \in \mathbb{Z}_+$, $m \in \{0,\dots,M_{\mathcal{X}}(k)\}$, $j \in \{1,\dots,d_{\mathcal{X}}^{k,m}\}$. Then for $f \in \mathcal{D}'_{k,m,j}(B_R)$, the following assertions are equivalent.*

(i) $f \in \mathcal{D}'_T(B_R)$.
(ii) $\mathcal{A}_j^{k,m}(f) \in \mathcal{D}'_{T,\natural}(B_R)$.
(iii) $\mathfrak{A}_{k,m,j}(f) \in \mathcal{D}'_{A(T),\natural}(-R,R)$.

Proof. It suffices to use (11.106), (11.108), Theorem 11.3(i), (viii), and Theorem 11.5(i). □

Proposition 16.6. *Let $r(T) = 0$ and suppose that the set $\mathcal{O}$ is connected and $\mathcal{O} \subset \mathfrak{X}$. Then a distribution $f \in \mathcal{D}'(\mathcal{O})$ belongs to $\mathcal{D}'_T(\mathcal{O})$ if and only if*

$$f^{k,m,j} = \sum_{\lambda\in\mathcal{Z}_T}\sum_{\eta=0}^{n(\lambda,T)} a_{\lambda,\eta,k,m,j}\Phi_{\lambda,\eta,k,m,j} + b_{\lambda,\eta,k,m,j}\Psi_{\lambda,\eta,k,m,j}$$

for all k, m, j, *where* $a_{\lambda,\eta,k,m,j}, b_{\lambda,\eta,k,m,j} \in \mathbb{C}$, *and* $b_{\lambda,\eta,k,m,j} = 0$ *in the case where* $0 \in \mathcal{O}$.

Proof. The argument is quite parallel to the proof of Proposition 15.6 (see (11.42), Remark 11.1, Theorem 11.2(i), and Proposition 11.11). □

16.2 Uniqueness Results

Let $\mathcal{X}$ be the same as in Sect. 16.1. Here we want to establish uniqueness theorems for the class $\mathcal{D}'_T(\mathcal{O})$, where $T \in \mathcal{E}'_\natural(\mathfrak{X})$, and $\mathcal{O}$ is a ζ domain in $\mathcal{X}$ with $\zeta = r(T)$.

We start with the case $\mathcal{O} = B_R$.

Theorem 16.1. *Let* $\nu, s \in \mathbb{Z}$, $s \geqslant \max\{0, 2[(1-\nu)/2]\}$. *Take a distribution* $T \in (\mathcal{D}'_\natural \cap \mathcal{M}^\nu)(\mathcal{X})$ *such that* $0 < r(T) < \pi/2$. *Assume that* $R > r(T)$, $f \in \mathcal{D}'_T(B_R)$, *and*

$$f = 0 \quad \textit{in } B_{r(T)}. \tag{16.24}$$

Then the following are true.

(i) *If* $f \in L_s^{1,\mathrm{loc}}(B_R)$, *then* $f^{k,m,j} = 0$ *in* B_R *for all* $0 \leqslant k \leqslant s+\nu+1$, $m \in \{0, \dots, M_\mathcal{X}(k)\}$, *and* $j \in \{1, \dots, d_\mathcal{X}^{k,m}\}$.
(ii) *If* $f \in C^s(B_R)$, *then* $f^{k,m,j} = 0$ *in* B_R *for all* $0 \leqslant k \leqslant s+\nu+2$, $m \in \{0, \dots, M_\mathcal{X}(k)\}$, *and* $j \in \{1, \dots, d_\mathcal{X}^{k,m}\}$.

Several remarks are in order here. The classes $L_s^{1,\mathrm{loc}}$ and $\mathcal{M}^\nu$ were introduced in Sect. 1.2. Every distribution $T \in \mathcal{E}'_\natural(\mathfrak{X})$ belongs to $\mathcal{M}^\nu(\mathcal{X})$ for some $\nu \in \mathbb{Z}$ (see Theorem 11.2 and Propositions 11.11–11.13). Next, the radius $r(T)$ in condition (16.24) cannot be decreased (see Theorem 16.4 below). Finally, the dependence between the order of smoothness of the distribution f and the set of zero coefficients in its Fourier expansion (11.9) is also precise (see Remark 16.1 below).

To prove Theorem 16.1 we require two lemmas.

Lemma 16.1. *Let* $T \in \mathcal{M}_\natural^0(\mathcal{X})$ *with* $0 < r(T) < \pi/2$. *Suppose that* $R > r(T)$, $f \in (\mathcal{D}'_T \cap L_\natural^{1,\mathrm{loc}})(B_R)$, *and* $f = 0$ *in* $B_{r(T)}$. *Then* $f = 0$ *in* B_R.

Proof. It suffices to consider the case where $R \leqslant \pi/2$. Owing to Theorem 11.2 and Propositions 11.11–11.13, there exists a function $\psi \in (\mathcal{E}'_\natural \cap C^3)(\mathfrak{X})$ such that $r(\psi) = r(T)$ and $T = P(L)\psi$ for some polynomial P. In view of (11.92) and Propositions 11.13(i) and 6.14,

$$\Lambda(\psi) \in \big(\mathcal{E}'_\natural \cap C\big)(-\pi/2, \pi/2).$$

Next, without loss of generality we can assume $f \in C_T^q(B_R)$, where $q = 2\deg P + 2\alpha_\mathcal{X} + 4$ (see the beginning of the proof of Lemma 15.2). Set

$$F = \mathfrak{A}_{0,0,1}\big(P(L)f\big).$$

Then by Theorem 11.3(i), (ii), (vii), $F \in C_\natural(-R, R)$, $F = 0$ on $(-r(\psi), r(\psi))$, and

$$F * \Lambda(\psi) = 0$$

on $(r(\psi) - R, R - r(\psi))$. Now using Theorems 13.1(i) and 11.3(i) and taking into account that $P(L)$ is an elliptical operator, we arrive at the desired assertion. □

Lemma 16.2. *Let $T \in \mathcal{M}^1_\natural(\mathcal{X})$ with $0 < r(T) < \pi/2$. Suppose that $R \in (r(T), \pi/2]$ and let $f \in C_T(B_R)$ possess the following properties:*

(1) *f has the form $f(p) = \varphi(\varrho)Y(\sigma)$, where $Y \in \mathcal{H}^{k,m}_{\mathcal{X}}$ for some $k \in \{0, 1, 2, 3\}$ and $m \in \{0, \dots, M_{\mathcal{X}}(k)\}$;*
(2) *$f = 0$ in $B_{r(T)}$.*

Then $f = 0$ in B_R.

Proof. If $k = 0$, then the assertion of Lemma 16.2 is an immediate consequence of Lemma 16.1. Assume that $k \geqslant 1$. For $\varkappa \in \{1, 2, 3\}$ and $\mu \in \{0, \dots, M_{\mathcal{X}}(\varkappa)\}$, we set

$$U_{\varkappa,\mu}(p) = u_{\varkappa,\mu}(\varrho), \quad p \in B_R,$$

where

$$u_{1,0}(\varrho) = \int_0^\varrho \frac{\varphi(\xi)}{1+\xi^2}\,\mathrm{d}\xi, \tag{16.25}$$

$$u_{2,0}(\varrho) = \int_0^\varrho \frac{\eta}{(1+\eta^2)^{\mathcal{N}_{\mathcal{X}}(2)}} \int_0^\eta \frac{(1+\xi^2)^{\mathcal{N}_{\mathcal{X}}(2)-2}\varphi(\xi)}{\xi}\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{16.26}$$

$$u_{2,1}(\varrho) = \int_0^\varrho \eta\big(1+\eta^2\big)^{\beta_{\mathcal{X}}-1} \int_0^\eta \frac{\varphi(\xi)}{\xi(1+\xi^2)^{\beta_{\mathcal{X}}+1}}\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{16.27}$$

$$\begin{aligned} u_{3,0}(\varrho) = &\int_0^\varrho \frac{\zeta}{(1+\zeta^2)^{\mathcal{N}_{\mathcal{X}}(2)}} \int_0^\zeta \frac{\eta}{(1+\eta^2)^{\mathcal{N}_{\mathcal{X}}(3)-\mathcal{N}_{\mathcal{X}}(2)+1}} \\ &\times \int_0^\eta \frac{(1+\xi^2)^{\mathcal{N}_{\mathcal{X}}(3)-2}\varphi(\xi)}{\xi^2}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta, \end{aligned} \tag{16.28}$$

$$u_{3,1}(\varrho) = \int_0^\varrho \frac{\zeta}{(1+\zeta^2)^2} \int_0^\zeta \eta\big(1+\eta^2\big)^{\beta_{\mathcal{X}}} \int_0^\eta \frac{\varphi(\xi)}{\xi^2(1+\xi^2)^{\beta_{\mathcal{X}}+1}}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta. \tag{16.29}$$

Now define

$$V_{\varkappa,\mu}(p) = (U_{\varkappa,\mu} \times T)(p), \quad p \in B_{R-r(T)}. \tag{16.30}$$

Clearly, $V_{\varkappa,\mu} \in C^\varkappa_\natural(B_{R-r(T)})$. We claim that

$$V_{k,m}(p) = 0, \quad p \in B_{R-r(T)}. \tag{16.31}$$

For each of the spaces $\mathcal{X}$, we have the following relations (see the proof of Proposition 16.2):

- $\mathcal{X} = \overline{\mathbb{R}^n}$ or $\mathcal{X} = \mathbb{P}^n_{\mathbb{R}}$:

$$(-A_1 - \mathrm{i}A_2)^\varkappa V_{\varkappa,0} = \big((A_1 + \mathrm{i}A_2)^\varkappa U_{\varkappa,0}\big) \times T; \tag{16.32}$$

- $\mathcal{X} = \mathbb{P}^n_{\mathbb{C}}$:

$$A_2^{\varkappa} V_{\varkappa,0} = \left(A_2^{\varkappa} U_{\varkappa,0}\right) \times T, \tag{16.33}$$
$$A_2 A_{n+1} V_{2,1} = (A_2 A_{n+1} U_{2,1}) \times T, \tag{16.34}$$
$$A_{n+1} A_2^2 V_{3,1} = \left(A_{n+1} A_2^2 U_{3,1}\right) \times T; \tag{16.35}$$

- $\mathcal{X} = \mathbb{P}^n_{\mathbb{Q}}$:

$$A_1^{\varkappa} V_{\varkappa,0} = \left(A_1^{\varkappa} U_{\varkappa,0}\right) \times T, \tag{16.36}$$
$$(A_{3n+2} A_1 - A_2 A_{3n+1}) V_{2,1} = \left((A_{3n+2} A_1 - A_2 A_{3n+1}) U_{2,1}\right) \times T, \tag{16.37}$$
$$\left(A_{3n+2} A_1^2 - A_2 A_{3n+1} A_1\right) V_{3,1} = \left(\left(A_{3n+2} A_1^2 - A_2 A_{3n+1} A_1\right) U_{3,1}\right) \times T; \tag{16.38}$$

- $\mathcal{X} = \mathbb{P}^2_{\mathbb{C}a}$:
$$(A_1 + \mathrm{i}A_2)^{\varkappa} V_{\varkappa,\mu} = \left((A_1 + \mathrm{i}A_2)^{\varkappa} U_{\varkappa,\mu}\right) \times T. \tag{16.39}$$

Put
$$v_{\varkappa,\mu}(\varrho) = V_{\varkappa,\mu}(p)|_{|p|=\varrho}.$$

By means of (11.16), (16.1)–(16.6), (16.14), (16.15), (16.22), and (16.25)–(16.29) we can write (16.32)–(16.39) in the following form:

- $\mathcal{X} = \overline{\mathbb{R}^n}$ or $\mathcal{X} = \mathbb{P}^n_{\mathbb{R}}$:

$$\left(D\left(-\varkappa + 1, 1 - \mathcal{N}_{\mathcal{X}}(\varkappa)\right) D\left(-\varkappa + 2, 1 - \mathcal{N}_{\mathcal{X}}(\varkappa - 1)\right) \cdots D(0,0) v_{\varkappa,0}\right)(\varrho)$$
$$\times \frac{(x_1 + \mathrm{i}x_2)^{\varkappa}}{\varrho^{\varkappa}} = \left(\varphi(\varrho) \frac{(x_1 + \mathrm{i}x_2)^{\varkappa}}{\varrho^{\varkappa}}\right) \times T;$$

- $\mathcal{X} = \mathbb{P}^n_{\mathbb{C}}$:

$$\left(D(-\varkappa + 1, -\varkappa + 1) D(-\varkappa + 2, -\varkappa + 2) \cdots D(0,0) v_{\varkappa,0}\right)(\varrho) \frac{\overline{z}_2^{\varkappa}}{\varrho^{\varkappa}}$$
$$= \left(\varphi(\varrho) \frac{\overline{z}_2^{\varkappa}}{\varrho^{\varkappa}}\right) \times T,$$
$$\left(D(-1,0) D(0,0) v_{2,1}\right)(\varrho) \frac{z_1 \overline{z}_2}{\varrho^2} = \left(\varphi(\varrho) \frac{z_1 \overline{z}_2}{\varrho^2}\right) \times T,$$
$$\left(D(-2,0) D(-1,-1) D(0,0) v_{3,1}\right)(\varrho) \frac{z_1 \overline{z}_2^2}{\varrho^3} = \left(\varphi(\varrho) \frac{z_1 \overline{z}_2^2}{\varrho^3}\right) \times T;$$

- $\mathcal{X} = \mathbb{P}^n_{\mathbb{Q}}$:

$$\left(D(-\varkappa + 1, -\varkappa + 1) D(-\varkappa + 2, -\varkappa + 2) \cdots D(0,0) v_{\varkappa,0}\right)(\varrho) P_{\varkappa,0}(\sigma)$$
$$= \left(\varphi(\varrho) P_{\varkappa,0}(\sigma)\right) \times T,$$
$$\left(D(-1,1) D(0,0) v_{2,1}\right)(\varrho) P_{2,1}(\sigma) = \left(\varphi(\varrho) P_{2,1}(\sigma)\right) \times T,$$
$$\left(D(-2,1) D(-1,-1) D(0,0) v_{3,1}\right)(\varrho) P_{3,1}(\sigma) = \left(\varphi(\varrho) P_{3,1}(\sigma)\right) \times T;$$

- $\mathcal{X} = \mathbb{P}^2_{\mathbb{C}a}$:

$$\big(D(0,0)v_{1,0}\big)(\varrho)R_{1,0}(\sigma) = \big(\varphi(\varrho)R_{1,0}(\sigma)\big) \times T,$$

$$\big(D(-1,-1)D(0,0)v_{2,0}\big)(\varrho)R_{2,0}(\sigma) + \frac{3}{4}\big(D(-1,3)D(0,0)v_{2,0}\big)(\varrho)R_{2,1}(\sigma)$$
$$= \left(\varphi(\varrho)R_{2,0}(\sigma) + \frac{3}{4}\big(D(-1,3)D(0,0)u_{2,0}\big)(\varrho)R_{2,1}(\sigma)\right) \times T,$$

$$\big(D(-2,2)D(-1,-1)D(0,0)v_{3,0}\big)(\varrho)R_{3,0}(\sigma)$$
$$+ \big(D(-2,3)D(-1,-1)D(0,0)v_{3,0}\big)(\varrho)R_{3,1}(\sigma)$$
$$= \big(\varphi(\varrho)R_{3,0}(\sigma) + \big(D(-2,3)D(-1,-1)D(0,0)u_{3,0}\big)(\varrho)R_{3,1}(\sigma)\big) \times T,$$

$$\big(D(-1,-1)D(0,0)v_{2,1}\big)(\varrho)R_{2,0}(\sigma) + \frac{3}{4}\big(D(-1,3)D(0,0)v_{2,1}\big)(\varrho)R_{2,1}(\sigma)$$
$$= \left(\big(D(-1,-1)D(0,0)u_{2,1}\big)(\varrho)R_{2,0}(\sigma) + \frac{3}{4}\varphi(\varrho)R_{2,1}(\sigma)\right) \times T,$$

$$\big(D(-2,-2)D(-1,-1)D(0,0)v_{3,1}\big)(\varrho)R_{3,0}(\sigma)$$
$$+ \big(D(-2,3)D(-1,-1)D(0,0)v_{3,1}\big)(\varrho)R_{3,1}(\sigma)$$
$$= \big(\big(D(-2,-2)D(-1,-1)D(0,0)u_{3,1}\big)(\varrho)R_{3,0}(\sigma) + \varphi(\varrho)R_{3,1}(\sigma)\big) \times T.$$

Since $\varphi(\varrho)Y(\sigma) \in \mathcal{D}'_T(B_R)$, these relations yield

$$D\big(-\varkappa+1, 1-\mathcal{N}_{\mathcal{X}}(\varkappa)\big)D\big(-\varkappa+2, 1-\mathcal{N}_{\mathcal{X}}(\varkappa-1)\big)\cdots D(0,0)v_{k,m} = 0$$

if $k = \varkappa$ and $m = 0$,

$$D(-1,\beta_{\mathcal{X}})D(0,0)v_{k,m} = 0$$

if $k = 2$ and $m = 1$, and

$$D(-2,\beta_{\mathcal{X}})D(-1,-1)D(0,0)v_{k,m} = 0$$

if $k = 3$ and $m = 1$ (see Proposition 16.3(i)). Hence,

$$v_{k,m}(\varrho) = c_1, \quad k=1,\ m=0, \tag{16.40}$$

$$v_{k,m}(\varrho) = \frac{c_1}{(1+\varrho^2)^{\mathcal{N}_X(2)-1}} + c_2, \quad k=2,\ m=0, \tag{16.41}$$

$$v_{k,m}(\varrho) = \frac{c_1}{(1+\varrho^2)^{\mathcal{N}_X(3)-1}} + \frac{c_2}{(1+\varrho^2)^{\mathcal{N}_X(2)-1}} + c_3, \quad k=3,\ m=0, \tag{16.42}$$

$$v_{k,m}(\varrho) = \begin{cases} c_1\big(1+\varrho^2\big)^{\beta_{\mathcal{X}}} + c_2, & k=2,\ m=1,\ \beta_{\mathcal{X}} \neq 0, \\ c_1 \log\big(1+\varrho^2\big) + c_2, & k=2,\ m=1,\ \beta_{\mathcal{X}} = 0, \end{cases} \tag{16.43}$$

$$v_{k,m}(\varrho) = \begin{cases} c_1\big(1+\varrho^2\big)^{\beta_{\mathcal{X}}} + c_2\big(1+\varrho^2\big)^{-1} + c_3, & k=3,\ m=1,\ \beta_{\mathcal{X}} \neq 0, \\ c_1 \log\big(1+\varrho^2\big) + c_2\big(1+\varrho^2\big)^{-1} + c_3, & k=3,\ m=1,\ \beta_{\mathcal{X}} = 0, \end{cases} \tag{16.44}$$

where c_1, c_2, c_3 are complex constants. Next, apply L^j, $j \in \{0, \dots, k-1\}$, to (16.30) with $\varkappa = k$ and $\mu = m$. Since $U_{k,m} \in C^k(B_R)$, $U_{k,m} = 0$ in $B_{r(T)}$, and $T \in \mathcal{M}_\natural^1(\mathcal{X})$, we get

$$\big(L^j V_{k,m}\big)(0) = 0, \quad j \in \{0, \dots, k-1\}. \tag{16.45}$$

If $k = 1$, then (16.45) and (16.40) give $V_{1,0} = 0$. Suppose that $k \geqslant 2$. Using the equalities

$$\begin{aligned} L\big((1+\varrho^2)^{-c}\big) &= -4c(\rho_{\mathcal{X}} + c)\big(1+\varrho^2\big)^{-c} \\ &\quad + 4c(\beta_{\mathcal{X}} + c)\big(1+\varrho^2\big)^{-c+1}, \quad c \in \mathbb{R}^1, \\ L\big(\log\big(1+\varrho^2\big)\big) &= 4\rho_{\mathcal{X}} - 4\beta_{\mathcal{X}}\big(1+\varrho^2\big) \end{aligned}$$

(see (11.18)), we conclude from (16.41)–(16.44) that $v_{k,m}(\varrho) = 0$ for all $\varrho \in [0, \tan(R - r(T)))$. Thereby (16.31) is established. In view of Lemma 16.1 and (16.31), $U_{k,m} = 0$ in B_R. But then $f = 0$ in B_R, as contended. □

Corollary 16.1. *Let $T \in \mathcal{M}_\natural^1(\mathcal{X})$ with $r(T) \in (0, \pi/2)$. Assume that $R \in (r(T), \pi/2]$, $f \in C_T^s(B_R)$, and $f = 0$ in $B_{r(T)}$. Then $f^{k,m,j} = 0$ in B_R for all $0 \leqslant k \leqslant s+3$, $0 \leqslant m \leqslant M_{\mathcal{X}}(k)$, and $1 \leqslant j \leqslant d_{\mathcal{X}}^{k,m}$.*

Proof. In the case $s = 0$ this follows from Lemma 16.2 (see Proposition 16.3(i) and (11.10)). Suppose that the statement of Corollary 16.1 is valid for $0 \leqslant s \leqslant l-1$ with some $l \in \mathbb{N}$. We shall prove it for $s = l$. Let $f \in C_T^l(B_R)$ and $f = 0$ in $B_{r(T)}$. Take $k \in \{0, \dots, l+3\}$, $m \in \{0, \dots, M_{\mathcal{X}}(k)\}$, and $j \in \{1, \dots, d_{\mathcal{X}}^{k,m}\}$. Owing to Proposition 16.3(i) and (11.10), $f^{k,m,j} \in C_T^l(B_R)$ and $f^{k,m,j} = 0$ in $B_{r(T)}$. By the induction assumption, $f^{k,m,j} = 0$ in B_R for $k \leqslant l+2$. Next, if $k = l+3$ and $m \leqslant M_{\mathcal{X}}(l+2)$, then in view of Proposition 16.4(iii),

$$\big(D\big(l+3+2\alpha_{\mathcal{X}}, \mathcal{N}_{\mathcal{X}}(l+3) + \gamma_{\mathcal{X}} - m\big) f_{l+3,m,j}\big)(\varrho) Y_1^{l+2,m}(\sigma) \in C_T^{l-1}(B_R).$$

In addition, we have

$$\big(D\big(l+3+2\alpha_{\mathcal{X}}, \mathcal{N}_{\mathcal{X}}(l+3) + \gamma_{\mathcal{X}} - m\big) f_{l+3,m,j}\big)(\varrho) = 0, \quad 0 \leqslant \varrho \leqslant \tan r(T).$$

Again by the induction hypothesis we infer that $f^{l+3,m,j} = 0$ in B_R. Analogously, thanks to Proposition 16.4(iv), $f^{l+3,m,j} = 0$ in B_R for $M_{\mathcal{X}}(l+2) < m \leqslant M_{\mathcal{X}}(l+3)$, and the proof is finished. □

Proof of Theorem 16.1. Examining the proof of Lemma 16.2 and Corollary 16.1, we conclude that the assertion of Theorem 16.1 holds for $\nu = 0, 1$. Using Theorem 11.2 and Proposition 16.6 and repeating the arguments in the proof of Lemma 15.3 and Theorem 15.4, we obtain the required result. □

Corollary 16.2. *Let $T \in \mathcal{E}_\natural'(\mathfrak{X})$ with $r(T) > 0$. Fix an integer $s \geqslant \max\{0, 2[(2\alpha_{\mathcal{X}} + 5 - [-c_2])/2]\}$, where $c_2 \in \mathbb{R}^1$ is the constant from the estimate*

$$\big|\widetilde{T}(\lambda)\big| \leqslant c_1(1+|\lambda|)^{c_2} e^{r(T)|\operatorname{Im}\lambda|}, \quad \lambda \in \mathbb{C}. \tag{16.46}$$

Assume that $R > r(T)$, $f \in C_T^s(B_R)$, *and* $f = 0$ *in* $B_{r(T)}$. *Then* $f^{k,m,j} = 0$ *in* B_R *for all* $0 \leqslant k \leqslant s - 2\alpha_{\mathcal{X}} - 2 + [-c_2]$, $0 \leqslant m \leqslant M_{\mathcal{X}}(k)$, *and* $1 \leqslant j \leqslant d_{\mathcal{X}}^{k,m}$.

Proof. We see from (16.46) and Propositions 11.11–11.13 that $T \in \mathcal{M}^{\nu}(\mathcal{X})$ with $\nu = -2\alpha_{\mathcal{X}} - 4 + [-c_2]$. Therefore, Corollary 16.2 is an immediate consequence of Theorem 16.1(ii). □

We establish now analogues of Theorems 15.1 and 15.5.

Theorem 16.2. *Let* $T \in \mathcal{E}'_{\natural}(\mathfrak{X})$ *with* $r(T) > 0$, *and let* $\mathcal{O}$ *be a* ζ *domain in* $\mathcal{X}$ *with* $\zeta = r(T)$. *Suppose that* $f \in \mathcal{D}'_T(\mathcal{O})$ *and* $f = 0$ *in some open ball* B *of radius* $r(T)$ *such that* Cl $B \subset \mathcal{O}$. *Then the following statements hold.*

(i) *If* $f = 0$ *in some open ball with the radius exceeding* $r(T)$, *then* $f = 0$ *in* $\mathcal{O}$.
(ii) *If* $f \in C_T^{\infty}(\mathcal{O})$, *then* $f = 0$ *in* $\mathcal{O}$.
(iii) *If* $T = T_1 + T_2$, *where* $T_1 \in \mathcal{D}_{\natural}(\mathfrak{X})$, $T_2 \in \mathcal{E}'_{\natural}(\mathfrak{X})$, *and* $r(T_2) < r(T)$, *then* $f = 0$ *in* $\mathcal{O}$.

Proof. These results can be obtained in the same way as Theorem 14.2 with attention to Corollary 16.2 and Theorem 11.3. □

We note that assumptions in Theorem 16.2 cannot be relaxed either (see Theorem 16.4 below).

Theorem 16.3. *Let* $T \in \mathcal{E}'_{\natural}(\mathfrak{X})$ *with* $r(T) > 0$, *and let* $\mathcal{O}$ *be a* ζ *domain in* $\mathcal{X}$ *with* $\zeta = r(T)$ *containing the ball* $\mathring{B}_{r(T)}$. *Suppose that* $f \in \mathcal{D}'_T(\mathcal{O})$, $f = 0$ *in* $B_{r(T)}$, *and* $f \in C^{\infty}(\mathcal{O}_1)$ *for some open subset* $\mathcal{O}_1$ *of* $\mathcal{O}$ *such that* $S_{r(T)}^{+} \subset \mathcal{O}_1$, *where* $S_{r(T)}^{+} = \{p = (p_1, \dots, p_{a_{\mathcal{X}}}) \in S_{r(T)} : \ p_1 \geqslant 0\}$. *Then* $f = 0$ *in* $\mathcal{O}$.

Proof. We can suppose that $T \in (\mathcal{E}'_{\natural} \cap C^{2\alpha_{\mathcal{X}}+1})(\mathfrak{X})$ and $\mathcal{O} = B_R$, where $r(T) < R \leqslant \pi/2$ (see Theorem 11.2 and the proof of Theorem 15.5). Let $\lambda \in \mathcal{Z}_T$ and $\eta \in \{0, \dots, n(\lambda, T)\}$. Using Theorem 11.2, we define $T^{\lambda,\eta} \in \mathcal{E}'_{\natural}(\mathfrak{X})$ by

$$\widetilde{T^{\lambda,\eta}}(z) = \widetilde{T}(z)\big(z^2 - \lambda^2\big)^{-\eta-1}, \quad z \in \mathbb{C}.$$

In view of Proposition 11.13, $T^{\lambda,\eta} \in C(\mathfrak{X})$. In addition, we have from Proposition 11.11 the following relations: (1) $(L+\lambda^2-\rho_{\mathcal{X}}^2)T^{\lambda,\eta+1} = -T^{\lambda,\eta}$ ($n(\lambda, T) \geqslant 1$, $\eta \in \{0, \dots, n(\lambda, T) - 1\}$); (2) $(L + \lambda^2 - \rho_{\mathcal{X}}^2)T^{\lambda,0} = -T$. Hence,

$$\big(L + \lambda^2 - \rho_{\mathcal{X}}^2\big)^{\eta+1} F = 0 \quad \text{in } B_{R-r(T)}, \tag{16.47}$$

where $F = f \times T^{\lambda,\eta}$. Next, let $\varrho = \tan r(T)$. For $t \in (0, \varrho^{-1})$, we put $a_{t,\mathcal{X}} = -\varphi_{t,1}$ if $\mathcal{X} \neq \mathbb{P}_{\mathbb{C}a}^2$ and $a_{t,\mathcal{X}} = \varphi_t$ if $\mathcal{X} = \mathbb{P}_{\mathbb{C}a}^2$, where the mappings $\varphi_{t,1}$ and φ_t are defined in Sect. 16.1. For each of the spaces $\mathcal{X}$, the ball $a_{t,\mathcal{X}} B_{r(T)}$ has the following form:

- $\mathcal{X} = \overline{\mathbb{R}^n}$:

$$a_{t,\mathcal{X}} B_{r(T)} = \left\{ x \in \mathbb{R}^n : \left(x_1 - \frac{t(1+\varrho^2)}{1-\varrho^2 t^2} \right)^2 + \sum_{l=2}^{n} x_l^2 < \left(\frac{\varrho(1+t^2)}{1-\varrho^2 t^2} \right)^2 \right\};$$

- $\mathcal{X} = \mathbb{P}^n_{\mathbb{R}}$:

$$a_{t,\mathcal{X}} B_{r(T)} = \left\{ x \in \mathbb{R}^n : \left(x_1 - \frac{t(1+\varrho^2)}{1-\varrho^2 t^2} \right)^2 + \frac{1+t^2}{1-\varrho^2 t^2} \sum_{l=2}^{n} x_l^2 < \left(\frac{\varrho(1+t^2)}{1-\varrho^2 t^2} \right)^2 \right\};$$

- $\mathcal{X} = \mathbb{P}^n_{\mathbb{C}}$:

$$a_{t,\mathcal{X}} B_{r(T)} = \left\{ z \in \mathbb{C}^n : \left(\operatorname{Re} z_1 - \frac{t(1+\varrho^2)}{1-\varrho^2 t^2} \right)^2 + (\operatorname{Im} z_1)^2 \right. \\ \left. + \frac{1+t^2}{1-\varrho^2 t^2} \sum_{l=2}^{n} |z_l|^2 < \left(\frac{\varrho(1+t^2)}{1-\varrho^2 t^2} \right)^2 \right\};$$

- $\mathcal{X} = \mathbb{P}^n_{\mathbb{Q}}$:

$$a_{t,\mathcal{X}} B_{r(T)} = \left\{ z \in \mathbb{C}^{2n} : \left(\operatorname{Re} z_1 - \frac{t(1+\varrho^2)}{1-\varrho^2 t^2} \right)^2 + (\operatorname{Im} z_1)^2 + |z_{n+1}|^2 \right. \\ \left. + \frac{1+t^2}{1-\varrho^2 t^2} \sum_{l=2}^{n} \left(|z_l|^2 + |z_{n+l}|^2 \right) < \left(\frac{\varrho(1+t^2)}{1-\varrho^2 t^2} \right)^2 \right\};$$

- $\mathcal{X} = \mathbb{P}^2_{\mathbb{C}a}$:

$$a_{t,\mathcal{X}} B_{r(T)} = \left\{ x \in \mathbb{R}^{16} : \left(x_1 - \frac{t(1+\varrho^2)}{1-\varrho^2 t^2} \right)^2 + \sum_{l=1}^{7} x_{2l+1}^2 \right. \\ \left. + \frac{1+t^2}{1-\varrho^2 t^2} \sum_{l=1}^{n} x_{2l}^2 < \left(\frac{\varrho(1+t^2)}{1-\varrho^2 t^2} \right)^2 \right\}.$$

Therefore, there are $\delta > 0$ and a neighborhood $\mathcal{U}$ of the unity in $K_{\mathcal{X}}$ such that $k a_{t,\mathcal{X}} \mathring{B}_{r(T)} \subset B_{r(T)} \cup \mathcal{O}_1$ for all $t \in (0, \delta)$ and $k \in \mathcal{U}$. Then (16.47) and the proof of Theorem 15.5 show that $F = 0$ in $B_{R-r(T)}$. Now Theorem 11.3 and the argument of Theorem 9.9 imply $f = 0$. □

16.3 Series Development Theorems

Here we discuss analogues of the results of Sects. 14.3–14.5 for the space $\mathcal{X}$. Let $T \in \text{Inv}_+(\mathcal{X})$. For all $k \in \mathbb{Z}_+$, $m \in \{0, \ldots, M_{\mathcal{X}}(k)\}$, and $j \in \{1, \ldots, d_{\mathcal{X}}^{k,m}\}$, we put

$$\zeta_{T,k,m,j} = -\mathfrak{A}_{k,m,j}^{-1}\big(\zeta'_{\Lambda(T)}\big|_{(-\pi/2,\pi/2)}\big)$$

(see (11.92)).

Theorem 16.4.

(i) $\zeta_{T,k,m,j} \in (\mathcal{D}'_T \cap \mathcal{D}'_{k,m,j})(\mathfrak{X})$ *and*

$$\mathcal{A}_j^{k,m}(\zeta_{T,k,m,j}) = \zeta_{T,0,0,1}.$$

(ii) *If* $r(T) > 0$, *then* $\zeta_{T,k,m,j} = 0$ *in* $B_{r(T)}$ *and* $S_{r(T)} \subset \text{supp}\ \zeta_{T,k,m,j}$.

(iii) *If* $T \in \mathfrak{M}(\mathcal{X})$, *then*

$$\zeta_{T,k,m,j} = \sum_{\eta=0}^{n(0,T)} \frac{a_{2(n(0,T)-\eta)}^{0,0}(\widetilde{T})}{(2\eta)!}\Phi_{0,\eta,k,m,j} + 2\sum_{\lambda\in\mathcal{Z}_T\setminus\{0\}}\ \sum_{\eta=0}^{n(\lambda,T)} a_{n(\lambda,T)}^{\lambda,\eta}(\widetilde{T})(\eta\Phi_{\lambda,\eta-1,k,m,j} + \lambda\Phi_{\lambda,\eta,k,m,j}),$$

where the series converges in $\mathcal{D}'(\mathfrak{X})$, *and the first sum is set to be equal to zero if* $0 \notin \mathcal{Z}_T$.

(iv) *If* $T \in \mathfrak{M}(\mathcal{X})$ *and* $s \in \mathbb{Z}_+$, *then there is a constant* $\sigma = \sigma(s, T) > 0$ *such that* $\zeta_{T,k,m,j} \in C_T^s(\mathfrak{X})$ *for* $k > \sigma$.

Proof. We can essentially imitate the same arguments as in the proof of Theorem 14.7. However, we now use Proposition 16.1, (11.54), and Theorem 11.3. □

Remark 16.1. Theorem 16.4 and the proof of Theorem 14.9 show that the assertions of Theorem 16.1 cannot be reinforced in the general case.

Theorem 16.5. *For* $T \in \text{Inv}_+(\mathcal{X})$ *such that* $r(T) > 0$, *the following are true.*

(i) *If* $0 < r \leqslant r(T) < R \leqslant \pi/2$ *and* $f \in (\mathcal{D}'_T \cap \mathcal{D}'_{k,m,j})(B_R)$, *then* $f = 0$ *in* B_r *if and only if*

$$f = \mathfrak{A}_{k,m,j}^{-1}\big((\zeta'_{\Lambda(T)} * u)\big|_{(-R,R)}\big) \tag{16.48}$$

for some $u \in \mathcal{E}'_\natural(\mathbb{R}^1)$ *with* supp $u \subset [r - r(T), r(T) - r]$.

(ii) *If* $R \in (r(T), \pi/2]$ *and* $f \in (C_T^\infty \cap \mathcal{D}'_{k,m,j})(B_R)$, *then* $(Df)(0) = 0$ *for each differential operator* D *if and only if* (16.48) *holds for some* $u \in \mathcal{D}_\natural(\mathbb{R}^1)$ *with* supp $u \subset [-r(T), r(T)]$.

The proof is similar to the proof of Theorem 14.10, only instead of Theorem 9.3, one applies Theorem 11.3.

Remark 16.2. In view of Theorem 11.5(viii), for $a = \pi/2 - r(T) + r$, we have

$$\mathfrak{A}^{-1}_{k,m,j}\big((\zeta'_{\Lambda(T)} * u)\big|_{(-a,a)}\big) = \zeta_{T,k,m,j} \times U,$$

where $U = -\Lambda^{-1}(u)$ and supp $U \subset \mathring{B}_{r(T)-r}$. So, (16.48) is an analogue of equality (15.25).

Now we present descriptions for solutions of convolution equations on domains in $\mathcal{X}$.

Theorem 16.6.

(i) *Let* $T \in \mathrm{Inv}_+(\mathcal{X})$ *with* $r(T) > 0$ *and* $R \in (r(T), \pi/2]$. *Then a distribution* $f \in \mathcal{D}'(B_R)$ *belongs to* $\mathcal{D}'_T(B_R)$ *if and only if for all* $k \in \mathbb{Z}_+$, $m \in \{0, \dots, M_{\mathcal{X}}(k)\}$, *and* $j \in \{1, \dots, d^{k,m}_{\mathcal{X}}\}$, *there is* $v_{k,m,j} \in \mathcal{E}'_{\natural}(\mathbb{R}^1)$ *such that* supp $v_{k,m,j} \subset [-r(T), r(T)]$ *and*

$$f^{k,m,j} = \mathfrak{A}^{-1}_{k,m,j}\big((\zeta'_{\Lambda(T)} * v_{k,m,j})\big|_{(-R,R)}\big).$$

(ii) *Let* $T \in \mathfrak{M}(\mathcal{X})$, $R \in (r(T), \pi/2]$, *and* $f \in \mathcal{D}'(B_R)$. *Then in order that* $f \in \mathcal{D}'_T(B_R)$, *it is necessary and sufficient that for all* k, m, j,

$$f^{k,m,j} = \sum_{\lambda \in \mathcal{Z}_T} \sum_{\eta=0}^{n(\lambda,T)} \alpha_{\lambda,\eta,k,m,j}\, \Phi_{\lambda,\eta,k,m,j},$$

where $\alpha_{\lambda,\eta,k,m,j} \in \mathbb{C}$, *and the series converges in* $\mathcal{D}'(B_R)$.

Proof. Similar results in the one-dimensional case were obtained in Theorems 13.13 and 13.14. Once Theorem 11.3 has been established, we arrive at the desired conclusion reasoning as in the proof of Theorem 14.16. □

Theorem 16.7. *Let* $T \in \mathfrak{N}(\mathcal{X})$, $0 \leqslant r < R \leqslant \pi/2$, $R - r > 2r(T)$, *and let* $f \in \mathcal{D}'(B_{r,R})$. *For* f *to belong to* $\mathcal{D}'_T(B_{r,R})$, *it is necessary and sufficient that for all* k, m, j,

$$f^{k,m,j} = \sum_{\lambda \in \mathcal{Z}_T} \sum_{\eta=0}^{n(\lambda,T)} \alpha_{\lambda,\eta,k,m,j}\, \Phi_{\lambda,\eta,k,m,j} + \beta_{\lambda,\eta,k,m,j}\, \Psi_{\lambda,\eta,k,m,j}, \qquad (16.49)$$

where $\alpha_{\lambda,\eta,k,m,j}, \beta_{\lambda,\eta,k,m,j} \in \mathbb{C}$, *and the series converges in* $\mathcal{D}'(B_{r,R})$.

Proof. Necessity. First, assume that $T \in (\mathfrak{N} \cap \mathfrak{R})(\mathcal{X})$, $f \in \mathcal{D}'_T(B_{r,R})$. Fix $\lambda \in \mathcal{Z}_T$ and $\eta \in \{0, \dots, n(\lambda, T)\}$. By Theorem 11.2 we define $T_{\lambda,\eta} \in \mathcal{E}'_{\natural}(\mathfrak{X})$ according to the rule

$$\widetilde{T_{\lambda,\eta}}(z) = b^{\lambda,\eta}\big(\widetilde{T}, z\big), \quad z \in \mathbb{C},$$

where $b^{\lambda,\eta}(\widetilde{T}, z)$ is given by (6.20). We conclude from (11.72) and Proposition 6.8 that the equalities of Proposition 9.13 hold true if we replace the operator $\Delta + \lambda^2$ by the operator $L + \lambda^2 - \rho_{\mathcal{X}}^2$. This, together with Proposition 16.3(i), gives

$$\left(L + \lambda^2 - \rho_{\mathcal{X}}^2\right)^{n(\lambda,T)+1}\left(f^{k,m,j} \times T_{\lambda,0}\right) = 0.$$

Referring to Proposition 16.6, we get

$$f^{k,m,j} \times T_{\lambda,0} = \sum_{\eta=0}^{n(\lambda,T)} a_{\lambda,\eta,k,m,j}(T, f)\Phi_{\lambda,\eta,k,m,j} + b_{\lambda,\eta,k,m,j}(T, f)\Psi_{\lambda,\eta,k,m,j} \tag{16.50}$$

for some complex constants $a_{\lambda,\eta,k,m,j}(T, f)$ and $b_{\lambda,\eta,k,m,j}(T, f)$. Proposition 11.14 and the argument in the proof of Proposition 9.11(iii) show that

$$\sum_{\lambda \in \mathcal{Z}_T} T_{\lambda,0} = \delta_0 \tag{16.51}$$

in the space $\mathcal{D}'(\mathfrak{X})$. By (16.50) and (16.51),

$$f^{k,m,j} = \sum_{\lambda \in \mathcal{Z}_T} \sum_{\eta=0}^{n(\lambda,T)} a_{\lambda,\eta,k,m,j}(T, f)\Phi_{\lambda,\eta,k,m,j} + b_{\lambda,\eta,k,m,j}(T, f)\, \Psi_{\lambda,\eta,k,m,j} \tag{16.52}$$

in $\mathcal{D}'(B_{r+r(T),R-r(T)})$. Then for some $c > 0$ independent of λ and η,

$$|a_{\lambda,\eta,k,m,j}(T, f)| + |b_{\lambda,\eta,k,m,j}(T, f)| \leqslant (2 + |\lambda|)^c,$$

whence the series on the right-hand side of (16.52) converges in $\mathcal{D}'(B_{r,R})$ (see (11.53), (11.55), (11.42), and [225, Part III, the proof of Lemma 2.7]). It follows that (16.49) is valid in the case under consideration (see Theorem 11.3 and the proof of Theorem 14.11). Now let $T \in \mathfrak{N}(X)$ and $f \in \mathcal{D}'_T(B_{r,R})$. If the degree of a polynomial P is large enough, then $P(L)T \in (\mathfrak{N} \cap \mathfrak{R})(\mathcal{X})$ because of Proposition 8.3. In addition, $f \in \mathcal{D}'_{P(L)T}(B_{r,R})$. By the above we obtain (16.49) in general using (11.72), Proposition 16.1, and Remark 11.1.

Sufficiency. Suppose that $f^{k,m,j}$ can be represented in the form (16.49) for all $k \in \mathbb{Z}_+$, $m \in \{0, \dots, M_{\mathcal{X}}(k)\}$, and $j \in \{1, \dots, d_{\mathcal{X}}^{k,m}\}$. Then $f^{k,m,j} \in \mathcal{D}'_T(B_{r,R})$ in view of Proposition 16.1. It remains only to apply Proposition 16.3(ii). □

To close we note that the technique developed in Chaps. 11 and 14 makes it possible to establish many other results for mean periodic functions on $\mathcal{X}$ analogous to those given in Sects. 14.3–14.5. We leave for the reader to reconsider the formulations and the proofs.

Chapter 17
Mean Periodicity on Phase Space and the Heisenberg Group

In Chaps. 13 and 14 we developed the theory of mean periodicity on the translation groups $\mathbb{R}^n$, $n \geqslant 1$. The most natural generalization of the translation groups are nilpotent groups. The Heisenberg group H^n is a principal model for nilpotent groups, and results obtained for H^n may suggest results that hold more generally for this important class of Lie groups.

In the case of the Heisenberg group it is very hard to study mean periodicity for functions of arbitrary growth. On the other hand, one can obtain interesting results for functions satisfying certain growth conditions. Our point of view here is to consider functions on H^n which are 2π-periodic in the t variable. They arise as functions on the quotient H^n/Γ where $\Gamma = \{(0, 2\pi k) \in H^n : k \in \mathbb{Z}\}$. The group $H^n_{\text{red}} = H^n/\Gamma$ is called the *reduced Heisenberg group*. The group convolution $*$ on H^n_{red} can be transferred to $\mathbb{C}^n$ as a nonstandard convolution. Given $f \in \mathcal{D}'(H^n_{\text{red}})$ and $T \in \mathcal{E}'(H^n_{\text{red}})$, we have

$$(f * T)_k = f_k \star_{-k} T_k, \quad k \in \mathbb{Z},$$

where f_k is the partial Fourier transform of f in the t variable, and $f_k \star_{-k} T_k$ is the $(-k)$-*twisted convolution*,

$$\langle f_k \star_{-k} T_k, \varphi\rangle = \big\langle f_k(z), \big\langle T_k(w), \varphi(z+w)\mathrm{e}^{-\frac{ik}{2}\,\mathrm{Im}\langle z,w\rangle_{\mathbb{C}}}\big\rangle\big\rangle, \quad \varphi \in \mathcal{D}\big(H^n_{\text{red}}\big).$$

Thus, the study of mean periodicity on H^n_{red} is reduced to the case of the phase space $\mathbb{C}^n$.

In Sect. 17.1 we prove some preliminary results concerning mean periodic functions on $\mathbb{C}^n$. Section 17.2 is devoted to phase-space analogues of John's uniqueness theorem and related questions. Finally, in Sect. 17.3 we study the kernel of the operator $f \to f \star T$. In particular, we show that for a broad class of distributions T, any smooth function in the kernel has an expansion in terms of eigenfunctions of the special Hermite operator $\mathfrak{L}$ satisfying the same convolution equation.

V.V. Volchkov, V.V. Volchkov, *Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group,* Springer Monographs in Mathematics,

17.1 Background Material

The present chapter is devoted to a study of twisted mean periodic functions on $\mathbb{C}^n$ in the spirit of the results obtained in Chaps. 13–16. In what follows it is assumed that $n \geqslant 2$. The case $n = 1$ is treated with minor modifications, and we leave it for the reader. Our notation is based on Chap. 12. We shall use it with no further references to Chap. 12.

Let $T \in \mathcal{E}'_\natural(\mathbb{C}^n)$, $T \neq 0$, and let $\mathcal{O}$ be an open subset of $\mathbb{C}^n$ such that the set $\mathcal{O}_T = \{z \in \mathbb{C}^n : z + \mathring{B}_{r(T)} \subset \mathcal{O}\}$ is nonempty. Following Sect. 14.1, we put

$$\mathcal{D}'_T(\mathcal{O}) = \left\{f \in \mathcal{D}'(\mathcal{O}) \colon f \star T = 0 \text{ in } \mathcal{O}_T\right\}, \tag{17.1}$$

$$C^s_T(\mathcal{O}) = \left(\mathcal{D}'_T \cap C^s\right)(\mathcal{O}), s \in \mathbb{Z}_+ \cup \{\infty\}, \qquad C_T(\mathcal{O}) = C^0_T(\mathcal{O}), \tag{17.2}$$

$$\mathrm{RA}_T(\mathcal{O}) = \left(\mathcal{D}'_T \cap \mathrm{RA}\right)(\mathcal{O}). \tag{17.3}$$

If the set $\mathcal{O}$ is $\mathrm{U}(n)$-invariant, we define

$$\mathcal{D}'_{T,\natural}(\mathcal{O}) = \left(\mathcal{D}'_T \cap \mathcal{D}'_\natural\right)(\mathcal{O}), \qquad C^s_{T,\natural}(\mathcal{O}) = \left(C^s_T \cap \mathcal{D}'_\natural\right)(\mathcal{O}).$$

In this section we shall establish the properties of classes (17.1)–(17.3) analogous to those given in Sect. 16.1.

We set

$$\mathcal{Z}_T = \left\{\lambda \in \mathcal{Z}\big(\widetilde{T}\big) : \operatorname{Re}\lambda \geqslant 0, \mathrm{i}\lambda \notin (0, +\infty)\right\},$$

$$n(\lambda, T) = \begin{cases} n_\lambda\big(\widetilde{T}\big) - 1 & \text{if } \lambda \in \mathcal{Z}_T \backslash \{0\}, \\ n_\lambda\big(\widetilde{T}\big)/2 - 1 & \text{if } \lambda = 0 \in \mathcal{Z}_T. \end{cases}$$

Proposition 17.1. *Let $\lambda \in \mathbb{C}$, $\eta, p, q \in \mathbb{Z}_+$, and $l \in \{1, \dots, d(n, p, q)\}$. Then*

$$\phi_{\lambda,\eta,p,q,l} \star T = \sum_{\nu=0}^{\eta} \binom{\eta}{\nu}_\lambda \widetilde{T}^{\langle \eta-\nu\rangle}(\lambda)\phi_{\lambda,\nu,p,q,l} \quad \text{in } \mathbb{C}^n$$

and

$$\psi_{\lambda,\eta,p,q,l} \star T = \sum_{\nu=0}^{\eta} \binom{\eta}{\nu}_\lambda \widetilde{T}^{\langle \eta-\nu\rangle}(\lambda)\psi_{\lambda,\nu,p,q,l} \quad \text{in } \mathbb{C}^n \backslash \mathring{B}_{r(T)}.$$

In particular, if $\lambda \in \mathcal{Z}_T$ and $\eta \in \{0, \dots, n(\lambda, T)\}$, then $\phi_{\lambda,\eta,p,q,l} \in \mathrm{RA}_T(\mathbb{C}^n)$ and $\psi_{\lambda,\eta,p,q,l} \in \mathrm{RA}_T(\mathbb{C}^n \backslash \{0\})$.

Proof. This statement can be obtained in the same way as Proposition 16.1 with attention to (12.35) and Proposition 12.13. □

Proposition 17.2. *Let $f \in \mathcal{D}'_T(\mathcal{O})$. Then*

$$\frac{\partial f}{\partial z_k} - \frac{\overline{z}_k}{4} f \in \mathcal{D}'_T(\mathcal{O}) \quad \text{and} \quad \frac{\partial f}{\partial \overline{z}_k} + \frac{z_k}{4} f \in \mathcal{D}'_T(\mathcal{O})$$

for all $k \in \{1, \dots, n\}$.

Proof. We can assume, by regularization, that $f \in C_T^\infty(\mathcal{O})$. Fix $w \in \mathcal{O}_T$. Set $a = (a_1, \dots, a_n)$, where $a_k = t \in \mathbb{R}^1$ and $a_j = 0$ if $j \neq k$. By the hypothesis and Proposition 12.1(vii) there is $\varepsilon > 0$ such that

$$\left(\left(f(z-a)\mathrm{e}^{-\frac{\mathrm{i}}{2}\operatorname{Im}\langle z,a\rangle_{\mathbb{C}}}\right) \star T\right)(w) = 0 \tag{17.4}$$

for $|t| < \varepsilon$. Differentiating (17.4) with respect to t and putting $t = 0$, we obtain

$$\frac{\partial f}{\partial z_k}(z) + \frac{\partial f}{\partial \overline{z}_k}(z) + \frac{\mathrm{i}}{2} f(z) \operatorname{Im} z_k \in C_T^\infty(\mathcal{O}). \tag{17.5}$$

Analogously, for $a_k = \mathrm{i}t$, $t \in \mathbb{R}^1$, and $a_j = 0$, $j \neq k$, one has

$$\frac{\partial f}{\partial z_k}(z) - \frac{\partial f}{\partial \overline{z}_k}(z) - \frac{1}{2} f(z) \operatorname{Im}(\mathrm{i}z_k) \in C_T^\infty(\mathcal{O}). \tag{17.6}$$

Combining (17.5) with (17.6), we complete the proof. □

In the next statements of this section we shall assume that the set $\mathcal{O}$ is $\mathrm{U}(n)$-invariant.

Proposition 17.3.

(i) *If $f \in \mathcal{D}'_T(\mathcal{O})$, then $f^{(p,q),k,l} \in \mathcal{D}'_T(\mathcal{O})$ for all $p, q \in \mathbb{Z}_+$ and $k, l \in \{1, \dots, d(n, p, q)\}$.*
(ii) *Let $f \in \mathcal{D}'(\mathcal{O})$, and let $f^{(p,q),l} \in \mathcal{D}'_T(\mathcal{O})$ for all p, q, l. Then $f \in \mathcal{D}'_T(\mathcal{O})$.*

Proof. It suffices to use (12.17) and (12.18). □

Proposition 17.4. *Let $D_i(s)$, $i = 1, 2$, be the differential operator given by* (12.19). *Suppose that $m \in \mathbb{N}$, $f \in C_T^m(\mathcal{O})$ and f has the form $f(z) = \varphi(\varrho)S_l^{p,q}(\sigma)$. Then*

(i) $(D_1(p+q)\varphi)(\varrho)S_k^{p,q+1}(\sigma) \in C_T^{m-1}(\mathcal{O})$ *for $p, q \in \mathbb{Z}_+$ and $k \in \{1, \dots, d(n, p, q+1)\}$;*
(ii) $(D_2(p+q)\varphi)(\varrho)S_k^{p+1,q}(\sigma) \in C_T^{m-1}(\mathcal{O})$ *for $p, q \in \mathbb{Z}_+$ and $k \in \{1, \dots, d(n, p+1, q)\}$;*
(iii) $(D_1(2-2n-p-q)\varphi)(\varrho)S_k^{p-1,q}(\sigma) \in C_T^{m-1}(\mathcal{O})$ *for $p \in \mathbb{N}$, $q \in \mathbb{Z}_+$, and $k \in \{1, \dots, d(n, p-1, q)\}$;*
(iv) $(D_2(2-2n-p-q)\varphi)(\varrho)S_k^{p,q-1}(\sigma) \in C_T^{m-1}(\mathcal{O})$ *for $p \in \mathbb{Z}_+$, $q \in \mathbb{N}$, and $k \in \{1, \dots, d(n, p, q-1)\}$.*

Proof. Put

$$P_1(z) = z_1^p \overline{z}_2^q, \qquad P_2(z) = z_1^{p-1}\overline{z}_2^q,$$
$$P_3(z) = \overline{z}_1 P_1(z) - \frac{p}{n+p+q-1}\varrho^2 P_2(z).$$

By Proposition 17.3(i), $F(z) = \varphi(\varrho)P_1(\sigma) \in C_T^m(\mathcal{O})$. A direct calculation gives

$$\frac{\partial F}{\partial z_1}(z) - \frac{\overline{z}_1}{4}F(z) = \frac{1}{2}\big(D_1(p+q)\varphi\big)(\varrho)P_3(\sigma) + \frac{p}{2(n+p+q-1)}\big(D_1(2-2n-p-q)\varphi\big)(\varrho)P_2(\sigma). \tag{17.7}$$

Since $P_2(z) \in \mathcal{H}_2^{n,p-1,q}$ and $P_3(z) \in \mathcal{H}_2^{n,p,q+1}$ (see (16.10)), from (17.7), Proposition 17.2, and Proposition 17.3(i) we obtain (i), (iii). In the same way, by means of the operator $\frac{\partial}{\partial \overline{z}_2} + \frac{z_2}{4}\,\mathrm{Id}$, we derive (ii), (iv). □

Proposition 17.5. *Let $R \in (r(T), +\infty]$, $p, q \in \mathbb{Z}_+$, and $l \in \{1, \dots, d(n,p,q)\}$. Then for $f \in \mathcal{D}'_{(p,q),l}(B_R)$, the following assertions are equivalent.*

(i) $f \in \mathcal{D}'_T(B_R)$.
(ii) $\mathcal{A}_l^{(p,q)}(f) \in \mathcal{D}'_{T,\natural}(B_R)$.
(iii) $\mathfrak{A}_{(p,q),l}(f) \in \mathcal{D}'_{A(T),\natural}(-R, R)$.

The proof depends on (12.76) and Theorems 12.3(i), (viii) and 12.5(i).

Proposition 17.6. *Let $r(T) = 0$ and suppose that the set $\mathcal{O}$ is connected. Then a distribution $f \in \mathcal{D}'(\mathcal{O})$ belongs to $\mathcal{D}'_T(\mathcal{O})$ if and only if*

$$f^{(p,q),l} = \sum_{\lambda \in \mathcal{Z}_T} \sum_{\eta=0}^{n(\lambda,T)} a_{\lambda,\eta,p,q,l}\,\phi_{\lambda,\eta,p,q,l} + b_{\lambda,\eta,p,q,l}\,\psi_{\lambda,\eta,p,q,l}$$

for all p, q, l, where $a_{\lambda,\eta,p,q,l}, b_{\lambda,\eta,p,q,l} \in \mathbb{C}$, and $b_{\lambda,\eta,p,q,l} = 0$ in the case where $0 \in \mathcal{O}$.

Proof. The assertion is proved similarly to Proposition 15.6 by using (12.35), Remark 12.3, Theorem 12.2(i), and Proposition 12.14. □

17.2 Phase Space Analogues of the Uniqueness Theorems

In this section we shall establish analogues of Theorems 14.1–14.5 for the twisted convolution equation on $\mathbb{C}^n$.

Introduce the class $\mathcal{M}^\nu(\mathbb{C}^n)$, $\nu \in \mathbb{Z}$, by analogy with Sect. 1.2. Namely, let $\mathcal{M}^0(\mathbb{C}^n)$ be the set of all complex-valued compactly supported measures on $\mathbb{C}^n$, and let $\mathcal{M}^1(\mathbb{C}^n)$ be the set of all distributions $f \in \mathcal{E}'(\mathbb{C}^n)$ such that $Df \in \mathcal{M}^0(\mathbb{C}^n)$ for each differential operator D on $\mathbb{C}^n$ of order at most one. If $\nu \in \mathbb{N}$ and ν is even (respectively ν is odd), we denote by $\mathcal{M}^\nu(\mathbb{C}^n)$ the set of all distributions $f \in \mathcal{E}'(\mathbb{C}^n)$ for which $\mathfrak{L}^{[\nu/2]}f \in \mathcal{M}^0(\mathbb{C}^n)$ (respectively $\mathfrak{L}^{[\nu/2]}f \in \mathcal{M}^1(\mathbb{C}^n)$). Finally, if $\nu < 0$ and ν is even (respectively ν is odd), we write $\mathcal{M}^\nu(\mathbb{C}^n)$ for the set of all distributions $f \in \mathcal{E}'(\mathbb{C}^n)$ such that $f = P(\mathfrak{L})u$ for some $u \in \mathcal{M}^0(\mathbb{C}^n)$ (respectively $u \in \mathcal{M}^1(\mathbb{C}^n)$) and some polynomial P of degree at most $[(1-\nu)/2]$.

By means of Theorem 12.2 and Propositions 12.14–12.16 it is easy to make sure that for every $T \in \mathcal{E}'_{(p,q),l}(\mathbb{C}^n)$, there exists $\nu \in \mathbb{Z}$ such that $T \in \mathcal{M}^\nu(\mathbb{C}^n)$.

Our first result can now be stated.

Theorem 17.1. *Let $\nu, s \in \mathbb{Z}$, $s \geqslant \max\{0, 2[(1-\nu)/2]\}$. Suppose that $T \in (\mathcal{E}'_\natural \cap \mathcal{M}^\nu)(\mathbb{C}^n)$ with $R > r(T) > 0$, $f \in \mathcal{D}'_T(B_R)$, and $f = 0$ in $B_{r(T)}$. Then the following assertions hold.*

(i) *If $f \in L_s^{1,\text{loc}}(B_R)$, then $f^{(p,q),l} = 0$ in B_R for $p+q \leqslant s+\nu+1$ and $1 \leqslant l \leqslant d(n,p,q)$.*
(ii) *If $f \in C^s(B_R)$, then $f^{(p,q),l} = 0$ in B_R for $p+q \leqslant s+\nu+2$ and $1 \leqslant l \leqslant d(n,p,q)$.*

We note that assumptions of Theorem 17.1 cannot be weakened in the general case (see Theorem 17.6 below and the proof of Theorem 14.9).

The proof of Theorem 17.1 will be based on the following lemmas.

Lemma 17.1. *Let $T \in (\mathcal{E}'_\natural \cap \mathcal{M}^0)(\mathbb{C}^n)$ with $r(T) > 0$. Assume that $R > r(T)$, $f \in (\mathcal{D}'_T \cap L_\natural^{1,\text{loc}})(B_R)$, and $f = 0$ in $B_{r(T)}$. Then $f = 0$ in B_R.*

Proof. For $z \in B_R$, we put

$$f_m(z) = g_m(|z|), \quad m \in \mathbb{N},$$

where the sequence g_m is determined by the recurrence relations

$$g_1(\varrho) = \frac{f_{(0,0),1}(\varrho)}{\sqrt{\omega_{2n-1}}} = \frac{1}{\omega_{2n-1}} \int_{\mathbb{S}^{2n-1}} f(\varrho\sigma)\, d\omega(\sigma), \tag{17.8}$$

$$g_{m+1}(\varrho) = e^{\varrho^2/4} \int_0^\varrho e^{-y^2/2} y^{1-2n} \int_0^y e^{x^2/4} x^{2n-1} g_m(x)\, dx\, dy, \quad m \in \mathbb{N}. \tag{17.9}$$

By (17.8) and (17.9), $f_m = 0$ in $B_{r(T)}$, $f_{m+1} \in C^{2m-1}(B_R)$, and

$$(-\mathfrak{L} - n\,\mathrm{Id}) f_{m+1} = f_m \quad \text{in } \mathcal{D}'(B_R).$$

Hence (see the proof of Lemma 14.1), $f_m \in C_T^{2m-3}(B_R)$ for $m \geqslant 2$. Then using Theorem 12.3 and the argument of Lemma 16.1, we obtain that $f_m = 0$ in B_R for all sufficiently large $m \in \mathbb{N}$. Combining this with (17.9), we arrive at the desired assertion. □

Lemma 17.2. *Let $T \in \mathcal{M}_\natural^1(\mathbb{C}^n)$ with $r(T) > 0$. Suppose that $R > r(T)$ and let $f \in C_T(B_R)$ possess the following properties:*

(1) *f has the form $f(z) = \varphi(\varrho)Y(\sigma)$, where $Y \in \mathcal{H}_2^{n,p,q}$ for some $p, q \in \mathbb{Z}_+$ such that $p+q \leqslant 3$;*

(2) $f = 0$ *in* $B_{r(T)}$.

Then f *vanishes identically in* B_R.

Proof. If $p = q = 0$, the statement of Lemma 17.2 follows directly from Lemma 17.1 because $\mathcal{M}^1_\natural(\mathbb{C}^n) \subset \mathcal{M}^0_\natural(\mathbb{C}^n)$. Assume that $p+q \geqslant 1$. For $\lambda, \mu \in \mathbb{Z}_+$ such that $1 \leqslant \lambda + \mu \leqslant 3$, we put

$$U_{\lambda,\mu}(z) = u_{\lambda,\mu}(\varrho), \quad z \in B_R,$$

where

$$u_{1,0}(\varrho) = \mathrm{e}^{-\varrho^2/4} \int_0^\varrho \varphi(\xi)\mathrm{e}^{\xi^2/4}\,\mathrm{d}\xi, \tag{17.10}$$

$$u_{0,1}(\varrho) = \mathrm{e}^{\varrho^2/4} \int_0^\varrho \varphi(\xi)\mathrm{e}^{-\xi^2/4}\,\mathrm{d}\xi, \tag{17.11}$$

$$u_{1,1}(\varrho) = \mathrm{e}^{-\varrho^2/4} \int_0^\varrho \eta e^{\eta^2/2} \int_0^\eta \frac{\varphi(\xi)\mathrm{e}^{-\xi^2/4}}{\xi}\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{17.12}$$

$$u_{2,0}(\varrho) = \mathrm{e}^{-\varrho^2/4} \int_0^\varrho \eta \int_0^\eta \frac{\varphi(\xi)\mathrm{e}^{\xi^2/4}}{\xi}\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{17.13}$$

$$u_{0,2}(\varrho) = \mathrm{e}^{\varrho^2/4} \int_0^\varrho \eta \int_0^\eta \frac{\varphi(\xi)\mathrm{e}^{-\xi^2/4}}{\xi}\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{17.14}$$

$$u_{2,1}(\varrho) = \mathrm{e}^{-\varrho^2/4} \int_0^\varrho \zeta \int_0^\zeta \eta\mathrm{e}^{\eta^2/2} \int_0^\eta \frac{\varphi(\xi)\mathrm{e}^{-\xi^2/4}}{\xi^2}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta, \tag{17.15}$$

$$u_{1,2}(\varrho) = \mathrm{e}^{\varrho^2/4} \int_0^\varrho \zeta \int_0^\zeta \eta\mathrm{e}^{-\eta^2/2} \int_0^\eta \frac{\varphi(\xi)\mathrm{e}^{\xi^2/4}}{\xi^2}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta, \tag{17.16}$$

$$u_{3,0}(\varrho) = \mathrm{e}^{-\varrho^2/4} \int_0^\varrho \zeta \int_0^\zeta \eta \int_0^\eta \frac{\varphi(\xi)\mathrm{e}^{\xi^2/4}}{\xi^2}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta, \tag{17.17}$$

$$u_{0,3}(\varrho) = \mathrm{e}^{\varrho^2/4} \int_0^\varrho \zeta \int_0^\zeta \eta \int_0^\eta \frac{\varphi(\xi)\mathrm{e}^{-\xi^2/4}}{\xi^2}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta. \tag{17.18}$$

Now define $V_{\lambda,\mu} \in C^{\lambda+\mu}_\natural(B_{R-r(T)})$ by

$$V_{\lambda,\mu}(z) = (U_{\lambda,\mu} \star T)(z), \quad z \in B_{R-r(T)}. \tag{17.19}$$

We claim that

$$V_{p,q}(z) = 0, \quad z \in B_{R-r(T)}. \tag{17.20}$$

One has

$$A^\lambda_{n+1} V_{\lambda,0} = \left(A^\lambda_{n+1} U_{\lambda,0}\right) \star T, \quad \lambda = 1, 2, 3, \tag{17.21}$$

$$A^\mu_1 V_{0,\mu} = \left(A^\mu_1 U_{0,\mu}\right) \star T, \quad \mu = 1, 2, 3, \tag{17.22}$$

$$A_2 A_{n+1} V_{1,1} = (A_2 A_{n+1} U_{1,1}) \star T, \tag{17.23}$$

$$A_2 A_{n+1}^2 V_{2,1} = \left(A_2 A_{n+1}^2 U_{2,1}\right) \star T, \tag{17.24}$$

$$A_{n+2} A_1^2 V_{1,2} = \left(A_{n+2} A_1^2 U_{1,2}\right) \star T, \tag{17.25}$$

where

$$A_k = \frac{\partial}{\partial z_k} - \frac{\overline{z}_k}{4}\,\mathrm{Id}, \quad A_{n+k} = \frac{\partial}{\partial \overline{z}_k} + \frac{z_k}{4}\,\mathrm{Id}, \quad 1 \leqslant k \leqslant n$$

(see the proof of Proposition 17.2). Let us transform equalities (17.21)–(17.25). It is easy to verify that for $\varkappa \in \mathbb{Z}_+$ and $u \in C^1(0, R - r(T))$, the identities

$$A_1\left(\overline{z}_1^{\varkappa} u(\varrho)\right) = \frac{\overline{z}_1^{\varkappa+1}}{2}(E_1 u)(\varrho), \tag{17.26}$$

$$A_2\left(z_1^{\varkappa} u(\varrho)\right) = \frac{z_1^{\varkappa}\overline{z}_2}{2}(E_1 u)(\varrho), \tag{17.27}$$

$$A_{n+1}\left(z_1^{\varkappa} u(\varrho)\right) = \frac{z_1^{\varkappa+1}}{2}(E_2 u)(\varrho), \tag{17.28}$$

$$A_{n+2}\left(\overline{z}_1^{\varkappa} u(\varrho)\right) = \frac{z_2\overline{z}_1^{\varkappa}}{2}(E_2 u)(\varrho) \tag{17.29}$$

hold, where

$$(E_1 u)(\varrho) = \frac{\mathrm{e}^{\varrho^2/4}}{\varrho}\frac{\mathrm{d}}{\mathrm{d}\varrho}\left(u(\varrho)\mathrm{e}^{-\varrho^2/4}\right), \tag{17.30}$$

$$(E_2 u)(\varrho) = \frac{-\mathrm{e}^{\varrho^2/4}}{\varrho}\frac{\mathrm{d}}{\mathrm{d}\varrho}\left(u(\varrho)\mathrm{e}^{\varrho^2/4}\right). \tag{17.31}$$

With the help (17.26)–(17.29) and (17.10)–(17.18) we can write (17.21)–(17.25) in the form

$$z_1^{\lambda}\left(E_2^{\lambda} v_{\lambda,0}\right)(\varrho) = \left(\varphi(\varrho)\left(\frac{z_1}{\varrho}\right)^{\lambda}\right) \star T, \quad \lambda = 1, 2, 3, \tag{17.32}$$

$$\overline{z}_1^{\mu}\left(E_1^{\mu} v_{0,\mu}\right)(\varrho) = \left(\varphi(\varrho)\left(\frac{\overline{z}_1}{\varrho}\right)^{\mu}\right) \star T, \quad \mu = 1, 2, 3, \tag{17.33}$$

$$z_1\overline{z}_2(E_1 E_2 v_{1,1})(\varrho) = \left(\varphi(\varrho)\frac{z_1\overline{z}_2}{\varrho^2}\right) \star T, \tag{17.34}$$

$$z_1^2\overline{z}_2\left(E_1 E_2^2 v_{2,1}\right)(\varrho) = \left(\varphi(\varrho)\frac{z_1^2\overline{z}_2}{\varrho^3}\right) \star T, \tag{17.35}$$

$$z_2\overline{z}_1^2\left(E_2 E_1^2 v_{1,2}\right)(\varrho) = \left(\varphi(\varrho)\frac{z_2\overline{z}_1^2}{\varrho^3}\right) \star T, \tag{17.36}$$

where

$$v_{\lambda,\mu}(\varrho) = V_{\lambda,\mu}(z)|_{|z|=\varrho}.$$

Recalling that $\varphi(\varrho)Y(\sigma) \in \mathcal{D}'_T(B_R)$ and applying Proposition 17.3(i), from (17.30)–(17.36) we obtain

$$v_{p,q}(\varrho) = \begin{cases} c_1 \mathrm{e}^{-\varrho^2/4}, & p=1,\ q=0, \\ c_1 \mathrm{e}^{\varrho^2/4}, & p=0,\ q=1, \\ c_1 \mathrm{e}^{\varrho^2/4} + c_2 \mathrm{e}^{-\varrho^2/4}, & p=1,\ q=1, \\ \big(c_1\varrho^2 + c_2\big)\mathrm{e}^{-\varrho^2/4}, & p=2,\ q=0, \\ \big(c_1\varrho^2 + c_2\big)\mathrm{e}^{\varrho^2/4}, & p=0,\ q=2, \\ c_1 \mathrm{e}^{\varrho^2/4} + c_2 \mathrm{e}^{-\varrho^2/4} + c_3\varrho^2\mathrm{e}^{-\varrho^2/4}, & p=2,\ q=1, \\ c_1 \mathrm{e}^{-\varrho^2/4} + c_2 \mathrm{e}^{\varrho^2/4} + c_3\varrho^2\mathrm{e}^{\varrho^2/4}, & p=1,\ q=2, \\ \big(c_1\varrho^4 + c_2\varrho^2 + c_3\big)\mathrm{e}^{-\varrho^2/4}, & p=3,\ q=0, \\ \big(c_1\varrho^4 + c_2\varrho^2 + c_3\big)\mathrm{e}^{\varrho^2/4}, & p=0,\ q=3, \end{cases}$$

with some $c_1, c_2, c_3 \in \mathbb{C}$. Next, according to (17.19),

$$\mathfrak{L}^j V_{p,q}(z) = \big(\mathfrak{L}^j U_{p,q} \star T\big)(z), \quad 0 \leqslant j \leqslant p+q-1. \tag{17.37}$$

When $z=0$, (17.37) becomes

$$\mathfrak{L}^j V_{p,q}(0) = 0, \quad 0 \leqslant j \leqslant p+q-1, \tag{17.38}$$

since $U_{p,q} \in C^{p+q}(B_R)$, $U_{p,q} = 0$ in $B_{r(T)}$, and $T \in \mathcal{M}^1_\natural(\mathbb{C}^n)$. By means of (12.20) and (12.21) we find that for $a \in \mathbb{R}^1$,

$$\mathfrak{L}\big(\mathrm{e}^{a\varrho^2}\big)(0) = -4an, \qquad \mathfrak{L}\big(\varrho^2\mathrm{e}^{a\varrho^2}\big)(0) = -4n, \qquad \mathfrak{L}\big(\varrho^4\mathrm{e}^{a\varrho^2}\big)(0) = 0,$$
$$\mathfrak{L}^2\big(\mathrm{e}^{a\varrho^2}\big)(0) = 16a^2n^2 + 16a^2n - n, \qquad \mathfrak{L}^2\big(\varrho^2\mathrm{e}^{a\varrho^2}\big)(0) = 32an(n+1),$$
$$\mathfrak{L}^2\big(\varrho^4\mathrm{e}^{a\varrho^2}\big)(0) = 32n(n+1).$$

Combining these relations with (17.38), we derive (17.20). Now Lemma 17.1 implies that $U_{p,q} = 0$ in B_R. Hence, $f = 0$ in B_R, and Lemma 17.2 is proved. □

Corollary 17.1. *Let $T \in \mathcal{M}^1_\natural(\mathbb{C}^n)$ with $r(T) > 0$. Suppose that $R > r(T)$, $f \in C^s_T(B_R)$, and $f = 0$ in $B_{r(T)}$. Then $f^{(p,q),l} = 0$ in B_R for all $p+q \leqslant s+3$ and $1 \leqslant l \leqslant d(n,p,q)$.*

Proof. If $s = 0$, this follows from Lemma 17.2 by means of Proposition 17.3(i) and (12.14). Assume that the assertion of Corollary 17.1 is true for $0 \leqslant s \leqslant j-1$ with some $j \in \mathbb{N}$. We shall prove it for $s = j$. Let $f \in C^j_T(B_R)$ and $f = 0$ in $B_{r(T)}$. Consider $f^{(p,q),l}$, where $p+q \leqslant j+3$ and $1 \leqslant l \leqslant d(n,p,q)$. Because of Proposition 17.3(i) and (12.14), $f^{(p,q),l} \in C^j_T(B_R)$ and $f^{(p,q),l} = 0$ in $B_{r(T)}$. We shall now distinguish three cases.

(a) The case $p+q \leqslant j+2$. In this situation $f^{(p,q),l} = 0$ in B_R by the induction hypothesis.

(b) The case $p+q=j+3$, $p \geqslant 1$. According to Proposition 17.4(iii),

$$\big(D_1(2-2n-p-q)f_{(p,q),l}\big)(\varrho)S_1^{p-1,q}(\sigma) \in C_T^{j-1}(B_R).$$

Furthermore, we have

$$\big(D_1(2-2n-p-q)f_{(p,q),l}\big)(\varrho)=0, \quad 0 \leqslant \varrho \leqslant r(T).$$

Again by induction assumption we conclude that $f^{(p,q),l}=0$ in B_R.

(c) The case $p=0, q=j+3$. As above, we infer from Proposition 17.4(iv) that $f^{(p,q),l}=0$ in B_R.

Thus, $f^{(p,q),l}$ vanishes in B_R for all $p+q \leqslant j+3$ and $1 \leqslant l \leqslant d(n,p,q)$, which proves Corollary 17.1. □

Proof of Theorem 17.1. The proof of Lemma 17.2 and Corollary 17.1 shows that the statement of Theorem 17.1 is valid for $\nu=0,1$. Using now Theorem 12.2, Proposition 17.6, and the arguments in Lemma 15.3 and Theorem 15.4, we complete the proof. □

Corollary 17.2. *Let $T \in \mathcal{E}'_\natural(\mathbb{C}^n)$, let $c_2 \in \mathbb{R}^1$ be the constant from the estimate*

$$\big|\widetilde{T}(\lambda)\big| \leqslant c_1(1+|\lambda|)^{c_2}\mathrm{e}^{r(T)|\operatorname{Im}\lambda|}, \quad \lambda \in \mathbb{C}, \tag{17.39}$$

and let $s \geqslant \max\{0, 2[(2n+4-[-c_2])/2]\}$. Assume that $R > r(T) > 0$, $f \in C_T^s(B_R)$, and $f=0$ in $B_{r(T)}$. Then $f^{(p,q),l}=0$ in B_R for all $p+q \leqslant s-2n-1+[-c_2]$ and $1 \leqslant l \leqslant d(n,p,q)$.

Proof. We see from (17.39) and Propositions 12.14–12.16 that T belongs to $\mathcal{M}^\nu(\mathbb{C}^n)$ with $\nu=-2n-3+[-c_2]$. Corollary 17.2 now follows from Theorem 17.1(ii). □

We present uniqueness theorems for the class $\mathcal{D}'_T(\mathcal{O})$, where $T \in \mathcal{E}'_\natural(\mathbb{C}^n)$ and $\mathcal{O}$ is a ζ domain in $\mathbb{C}^n$ with $\zeta=r(T)>0$.

Theorem 17.2. *Let T and $\mathcal{O}$ be as above, $f \in \mathcal{D}'_T(\mathcal{O})$, and let $f=0$ in some open ball B of radius $r(T)$ such that* Cl $B \subset \mathcal{O}$. *Then*

(i) *If $f=0$ in some open ball with the radius exceeding $r(T)$, then $f=0$ in $\mathcal{O}$.*
(ii) *If $f \in C_T^\infty(\mathcal{O})$, then $f=0$ in $\mathcal{O}$.*
(iii) *If $T=T_1+T_2$ where $T_1 \in \mathcal{D}_\natural(\mathbb{C}^n)$, $T_2 \in \mathcal{E}'_\natural(\mathbb{C}^n)$ and $r(T_2)<r(T)$, then $f=0$ in $\mathcal{O}$.*

Proof. Let $z_0 \in \mathbb{C}^n$ be fixed. By Proposition 12.1(vii) we have

$$\big(f(z+z_0)\mathrm{e}^{\frac{i}{2}\operatorname{Im}\langle z,z_0\rangle_{\mathbb{C}}}\big)\star T=(f\star T)(z+z_0)\mathrm{e}^{\frac{i}{2}\operatorname{Im}\langle z,z_0\rangle_{\mathbb{C}}}. \tag{17.40}$$

In the same way as in the proof of Theorem 14.2 we conclude from (17.40), Corollary 17.2, and Theorem 12.3 that our result is valid if $\mathcal{O}=z_0+B_R$, $R>r(T)$, and $f=0$ in $z_0+B_{r(T)}$. Hence, by the definition of a ζ domain, we obtain Theorem 17.2 in general. □

Theorem 17.3. *Let T and $\mathcal{O}$ be as in Theorem 17.2, and let $f \in \mathcal{D}'_T(\mathcal{O})$. Assume that $\mathcal{O}$ contains the ball $\mathring{B}_{r(T)}$, $f = 0$ in $B_{r(T)}$, and $f \in C^\infty(\mathcal{O}_1)$ for some open subset $\mathcal{O}_1$ of $\mathcal{O}$ such that $\{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z| = r(T)$ and $\operatorname{Re} z_1 \geqslant 0\} \subset \mathcal{O}_1$. Then $f = 0$ in $\mathcal{O}$.*

This statement is proved similarly to Theorems 14.3 and 16.3 by using Theorems 12.2 and 12.3.

In the case where $\mathcal{O} = \mathbb{C}^n$ we have the following analogs to Theorems 14.4 and 14.5.

Theorem 17.4. *Let $T_1, T_2 \in \mathcal{E}'_\natural(\mathbb{C}^n)$, $r(T_2) > 0$, and $T = T_1 \star T_2$. Suppose that $f \in \mathcal{D}'_T(\mathbb{C}^n)$ and $f = 0$ in $B_{r(T)}$. Then*

(i) *If $T_2 \in \operatorname{Inv}(\mathbb{C}^n)$ and $|\operatorname{Im}\lambda|(\log(2 + |\lambda|))^{-1} \to +\infty$ as $\lambda \to \infty$, $\lambda \in \mathcal{Z}_{T_2}$, then $f = 0$.*
(ii) *If $T_2 \in \mathfrak{E}(\mathbb{C}^n)$ and the order of the distribution f is finite, then $f = 0$.*

Theorem 17.5. *For $T \in \mathcal{E}'_\natural(\mathbb{C}^n)$ such that $r(T) > 0$, the following assertions hold.*

(i) *If $T \notin \operatorname{Inv}_+(\mathbb{C}^n)$, $f \in \mathcal{D}'_T(\mathbb{C}^n)$, and $f = 0$ in $B_{r(T)}$, then $f = 0$ in $\mathbb{C}^n$.*
(ii) *If $T \in \operatorname{Inv}_+(\mathbb{C}^n)$, then there is nonzero $f \in (\mathcal{D}'_T \cap \mathcal{D}'_\natural)(\mathbb{C}^n)$ vanishing in $B_{r(T)}$.*

The validity of these results follows from Theorems 13.2, 13.3, and 12.3 (see the proof of Theorem 14.5).

17.3 Characterizations of the Kernel of the Twisted Convolution Operator

The purpose of this section is to present analogues of Theorems 14.7, 14.10, 14.16, 14.17(i), and 14.24(i) for the phase space. We start by defining a distribution $\zeta_{T,(p,q),l}$ that is an analogue of $\zeta_{T,k,j}$ in Sect. 14.3.

Let $T \in \operatorname{Inv}_+(\mathbb{C}^n)$. For $p, q \in \mathbb{Z}_+$ and $l \in \{1, \dots, d(n, p, q)\}$, we set

$$\zeta_{T,(p,q),l} = -\mathfrak{A}^{-1}_{(p,q),l}\big(\zeta'_{\Lambda(T)}\big).$$

Theorem 17.6.

(i) *$\zeta_{T,(p,q),l} \in (\mathcal{D}'_T \cap \mathcal{D}'_{(p,q),l})(\mathbb{C}^n)$ and*

$$\mathcal{A}_l^{(p,q)}(\zeta_{T,(p,q),l}) = \zeta_{T,(0,0),1}.$$

(ii) *If $r(T) > 0$, then $\zeta_{T,(p,q),l} = 0$ in $B_{r(T)}$ and $S_{r(T)} \subset \operatorname{supp} \zeta_{T,(p,q),l}$.*
(iii) *If $R > r(T)$, $z \in \mathbb{C}^n$, $u \in \mathcal{E}'_\natural(\mathbb{C}^n)$, and $\zeta_{T,(p,q),l} \star u = 0$ in $z + B_R$, then $u = T \star v$ for some $v \in \mathcal{E}'_\natural(\mathbb{C}^n)$.*

(iv) *If* $T \in \mathfrak{M}(\mathbb{C}^n)$*, then*

$$\zeta_{T,(p,q),l} = \sum_{\eta=0}^{n(0,T)} \frac{a^{0,0}_{2(n(0,T)-\eta)}(\widetilde{T})}{(2\eta)!}\phi_{0,\eta,p,q,l} + 2\sum_{\lambda\in\mathcal{Z}_T\setminus\{0\}}\sum_{\eta=0}^{n(\lambda,T)} a^{\lambda,\eta}_{n(\lambda,T)}\big(\widetilde{T}\big)(\eta\phi_{\lambda,\eta-1,p,q,l} + \lambda\phi_{\lambda,\eta,p,q,l}),$$

where the series converges in $\mathcal{D}'(\mathbb{C}^n)$*, and the first sum is set to be equal to zero if* $0 \notin \mathcal{Z}_T$.

(v) *If* $T \in \mathfrak{M}(\mathbb{C}^n)$*,* $m \in \mathbb{Z}_+$*, and* $R \in (0, +\infty)$*, then there is a constant* $\sigma = \sigma(m, R, T)$ *such that* $\zeta_{T,(p,q),l} \in C^m_T(B_R)$ *for* $p + q > \sigma$*. The same is true with* $R = +\infty$*, provided that*

$$\frac{n(\lambda, T) + |\mathrm{Im}\lambda|}{\log(2 + |\lambda|)} \to 0 \quad \text{as } \lambda \to \infty, \lambda \in \mathcal{Z}_T.$$

This result is proved in the same manner as in the case of the distribution $\zeta_{T,k,j}$ (see Theorems 14.7 and 12.3 and Propositions 12.9 and 17.1).

Theorem 17.6 and the proof of Theorem 14.9 show that the statements of Theorem 17.1 cannot be strengthened in the general case.

Employing Theorem 12.3 and the arguments in the proof of Theorem 14.10, we obtain

Theorem 17.7. *For* $T \in \mathrm{Inv}_+(\mathbb{C}^n)$ *with* $r(T) > 0$*, we have the following statements.*

(i) *If* $0 < r \leqslant r(T) < R \leqslant +\infty$ *and* $f \in (\mathcal{D}'_T \cap \mathcal{D}'_{(p,q),l})(B_R)$*, then* $f = 0$ *in* B_r *if and only if*

$$f = \zeta_{T,(p,q),l} \star U \quad \text{in } B_R \tag{17.41}$$

for some $U \in \mathcal{E}'_\natural(\mathbb{C}^n)$ *with* $\mathrm{supp}\, U \subset \mathring{B}_{r(T)-r}$.

(ii) *If* $R \in (r(T), +\infty]$ *and* $f \in (C^\infty_T \cap \mathcal{D}'_{(p,q),l})(B_R)$*, then* $(Df)(0) = 0$ *for each differential operator* D *if and only if relation* (17.41) *holds for some* $U \in \mathcal{D}_\natural(\mathbb{C}^n)$ *with* $\mathrm{supp}\, U \subset \mathring{B}_{r(T)}$.

The following two results are analogues to Theorems 14.16, 14.17(i), and 14.24(i).

Theorem 17.8.

(i) *Let* $T \in \mathrm{Inv}_+(\mathbb{C}^n)$*,* $0 < r(T) < R \leqslant +\infty$*, and* $f \in \mathcal{D}'(B_R)$*. Then* $f \in \mathcal{D}'_T(B_R)$ *if and only if for all* $p, q \in \mathbb{Z}_+$ *and* $l \in \{1, \ldots, d(n, p, q)\}$*, there exists* $u_{p,q,l} \in \mathcal{E}'_\natural(\mathbb{C}^n)$ *such that* $\mathrm{supp}\, u_{p,q,l} \subset \mathring{B}_{r(T)}$ *and*

$$f^{(p,q),l} = \zeta_{T,(p,q),l} \star u_{p,q,l} \quad \text{in } B_R.$$

(ii) *Let* $T \in \mathfrak{M}(\mathbb{C}^n)$*,* $R \in (r(T), +\infty]$*, and* $f \in \mathcal{D}'(B_R)$*. For* f *to belong to*

$\mathcal{D}'_T(B_R)$, it is necessary and sufficient that for all p, q, l,

$$f^{(p,q),l} = \sum_{\lambda \in \mathcal{Z}_T} \sum_{\eta=0}^{n(\lambda,T)} \alpha_{\lambda,\eta,p,q,l} \phi_{\lambda,\eta,p,q,l},$$

where $\alpha_{\lambda,\eta,p,q,l} \in \mathbb{C}$, and the series converges in $\mathcal{D}'(B_R)$.

Theorem 17.9. *Let $T \in \mathfrak{N}(\mathbb{C}^n)$, $0 \leqslant r < R \leqslant +\infty$, $R - r > 2r(T)$, and let $f \in \mathcal{D}'(B_{r,R})$. For f to belong to $\mathcal{D}'_T(B_{r,R})$, it is necessary and sufficient that for all p, q, l,*

$$f^{(p,q),l} = \sum_{\lambda \in \mathcal{Z}_T} \sum_{\eta=0}^{n(\lambda,T)} \alpha_{\lambda,\eta,p,q,l}\, \phi_{\lambda,\eta,p,q,l} + \beta_{\lambda,\eta,p,q,l}\, \psi_{\lambda,\eta,p,q,l},$$

where $\alpha_{\lambda,\eta,p,q,l}, \beta_{\lambda,\eta,p,q,l} \in \mathbb{C}$, and the series converges in $\mathcal{D}'(B_{r,R})$.

Theorems 17.8 and 17.9 are proved similarly to Theorems 16.6 and 16.7.

We note that a number of other results in Sects. 14.3–14.5 also admit extensions to the case of the twisted convolution. The proofs can be carried out along the same lines (see Chap. 12). We leave the details for the reader.

In conclusion, we want to make a few comments on mean periodic functions on the Heisenberg groups. Using an appropriate version of the Fourier transformation, we can reduce convolution equations on H^n (respectively, on H^n_{red}) to twisted convolution equations and obtain for these cases analogues of several results presented above. We expound in greater detail on analogues of Theorem 17.2(i) for H^n_{red}.

Let T be a compactly supported distribution on H^n_{red}. Put

$$r(T) = \inf\{r > 0 : \text{supp } T \subset C_r\},$$

where

$$C_r = \left\{\{(z, t + 2\pi k) : k \in \mathbb{Z}\} \in H^n_{\text{red}} : z \in B_r, 0 \leqslant t < 2\pi\right\}.$$

For $f \in \mathcal{D}'(C_r)$ and $k \in \mathbb{Z}$, we define $f_k \in \mathcal{D}'(B_r)$ by

$$\langle f_k, \varphi\rangle = \left\langle f, \varphi(z)\mathrm{e}^{-\mathrm{i}kt}\right\rangle, \quad \varphi \in \mathcal{D}(B_r). \tag{17.42}$$

The symbol $*$ will stand for the group convolution on H^n_{red}.

Theorem 17.10. *Let T be a radial compactly supported distribution on H^n_{red}. Suppose that*

$$T_k \neq 0 \quad \text{and} \quad r(T_k) = r(T) \quad \text{for every } k \in \mathbb{Z}. \tag{17.43}$$

Then for all $R \in (r(T), +\infty]$ and $\varepsilon \in (0, R - r(T))$, we have

$$\left\{f \in \mathcal{D}'(C_R) : f * T = 0 \text{ in } C_{R-r(T)} \text{ and } f = 0 \text{ in } C_{r(T)+\varepsilon}\right\} = \{0\}.$$

Furthermore, for $R = +\infty$, this result remains valid if (17.43) *is replaced by the condition*

$$T_k \neq 0 \quad \textit{for every } k \in \mathbb{Z}. \tag{17.44}$$

Proof. Let $f \in \mathcal{D}'(C_R)$ and $k \in \mathbb{Z}$. By (1.53), (17.42), and (17.43),

$$(f * T)_k = f_k \star_{-k} T_k, \quad \text{in } B_{R-r(T_k)},$$

where the distribution on the right-hand side is defined by

$$\langle f_k \star_{-k} T_k, \varphi\rangle = \big\langle f_k(z), \big\langle T_k(w), \varphi(z+w)\mathrm{e}^{-\frac{ik}{2}\operatorname{Im}\langle z,w\rangle_{\mathbb{C}}}\big\rangle\big\rangle, \quad \varphi \in \mathcal{D}(B_{R-r(T_k)}).$$

Using now Theorem 17.2(i), we obtain the required result. □

It is easy to make sure that conditions (17.43) and (17.44) in Theorem 17.10 cannot be omitted. Next, in contrast to the situation for the phase space $\mathbb{C}^n$, there exist nonzero radial distributions $T \in \mathcal{E}'(H^n_{\mathrm{red}})$ such that $r(T) = 0$ and

$$\big\{f \in \mathcal{D}'(C_R) : f * T = 0 \textit{ in } C_R \textit{ and } f = 0 \textit{ in } C_r\big\} \neq \{0\}$$

for all $R > 0$, $r \in (0, R)$. We also note that the equalities in (17.43) are satisfied, for instance, if T has the form

$$\langle T, \varphi\rangle = \langle U(z), \varphi(z, 0)\rangle, \quad \varphi \in \mathcal{E}\big(H^n_{\mathrm{red}}\big),$$

where $U \in \mathcal{E}'_{\natural}(\mathbb{C}^n)$.

Part IV
Local Aspects of Spectral Analysis and the Exponential Representation Problem

This part continues our study of mean periodic functions. Here we indicate how the results of Part III can be applied to questions of spectral analysis and spectral synthesis for translation-invariant subspaces.

A translation-invariant subspace $V \subset \mathcal{E}(\mathbb{R}^n)$ is said to admit *spectral analysis* if V contains an exponential, i.e., if there exists $z \in \mathbb{C}^n$ such that $f(x) = e^{i\langle z,x\rangle_{\mathbb{C}}}$ belongs to V. If the exponential polynomials belonging to V are dense in V, we say that V admits *spectral synthesis*. In case every translation-invariant subspace admits spectral analysis (synthesis), we say that spectral analysis (synthesis) holds in $\mathcal{E}(\mathbb{R}^n)$. The *exponential representation problem* for V consists in the representation of an arbitrary element of V as an integral over the set of exponential polynomials belonging to V.

If V is a translation-invariant subspace of $\mathcal{E}(\mathbb{R}^n)$, then there exists a family $\mathcal{T} \subset \mathcal{E}'(\mathbb{R}^n)$ such that $f \in V$ if and only if $f * T = 0$ for all $T \in \mathcal{T}$. This follows from the Hahn–Banach theorem. Thus, the above problems are reduced to the study of systems of convolution equations of compact support in $\mathbb{R}^n$. The remarks presented below give an indication of the spirit which animates the study of these questions.

1. Schwartz [188] proved that spectral synthesis holds in $\mathcal{E}(\mathbb{R}^1)$.
2. Gurevich [102] discovered that translations were not sufficient to generate a spectral synthesis for dimensions greater than one. More precisely, there exists six distributions $T_1, \ldots, T_6 \in \mathcal{E}'(\mathbb{R}^n)$ $(n \geqslant 2)$ such that $\{z \in \mathbb{C}^n : \widehat{T_1}(z) = \cdots = \widehat{T_6}(z) = 0\} = \varnothing$ but $\{f \in \mathcal{D}'(\mathbb{R}^n) : f * T_1 = \cdots = f * T_6 = 0\} \neq \{0\}$.
3. Brown, Schreiber, and Taylor [42] showed that every translation-invariant rotation-invariant subspace of $\mathcal{E}(\mathbb{R}^n)$, $n \geqslant 2$, is spanned by the polynomial-exponential functions it contains.
4. Different exponential representations for solutions of linear partial differential equations with constant coefficients on $\mathbb{R}^n$ are given by Ehrenpreis [69, Chap. 7], and Palamodov [165, Chap. 6]. There are many works devoted to the exponential representation problem for systems of convolution equations (see, for example, Berenstein and Struppa [19]).
5. The complete reconstruction, or *deconvolution*, of f given $T \in \mathcal{T}$ and $f * T$ is of great interest. The theory of deconvolution has its roots in the work of Wiener [261] and Hörmander [125], and has been developed into a working theory by Berenstein et al. (see [16, 24, 25, 27, 28]).
6. Further developments deal with analogous questions for other spaces. In particular, many authors studied the case of symmetric spaces and the Heisenberg group (see, e.g., Berenstein [11], Berenstein and Gay [15], Berenstein and Zalcman [26], Thangavelu [210], and Wawrzrynczyk [251–255]).

In Part IV we study related problems for systems of convolution equations on domains of homogeneous spaces. The main difference with the above-mentioned results is that we do not have any longer the group of translations at our disposal. The absence of the group structure provides a serious complicating factor.

In Chaps. 18 and 19 we present results for Euclidean spaces. The case of symmetric spaces is investigated in Chaps. 20 and 21. Finally, results for the phase space and the Heisenberg group is briefly discussed in the comments.

Chapter 18
A New Look at the Schwartz Theory

The theory of mean periodic functions of Schwartz [188] is a study of overdetermined systems of homogeneous convolution equations on the real line. The main function of the present chapter is to construct local analogues of this theory. The point of view developed here and the results obtained are sufficiently different from the above-mentioned article.

Let $\mathcal{I}$ be a nonempty index set. We will deal with the system of convolution equations

$$(f * T_i)(t) = 0, \quad |t| < R - r(T_i), \ i \in \mathcal{I}, \tag{18.1}$$

where $f \in \mathcal{D}'(-R, R)$ is unknown, $T_i \in \mathcal{E}'(\mathbb{R}^1)\backslash\{0\}$, $\operatorname{supp} T_i \subset [-r(T_i), r(T_i)]$, and $r(T_i) < R \leqslant +\infty$ for all $i \in \mathcal{I}$. If $T_{i_0} \in (\mathrm{Inv}_+ \cap \mathrm{Inv}_-)(\mathbb{R}^1)$ for some $i_0 \in \mathcal{I}$, we present the characterization of f analogous to that given in Theorem 13.13. In the case where $T_{i_0} \in \mathfrak{M}(\mathbb{R}^1)$ and

$$\inf_{i \in \mathcal{I}} r(T_i) + r(T_\nu) < R \leqslant +\infty \quad \text{for all } \nu \in \mathcal{I},$$

we obtain the exponential representation

$$f = \sum_{\lambda \in \mathcal{Z}} \sum_{\eta=0}^{m(\lambda)} \gamma_{\lambda,\eta} (\mathrm{i}t)^\eta \mathrm{e}^{\mathrm{i}\lambda t}, \quad t \in (-R, R), \tag{18.2}$$

where $\mathcal{Z} = \bigcap_{i \in \mathcal{I}} \mathcal{Z}(\widehat{T_i})$, $m(\lambda) = \min_{i \in \mathcal{I}} m(\lambda, T_i)$, and the series in (18.2) converges in $\mathcal{D}'(-R, R)$. Decomposition (18.2) shows that f vanishes in $(-R, R)$ if $\mathcal{Z} = \varnothing$. It turns out that the same is true without the requirement $T_{i_0} \in \mathfrak{M}(\mathbb{R}^1)$ (Theorem 18.1). This result is a local version of the Schwartz theorem on spectral analysis. Mean periodic functions with respect to a couple of distributions are discussed in more detail. New effects were discovered: for example, for $R = r(T_1) + r(T_2)$, an important role is played by the rate at which the roots of the equations $\widehat{T_i}(z) = 0$, $i = 1, 2$, "come together" at infinity (see Theorem 18.8). In Sect. 18.3 we study analogues of Theorem 18.8 for system (18.1). Here, we find connections with division-type formulas for entire functions (see (18.42)). To close the chapter we will show

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how to reconstruct a distribution $f \in \mathcal{D}'(\mathbb{R}^1)$ from the knowledge of its convolutions $f * T_l$, $l \in \{1, \dots, m\}$, $m \in \mathbb{N}\backslash\{1\}$, for a broad class of distributions $T_1, \dots, T_m$. We give here an explicit reconstruction formula using the biorthogonal systems of Part II.

18.1 Localization of the Schwartz Theorems. The Effect of the Size of the Domain

Throughout this part $\mathcal{I}$ denotes a nonempty index set. Let $\mathfrak{T}(\mathbb{R}^1)$ be the set of all families $\mathcal{T} = \{T_i\}_{i\in\mathcal{I}}$ such that

$$T_i \in \mathcal{E}'\big(\mathbb{R}^1\big), \quad T_i \neq 0 \quad \text{and} \quad \operatorname{supp} T_i \subset [-r(T_i), r(T_i)]$$

for all $i \in \mathcal{I}$. For $\mathcal{T} \in \mathfrak{T}(\mathbb{R}^1)$, we set

$$r^*(\mathcal{T}) = \sup_{i\in\mathcal{I}} r(T_i), \qquad r_*(\mathcal{T}) = \inf_{i\in\mathcal{I}} r(T_i),$$

$$R_{\mathcal{T}} = r_*(\mathcal{T}) + r^*(\mathcal{T}), \qquad \mathcal{Z}(\mathcal{T}) = \bigcap_{i\in\mathcal{I}} \mathcal{Z}\big(\widehat{T}_i\big).$$

In the sequel we denote by $\mathfrak{T}_1(\mathbb{R}^1)$ the set of all families $\mathcal{T} \in \mathfrak{T}(\mathbb{R}^1)$ such that

$$0 < r_*(\mathcal{T}) = r(T_{i_1}) \quad \text{and} \quad r^*(\mathcal{T}) = r(T_{i_2}) \quad \text{for some } i_1, i_2 \in \mathcal{I}.$$

Next, let $\mathfrak{T}_2(\mathbb{R}^1)$ be the set of all $\mathcal{T} \in \mathfrak{T}_1(\mathbb{R}^1)$ such that $T_i \in \mathcal{D}(\mathbb{R}^1)$ for some $i \in \mathcal{I}$ satisfying $r(T_i) = r_*(\mathcal{T})$. Also we write $\mathcal{T} \in \mathfrak{T}_3(\mathbb{R}^1)$ if

$$\mathcal{T} \in \mathfrak{T}_1\big(\mathbb{R}^1\big) \quad \text{and} \quad T_i \in \mathcal{D}\big(\mathbb{R}^1\big)$$

for all $i \in \mathcal{I}$ such that $r(T_i) = r^*(\mathcal{T})$.

Assume that $\mathcal{T} \in \mathfrak{T}(\mathbb{R}^1)$ and

$$r(T_i) < R \leqslant +\infty \quad \text{for all } i \in \mathcal{I}. \tag{18.3}$$

Let us consider the system of convolution equations

$$(f * T_i)(t) = 0, \quad |t| < R - r(T_i),\ i \in \mathcal{I}, \tag{18.4}$$

with unknown $f \in \mathcal{D}'(-R, R)$. Denote by $\mathcal{D}'_{\mathcal{T}}(-R, R)$ the set of all distributions $f \in \mathcal{D}'(-R, R)$ such that (18.4) is satisfied. Also let

$$\mathcal{D}'_{\mathcal{T},\natural}(-R, R) = \big(\mathcal{D}'_{\mathcal{T}} \cap \mathcal{D}'_{\natural}\big)(-R, R),$$

$$C^m_{\mathcal{T}}(-R, R) = \big(\mathcal{D}'_{\mathcal{T}} \cap C^m\big)(-R, R), \qquad C^m_{\mathcal{T},\natural}(-R, R) = \big(C^m_{\mathcal{T}} \cap \mathcal{D}'_{\natural}\big)(-R, R),$$

$$\mathrm{QA}_{\mathcal{T}}(-R,R)=\big(\mathcal{D}'_{\mathcal{T}}\cap \mathrm{QA}\big)(-R,R),\qquad G^{\alpha}_{\mathcal{T}}(-R,R)=\big(\mathcal{D}'_{\mathcal{T}}\cap G^{\alpha}\big)(-R,R),$$

where $m\in\mathbb{Z}_+\cup\{\infty\}$ and $\alpha>0$.

In this section we shall study solutions of (18.4) for various $\mathcal{T}$ and R. To begin with, we prove the following uniqueness result.

Theorem 18.1. *Let $\mathcal{T}\in\mathfrak{T}(\mathbb{R}^1)$, $\mathcal{Z}(\mathcal{T})=\varnothing$, and assume that (18.3) is fulfilled. Then the following assertions hold.*

(i) *If $f\in\mathcal{D}'_{\mathcal{T}}(-R,R)$ and*

$$r_*(\mathcal{T})+r(T_i)<R\leqslant+\infty\quad \textit{for all } i\in\mathcal{I}, \tag{18.5}$$

then $f=0$.

(ii) *If $\mathcal{T}\in\mathfrak{T}_1(\mathbb{R}^1)$, $R=R_{\mathcal{T}}$ and $f\in C^\infty[-R,R]\cap\mathcal{D}'_{\mathcal{T}}(-R,R)$, then $f=0$.*

Proof. There is no loss of generality in assuming that $\mathcal{Z}(\widehat{T_i})\neq\varnothing$ for all $i\in\mathcal{I}$. Next, it is easy to dispense with the case where $R=+\infty$. In this case we take $U\in\mathcal{T}$ and for each $\lambda\in\mathcal{Z}(\widehat{U})$, select $V\in\mathcal{T}$ such that $\lambda\notin\mathcal{Z}(\widehat{V})$. Then $c_{\lambda,\eta}(U,f)=0$ for all $\lambda\in\mathcal{Z}(\widehat{U})$, $\eta\in\{0,\dots,m(\lambda,U)\}$, and $f\in\mathcal{D}'_{\mathcal{T}}(\mathbb{R}^1)$ because of Proposition 13.8(i). This, together with Proposition 13.6(ii), yields $f=0$.

To prove (i) and (ii) for $R<+\infty$, first suppose that $r_*(\mathcal{T})=r(U)$ for some $U\in\mathcal{T}$. Since $\mathcal{Z}(\mathcal{T})=\varnothing$, for each $\lambda\in\mathcal{Z}(\widehat{U})$, there is $V\in\mathcal{T}$ such that $\lambda\notin\mathcal{Z}(\widehat{V})$. As above, the required assertions now follow from Proposition 13.8(i), (ii) and Proposition 13.6(ii). Suppose now that $r_*(\mathcal{T})<r(T_i)$ for all $i\in\mathcal{I}$. Then for each $\varepsilon>0$, the set

$$A_\varepsilon=\{i\in\mathcal{I}: r(T_i)<r_*(\mathcal{T})+\varepsilon\}$$

is nonempty. First, consider the case where $r^*(\mathcal{T})=r(T)$ for some $T\in\mathcal{T}$. Let $\lambda\in\mathcal{Z}(\widehat{T})$, $\varepsilon=R-R_{\mathcal{T}}$, $f\in\mathcal{D}'_{\mathcal{T}}(-R,R)$, and assume that (18.5) is satisfied. We claim that

$$c_{\lambda,\eta}(T,f)=0\quad \text{for all } \eta\in\{0,\dots,m(\lambda,T)\}. \tag{18.6}$$

If there exists $i\in A_\varepsilon$ such that $\lambda\notin\mathcal{Z}(\widehat{T_i})$, this follows by Proposition 13.8(i) with $U=T$ and $V=T_i$. Otherwise, $\lambda\in\bigcap_{i\in A_\varepsilon}\mathcal{Z}(\widehat{T_i})$. Choose $j\in A_\varepsilon$ and $\mu\in\mathcal{I}$ so that $\lambda\notin\mathcal{Z}(\widehat{T_\mu})$. Applying Proposition 13.8(i) with $U=T_j$ and $V=T_\mu$, one obtains

$$c_{\lambda,\eta}(T_j,f)=0\quad \text{for all } \eta\in\{0,\dots,m(\lambda,T_j)\}.$$

Combining this with (13.21), (6.5), (8.22), and Proposition 8.5(iii), we conclude that

$$f*W_j=0\quad \text{in } \big(-R+r(T_j),R-r(T_j)\big), \tag{18.7}$$

where $W_j=T_j^{\lambda,m(\lambda,T_j)}$. Notice that $\lambda\notin\mathcal{Z}(\widehat{W_j})$ (see relation (8.23)). Using now (18.7), (8.22), and Proposition 13.8(i) with $U=T$ and $V=W_j$, we arrive at (18.6). Thus, $f=0$ in view of Proposition 13.6(ii). In order to complete the proof of the theorem, it remains to consider the case where $r(T_i)<r^*(\mathcal{T})<+\infty$ for all $i\in\mathcal{I}$. Take $T\in\mathcal{T}$ arbitrarily and suppose that $\lambda\in\mathcal{Z}(\widehat{T})$. If there exists $\mu\in\mathcal{I}$

such that $r(T_\mu) \leqslant r(T)$ and $\lambda \notin \mathcal{Z}(\widehat{T_\mu})$, the above argument shows that (18.6) holds. Otherwise, $\lambda \in \mathcal{Z}(\widehat{T_i})$ for each $i \in \mathcal{I}$ such that $r(T_i) \leqslant r(T)$. As $\mathcal{Z}(\mathcal{T}) = \varnothing$, one has $\lambda \notin \mathcal{Z}(\widehat{T_\mu})$ for some $\mu \in \mathcal{I}$ such that $r(T_\mu) > r(T)$. Setting $\varepsilon = R - r(T_\mu)$, we deduce as before that (18.7) is valid for some $j \in A_\varepsilon$ (see Proposition 13.8(i) with $U = T_j$ and $V = T_\mu$). Again, using Proposition 13.8(i) with $U = T$ and $V = W_j$, we obtain (18.6). This, together with Proposition 13.6(ii), completes the proof. □

It is clear from Proposition 13.1(ii) that the assertions of Theorem 18.1 fail without the assumption $\mathcal{Z}(\mathcal{T}) = \varnothing$. We note that if $\mathcal{Z}(\mathcal{T}) = \varnothing$ and $\mathcal{Z}(\widehat{T_i}) \neq \varnothing$ for all $i \in \mathcal{I}$, then there exists an injective mapping $\mathcal{I} \to \mathbb{C}$. The following statement shows that the value R in Theorem 18.1 cannot be decreased in general (see also Theorems 18.8 and 18.9 below).

Theorem 18.2. *For each $\eta > 0$, there exists $T_\eta \in (\mathcal{E}'_\natural \cap \mathfrak{N})(\mathbb{R}^1)$ such that the following results are true.*

(i) $r(T_\eta) = 1$ *and* $\mathcal{Z}(\widehat{T_\eta}) \subset \mathbb{R}^1$ *for all* η.
(ii) $\mathcal{Z}(\widehat{T_{\eta_1}}) \cap \mathcal{Z}(\widehat{T_{\eta_2}}) = \varnothing$ *for* $\eta_1 \neq \eta_2$.
(iii) *There exists nontrivial* $f \in \mathcal{D}'(-2, 2)$ *such that* $f * T_\eta = 0$ *in* $(-1, 1)$ *for all* η.

Proof. For $\eta > 0$, we define $T_\eta \in (\mathcal{E}'_\natural \cap \mathfrak{N})(\mathbb{R}^1)$ by letting

$$\widehat{T_\eta}(z) = 1 - \mathrm{e}^\eta \cos z, \quad z \in \mathbb{C}. \tag{18.8}$$

Also let $T \in \mathfrak{N}(\mathbb{R}^1)$ be defined by $\widehat{T}(z) = \cos z$. It follows by (18.8) and Theorem 6.3 that $\mathcal{Z}(\widehat{T_\eta}) \subset \mathbb{R}^1$ and

$$r(T_\eta) = r(T) = 1$$

for all η. In addition, it is clear from (18.8) that (ii) is satisfied. Next, let $f = \zeta_T$. Using Theorem 8.5 and (13.5), we see that

$$f * T_\eta = \zeta_T$$

in $\mathbb{R}^1$ for all η. According to Proposition 8.20(v), f has the required property. This concludes the proof. □

Thus, the assumption on R in Theorem 18.1(i) cannot be weakened in the general case. However, for a broad class of families $\mathcal{T} \in \mathfrak{T}(\mathbb{R}^1)$, Theorem 18.1 remains to be true with $R > r(\mathcal{T})$ (see Corollary 18.1 below).

To continue, for $\mathcal{T} \in \mathfrak{T}(\mathbb{R}^1)$ and $\lambda \in \mathcal{Z}(\mathcal{T})$, we define

$$m(\lambda, \mathcal{T}) = \min_{i \in \mathcal{I}} m(\lambda, T_i).$$

For the rest of this section, we assume that $i_0 \in \mathcal{I}$ is fixed. The following result characterizes some classes of solutions of (18.4).

Theorem 18.3. *Assume that $\mathcal{T} \in \mathfrak{T}(\mathbb{R}^1)$ and let $T = T_{i_0}$. Then the following assertions hold.*

(i) *Assume that (18.5) is satisfied, let $T \in \mathfrak{M}(\mathbb{R}^1)$, and let $f \in \mathcal{D}'(-R, R)$. For f to belong to $\mathcal{D}'_{\mathcal{T}}(-R, R)$, it is necessary and sufficient that*

$$f = \sum_{\lambda \in \mathcal{Z}(\mathcal{T})} \sum_{\eta=0}^{m(\lambda,\mathcal{T})} \gamma_{\lambda,\eta} \mathrm{e}^{\lambda,\eta}, \tag{18.9}$$

where $\gamma_{\lambda,\eta} \in \mathbb{C}$, and the series in (18.9) converges in $\mathcal{D}'(-R, R)$. If $\mathcal{T} \in (\mathfrak{T}_2 \cup \mathfrak{T}_3)(\mathbb{R}^1)$, this statement remains valid with $R \in [R_{\mathcal{T}}, +\infty]$.

(ii) *Suppose that (18.5) is fulfilled, let $T \in \mathfrak{M}(\mathbb{R}^1)$, and let $f \in C^\infty(-R, R)$. Then $f \in C^\infty_{\mathcal{T}}(-R, R)$ if and only if relation (18.9) holds, where the series converges in $\mathcal{E}(-R, R)$. If $\mathcal{T} \in \mathfrak{T}_1(\mathbb{R}^1)$, the same is true for $R \in [R_{\mathcal{T}}, +\infty]$.*

(iii) *Assume that (18.5) holds true and let $\alpha > 0$, $T \in \mathfrak{G}_\alpha(\mathbb{R}^1)$, $r(T) > 0$, $f \in C^\infty(-R, R) \cap G^\alpha[-r(T), r(T)]$. Then $f \in C^\infty_{\mathcal{T}}(-R, R)$ if and only if equality (18.9) holds with the series converging in $\mathcal{E}(-R, R)$. If $\mathcal{T} \in \mathfrak{T}_1(\mathbb{R}^1)$, this assertion remains true with $R \in [R_{\mathcal{T}}, +\infty]$.*

(iv) *Assume that (18.3) is satisfied, let $T \in \mathfrak{M}(\mathbb{R}^1)$, and suppose that $f \in \mathcal{D}'(-R, R)$. Then $f \in \mathrm{QA}_{\mathcal{T}}(-R, R)$ if and only if decomposition (18.9) holds, where the constants $\gamma_{\lambda,\eta}$ satisfy*

$$\max_{0 \leqslant \eta \leqslant m(\lambda,\mathcal{T})} |\gamma_{\lambda,\eta}| \leqslant M_q (1 + |\lambda|)^{-q} \quad \textit{for all } \lambda \in \mathcal{Z}(\mathcal{T}),\ q \in \mathbb{N},$$

and

$$\sum_{p=1}^{\infty} \frac{1}{\inf_{q \geqslant p} M_q^{1/q}} = +\infty. \tag{18.10}$$

(v) *Let $T \in \mathfrak{E}(\mathbb{R}^1)$, $f \in \mathcal{D}'(\mathbb{R}^1)$, and assume that f is of finite order on $\mathbb{R}^1$. Then $f \in \mathcal{D}'_{\mathcal{T}}(\mathbb{R}^1)$ if and only if relation (18.9) is valid with the series converging in $\mathcal{E}(\mathbb{R}^1)$. In particular, if $f \in \mathcal{D}'_{\mathcal{T}}(\mathbb{R}^1)$, then $f \in C^\infty_{\mathcal{T}}(\mathbb{R}^1)$.*

Proof. The sufficiency in (i)–(v) is a consequence of Theorems 13.14 and 13.15. Let us prove the necessity. If $T \in \mathfrak{M}(\mathbb{R}^1)$ and $f \in \mathcal{D}'_{\mathcal{T}}(-R, R)$, then it follows by Theorems 13.14(i) and 13.9(ii) that

$$f = \sum_{\lambda \in \mathcal{Z}(\widehat{T})} \sum_{\eta=0}^{m(\lambda,T)} c_{\lambda,\eta}(T, f) \mathrm{e}^{\lambda,\eta}, \tag{18.11}$$

where the series converges in $\mathcal{D}'(-R, R)$. In addition, $T \notin \mathcal{D}(\mathbb{R}^1)$ and

$$|c_{\lambda,\eta}(T, f)| \leqslant (2 + |\lambda|)^{\gamma}, \tag{18.12}$$

where $\gamma > 0$ is independent of λ, η (see Theorem 8.5 and Corollary 13.3). Using now Proposition 13.8 with $U = T$ and $V = T_i$, $i \in \mathcal{I}\backslash\{i_0\}$, we obtain the necessity in (i). The proof of the necessity in (ii)–(v) is based on the same idea by using Theorems 13.14(ii), (iii), 13.15, and 8.1(i) and Corollary 13.3(ii)–(v). □

It can be shown that the assumptions on R in Theorem 18.3(i), (ii) cannot be weakened in general (see Theorem 18.9). The question when R can be decreased requires further study. We shall now discuss it when only (18.3) is fulfilled, which is a most restrictive case.

Theorem 18.4. *Let $\mathcal{T} \in \mathfrak{T}(\mathbb{R}^1)$, $T = T_{i_0}$, and assume that for all $\lambda \in \mathcal{Z}(\widehat{T})$, $\nu \in \mathcal{I}\backslash\{i_0\}$, the following estimates hold:*

$$\sum_{\eta=0}^{m(\lambda,T)} \left|\widehat{T}_\nu^{(\eta)}(\lambda)\right| \leqslant M_{q,\nu}(2+|\lambda|)^{-q}, \quad q = 1, 2, \dots, \tag{18.13}$$

where the constants $M_{q,\nu} > 0$ do not depend on λ, and

$$\sum_{p=1}^{\infty} \frac{1}{\inf_{q \geqslant p} M_{q,\nu}^{1/q}} = +\infty. \tag{18.14}$$

Then assertions (i) *and* (ii) *of Theorem 18.3 remain valid, provided that* (18.5) *is replaced by* (18.3).

Proof. It is enough to prove the necessity. If $f \in \mathcal{D}'_{\mathcal{T}}(-R, R)$ and $T \in \mathfrak{M}(\mathbb{R}^1)$, then from Theorems 13.14(i) and 13.9(ii) we have equality (18.11) with coefficients $c_{\lambda,\eta}(T, f)$ satisfying (18.12). For $R < +\infty$, equality (18.11) gives extension of f on $\mathbb{R}^1$ to a distribution in $\mathcal{D}'_{\mathcal{T}}(\mathbb{R}^1)$ (see Proposition 8.17(i)). Using now (18.13), (18.14), and (13.5), we see from Proposition 8.18 that $f * T_\nu \in \mathrm{QA}(\mathbb{R}^1)$ for all $\nu \in \mathcal{I}\backslash\{i_0\}$. Because of (18.4) and Theorem 8.1(i), this yields $f \in \mathcal{D}'_{\mathcal{T}}(\mathbb{R}^1)$. Similarly, each $f \in C^\infty_{\mathcal{T}}(-R, R)$ admits extension on $\mathbb{R}^1$ to a function in $C^\infty_{\mathcal{T}}(\mathbb{R}^1)$ (see (18.11) and Proposition 8.17(ii)). Theorem 18.3 is thereby established. □

Theorem 18.4 admits the following obvious but important consequence.

Corollary 18.1. *Assume that $\mathcal{T}$ satisfies all the assumptions in Theorem 18.4 and suppose that $T \in \mathfrak{M}(\mathbb{R}^1)$ and (18.3) is fulfilled. Then $\mathcal{D}'_{\mathcal{T}}(-R, R) = \{0\}$ if $\mathcal{Z}_{\mathcal{T}} = \varnothing$.*

The proof follows at once from (18.9).

Conditions (18.13) and (18.14) are valid, in particular, if zeros of $\widehat{T}_\nu$, $\nu \in \mathcal{I}\backslash\{i_0\}$, approach sufficiently fast to zeros $\lambda \in \mathcal{Z}(\widehat{T})$ as $\lambda \to \infty$. Owing to the following result, we see that assumption (18.14) in Theorem 18.4 cannot be weakened.

Theorem 18.5. *Assume that $\mathcal{I}\backslash\{i_0\} \neq \varnothing$. Then for each sequence $\{M_q\}_{q=1}^\infty$ of positive numbers satisfying*

$$\sum_{p=1}^{\infty} \frac{1}{\inf_{q \geqslant p} M_q^{1/q}} < +\infty, \tag{18.15}$$

there exist $T_{i_0} \in (\mathcal{E}'_\natural \cap \mathfrak{N})(\mathbb{R}^1)$ *and* $T_\nu \in \mathcal{E}'_\natural(\mathbb{R}^1)\backslash\{0\}$, $\nu \in \mathcal{I}\backslash\{i_0\}$, *such that the family* $\mathcal{T} = \{T_i\}_{i \in \mathcal{I}}$ *possesses the following properties:*

(1) *for each* $\nu \in \mathcal{I}\backslash\{i_0\}$, *estimate* (18.13) *holds with* $M_{q,\nu} = M_q$*;*
(2) $r^*(\mathcal{T}) < +\infty$, $\mathcal{Z}_{\mathcal{T}} = \varnothing$, *and* $C^\infty_{\mathcal{T}}(-R, R) \neq \{0\}$ *for some* $R > r^*(\mathcal{T})$.

For a proof, we refer the reader to V.V. Volchkov [225], Part III, Theorem 1.10.

To continue, for $\mathcal{T} \in \mathfrak{T}(\mathbb{R}^1)$ and $R \in (0, +\infty]$, denote by $\mathcal{W}_{\mathcal{T},R}$ the set of all distributions $w \in \mathcal{E}'(\mathbb{R}^1)$ with the following properties:

(a) $\operatorname{supp} w \subset [-r(T), r(T)]$, where $T = T_{i_0}$;
(b) for each $\nu \in \mathcal{I}\backslash\{i_0\}$, there exist $w_{1,\nu}, w_{2,\nu} \in \mathcal{E}'(\mathbb{R}^1)$ such that

$$w_{1,\nu} = 0 \quad \text{in } \left\{t \in \mathbb{R}^1 \colon |t| > r(T) + r(T_\nu) - R\right\} \tag{18.16}$$

and

$$T * w_{2,\nu} + w_{1,\nu} = T_\nu * w. \tag{18.17}$$

Also let

$$\mathcal{W}_{\mathcal{T},R,\natural} = \mathcal{W}_{\mathcal{T},R} \cap \mathcal{E}'_\natural\left(\mathbb{R}^1\right).$$

We point out that the sets $\mathcal{W}_{\mathcal{T},R}$ and $\mathcal{W}_{\mathcal{T},R,\natural}$ depend on $i_0 \in \mathcal{I}$.

Theorem 18.6. *Let* $\mathcal{T} \in \mathfrak{T}(\mathbb{R}^1)$, $T = T_{i_0} \in (\mathrm{Inv}_+ \cap \mathrm{Inv}_-)(\mathbb{R}^1)$, *and assume that* (18.3) *is satisfied. Then the following statements are valid.*

(i) *If* $f \in \mathcal{D}'(-R, R)$, *then for* f *to belong to* $\mathcal{D}'_{\mathcal{T}}(-R, R)$, *it is necessary and sufficient that*

$$f = \zeta_T * w \quad \textit{in } (-R, R) \tag{18.18}$$

for some $w \in \mathcal{W}_{\mathcal{T},R}$.

(ii) $\mathcal{D}'_{\mathcal{T}}(-R, R) = \{0\}$ *if and only if for each* $w \in \mathcal{W}_{\mathcal{T},R}$, *there exists* $\varphi \in \mathcal{E}'(\mathbb{R}^1)$ *such that*

$$w = T * \varphi.$$

Proof. To show (i), first suppose $f \in \mathcal{D}'_{\mathcal{T}}(-R, R)$. Using Theorem 13.13, we obtain (18.18) for some $w \in \mathcal{E}'(\mathbb{R}^1)$ with $\operatorname{supp} w \subset [-r(T), r(T)]$. Let us extend f on $\mathbb{R}^1$ by formula (18.18). Then $f \in \mathcal{D}'_T(\mathbb{R}^1)$ and $f * T_\nu \in \mathcal{D}'_T(\mathbb{R}^1)$ for all $\nu \in \mathcal{I}$. However,

$$f * T_\nu = \zeta_T * w * T_\nu = 0$$

in $(r(T_\nu) - R, R - r(T_\nu))$. Theorem 13.4 shows that

$$f * T_\nu = \zeta_T * w_{1,\nu}$$

for some $w_{1,\nu} \in \mathcal{E}'(\mathbb{R}^1)$ satisfying (18.16). Hence,

$$\zeta_T * (w_{1,\nu} - T_\nu * w) = 0$$

and $w \in \mathcal{W}_{\mathcal{T}}$ because of Proposition 8.20(iv). Conversely, if (18.18) holds for some $w \in \mathcal{W}_{\mathcal{T},R}$, then (18.17), (18.16), and Proposition 8.20(v) imply that $f \in \mathcal{D}'_{\mathcal{T}}(-R, R)$, proving (i).

Part (ii) is evident from (i) and Proposition 8.20(v). This finishes the proof. □

18.2 Pairwise Mean Periodic Functions

Throughout the section we assume that

$$\mathcal{T} = \{T_1, T_2\}$$

is a family of nonzero distributions in $\mathcal{E}'(\mathbb{R}^1)$ such that $\operatorname{supp} T_\nu \subset [-r_\nu, r_\nu]$, where $r_\nu = r(T_\nu)$, $\nu \in \{1, 2\}$. As before, we set

$$\mathcal{Z}(\mathcal{T}) = \mathcal{Z}\big(\widehat{T}_1\big) \cap \mathcal{Z}\big(\widehat{T}_2\big)$$

and

$$r^*(\mathcal{T}) = \max\{r_1, r_2\}.$$

In this section we shall investigate the problem of existence of a nonzero solution of the system

$$(f * T_\nu)(t) = 0, \qquad |t| < R - r_\nu, \quad \nu = 1, 2,$$

where $r^*(\mathcal{T}) < R \leqslant +\infty$. By analogy with the previous section we define

$$\mathcal{D}'_{\mathcal{T}}(-R, R) = \big(\mathcal{D}'_{T_1} \cap \mathcal{D}'_{T_2}\big)(-R, R), \quad \mathrm{QA}_{\mathcal{T}}(-R, R) = \big(\mathcal{D}'_{\mathcal{T}} \cap \mathrm{QA}\big)(-R, R),$$

and

$$C^m_{\mathcal{T}}(-R, R) = \big(\mathcal{D}'_{\mathcal{T}} \cap C^m\big)(-R, R)$$

for $m \in \mathbb{Z}_+$ or $m = \infty$.

Theorem 18.1(i) shows that $\mathcal{D}'_{\mathcal{T}}(-R, R) = \{0\}$, provided that $R > r_1 + r_2$ and $\mathcal{Z}(\mathcal{T}) = \varnothing$. If $\mathcal{Z}(\mathcal{T}) \neq \varnothing$, then for each $\lambda \in \mathcal{Z}(\mathcal{T})$, the function $f(t) = e^{i\lambda t}$ is in $(\mathcal{D}'_{\mathcal{T}} \cap \mathrm{RA})(\mathbb{R}^1)$. Therefore, we restrict ourselves to only the case where

$$\mathcal{Z}(\mathcal{T}) = \varnothing \quad \text{and} \quad r^*(\mathcal{T}) < R \leqslant r_1 + r_2.$$

Theorem 18.7. *Let $\mathcal{Z}(\mathcal{T}) = \varnothing$, $T_1 \in (\mathrm{Inv}_+ \cap \mathrm{Inv}_-)(\mathbb{R}^1)$, and $R = r_1 + r_2 > r^*(\mathcal{T})$. Assume that there exists a sequence $\zeta_1, \zeta_2, \dots$ of complex numbers such that*

$$(2 + |\zeta_k|)^\alpha \big(\big|\widehat{T}_1(\zeta_k)\big| + \big|\widehat{T}_2(\zeta_k)\big|\big) \to 0 \quad \text{as } k \to \infty \tag{18.19}$$

for each $\alpha > 0$, and

$$|\operatorname{Im} \zeta_k| \leqslant c \log(2 + |\zeta_k|), \tag{18.20}$$

where the constant $c > 0$ is independent of k. Then $\mathcal{D}'_{\mathcal{T}}(-R, R) = \{0\}$.

Proof. Let $f \in \mathcal{D}'_{\mathcal{T}}(-R, R)$. Theorem 18.6(i) shows that (18.18) holds with $T = T_1$ and $w \in \mathcal{W}_{\mathcal{T},R}$. Owing to (18.17), (18.16), and Corollary 6.2,

$$\widehat{T}_1(\zeta_k)\widehat{w}_{2,2}(\zeta_k) + \widehat{w}_{1,2}(\zeta_k) = \widehat{T}_2(\zeta_k)\widehat{w}(\zeta_k)$$

for all $k \in \mathbb{N}$, where $\widehat{w}_{1,2}$ is a polynomial. Relations (18.19) and (18.20) and Theorem 6.3 ensure us that $\widehat{w}_{1,2}(\zeta_k) \to 0$ and $\zeta_k \to \infty$ as $k \to \infty$. Thus, $\widehat{w}_{1,2} = 0$ and

$$T_1 * w_{2,2} = T_2 * w$$

(see (18.17)). Bearing in mind that $\mathcal{Z}(\mathcal{T}) = \varnothing$ and $T_1 \in \mathrm{Inv}(\mathbb{R}^1)$, we see from Theorem 6.4 that $w = T_1 * \varphi$ for some $\varphi \in \mathcal{E}'(\mathbb{R}^1)$. Now it follows from (18.18) and Proposition 8.20(i) that $f = 0$. Hence the theorem. □

We shall now show that the assumptions in Theorem 18.7 cannot be considerably weakened.

To begin with, we introduce some notion. Let $f_j : \mathbb{C} \to \mathbb{C}$, $j = 1, 2$, be nonzero entire functions. We shall write $\mathcal{Z}(f_1) \approx \mathcal{Z}(f_2)$ if for each $\alpha > 0$, there exists $\lambda \in \mathcal{Z}(f_1)$ such that $|f_2(\lambda)| < (2 + |\lambda|)^{-\alpha}$. Otherwise, we shall write $\mathcal{Z}(f_1) \not\approx \mathcal{Z}(f_2)$.

Suppose that $T_1 \in \mathfrak{N}(\mathbb{R}^1)$ and $\lambda \in \mathcal{Z}(\widehat{T}_1)$. By the definition of $\mathfrak{N}(\mathbb{R}^1)$ one infers that the estimates

$$|\mathrm{Im}\,\lambda| + \frac{1}{|\widehat{T}_1^{(n_\lambda)}(\lambda)|} \leqslant (2 + |\lambda|)^{\gamma_1}, \quad n_\lambda \leqslant \gamma_2, \tag{18.21}$$

hold with $\gamma_1, \gamma_2 > 0$ independent of λ (here and below we write n_λ instead of $n_\lambda(\widehat{T}_1)$). Using Theorem 6.3, we see from (18.21) and Proposition 6.10 that for all $\lambda \in \mathcal{Z}(\widehat{T}_1)$ and $\eta \in \{0, \dots, m(\lambda, T_1)\}$, the estimate

$$\left|\widehat{T}_2^{(\eta)}(\lambda)\right| \leqslant (2 + |\lambda|)^{\gamma_3} \tag{18.22}$$

holds with $\gamma_3 > 0$ independent of λ, η. For $\lambda \in \mathcal{Z}(\widehat{T}_1) \backslash \mathcal{Z}(\widehat{T}_2)$, we now define the sequence $\omega_{\lambda,\eta} = \omega_{\lambda,\eta}(T_1, T_2) \in \mathbb{C}$, $\eta \in \{0, \dots, n_\lambda - 1\}$, as follows. Put

$$\omega_{\lambda,n_\lambda-1} = \frac{a^{\lambda,n_\lambda-1}_{n_\lambda-1}(\widehat{T}_1)}{\widehat{T}_2(\lambda)}. \tag{18.23}$$

Next, if $n_\lambda \geqslant 2$, we set

$$\omega_{\lambda,\eta} = \frac{a^{\lambda,\eta}_{n_\lambda-1}(\widehat{T}_1)}{\widehat{T}_2(\lambda)} - \sum_{\nu=\eta+1}^{n_\lambda-1} \binom{\nu}{\eta} \frac{\widehat{T}_2^{(\nu-\eta)}(\lambda)\omega_{\lambda,\nu}}{\widehat{T}_2(\lambda)}, \quad \eta \in \{0, \dots, n_\lambda - 2\}. \tag{18.24}$$

Assume now that $\mathcal{Z}(\widehat{T}_1) \not\approx \mathcal{Z}(\widehat{T}_2)$. Then relations (18.21)–(18.24) imply that

$$|\omega_{\lambda,\eta}(T_1, T_2)| \leqslant (2 + |\lambda|)^{\gamma_4}, \tag{18.25}$$

where $\gamma_4 > 0$ is independent of λ, η. Define $\Omega_{T_1,T_2} \in \mathcal{D}'(\mathbb{R}^1)$ by the formula

$$\Omega_{T_1,T_2} = \sum_{\lambda\in\mathcal{Z}(\widehat{T}_1)} \sum_{\eta=0}^{m(\lambda,T_1)} \omega_{\lambda,\eta}(T_1,T_2)e^{\lambda,\eta}. \tag{18.26}$$

It follows from Proposition 8.17(i) and (18.25) that the series in (18.26) converges in $\mathcal{D}'(\mathbb{R}^1)$. By (13.5), Theorem 8.5, and the definition of $\omega_{\lambda,\eta}(T_1,T_2)$ we see that

$$\Omega_{T_1,T_2} \in \mathcal{D}'_{T_1}(\mathbb{R}^1) \quad \text{and} \quad \Omega_{T_1,T_2} * T_2 = \zeta_{T_1} \quad \text{in } \mathbb{R}^1. \tag{18.27}$$

Theorem 18.8. *Let $\mathcal{Z}(\mathcal{T}) = \varnothing$ and $T_1 \in \mathfrak{N}(\mathbb{R}^1)$. Then the following statements are valid.*

(i) *If $R = r_1 + r_2 > r^*(\mathcal{T})$ and $\mathcal{Z}(\widehat{T}_1) \approx \mathcal{Z}(\widehat{T}_2)$, then $\mathcal{D}'_{\mathcal{T}}(-R,R) = \{0\}$. This occurs, in particular, if $T_2 \in \mathcal{D}(\mathbb{R}^1)$.*

(ii) *Assume that $r^*(\mathcal{T}) < R \leqslant r_1 + r_2$, $\mathcal{Z}(\widehat{T}_1) \not\approx \mathcal{Z}(\widehat{T}_2)$, and let $f \in \mathcal{D}'(-R,R)$. Then $f \in \mathcal{D}'_{\mathcal{T}}(-R,R)$ if and only if*

$$f = \Omega_{T_1,T_2} * u \quad \text{in } (-R,R) \tag{18.28}$$

for some $u \in \mathcal{E}'(\mathbb{R}^1)$ with $\operatorname{supp} u \subset [R - r_1 - r_2, r_1 + r_2 - R]$. In particular, $C^\infty_{\mathcal{T}}(-R,R) \neq \{0\}$ if $r^*(\mathcal{T}) < R < r_1 + r_2$.

(iii) *If $r^*(\mathcal{T}) < R = r_1 + r_2$, and $\mathcal{Z}(\widehat{T}_1) \not\approx \mathcal{Z}(\widehat{T}_2)$, then there exists $m = m(\mathcal{T}) \in \mathbb{Z}_+$ such that $C^m_{\mathcal{T}}(-R,R) = \{0\}$.*

Proof. Part (i) is immediate from Theorem 18.7. To prove (ii), first suppose that $f \in \mathcal{D}'_{\mathcal{T}}(-R,R)$. It follows by Theorem 13.16(i) that f admits extension on $\mathbb{R}^1$ to distribution in $\mathcal{D}'_{T_1}(\mathbb{R}^1)$. Then $f * T_2 \in \mathcal{D}'_{T_1}(\mathbb{R}^1)$ and $f * T_2 = 0$ in $(-r_1, r_1)$.

Owing to Theorem 13.4(i),

$$f * T_2 = \zeta_{T_1} * u \quad \text{in } \mathbb{R}^1 \tag{18.29}$$

for some $u \in \mathcal{E}'(\mathbb{R}^1)$ with $\operatorname{supp} u \subset [R-r_1-r_2, r_1+r_2-R]$. By (18.27) and (18.29) we obtain

$$f * T_2 = \Omega_{T_1,T_2} * T_2 * u \quad \text{in } \mathbb{R}^1,$$

so

$$f - \Omega_{T_1,T_2} * u \in \mathcal{D}'_{\mathcal{T}}(\mathbb{R}^1).$$

In view of Theorem 18.1(i), this yields (18.28). In addition, Proposition 8.20(v), (18.27), and (18.28) imply that $\Omega_{T_1,T_2} * u$ in $\mathcal{D}'_{\mathcal{T}}(-R,R)$ for each $u \in \mathcal{E}'(\mathbb{R}^1)$ with $\operatorname{supp} u \subset [R - r_1 - r_2, r_1 + r_2 - R]$. If $R < r_1 + r_2$, we can choose $u \in \mathcal{D}(\mathbb{R}^1)$ with this property such that

$$(-R,R) \cap \operatorname{supp}(\Omega_{T_1,T_2} * u) \neq \varnothing$$

(see (18.27), (13.5) and Theorem 13.9). Thus, (ii) is proved.

Assume now that $r^*(\mathcal{T}) < R = r_1 + r_2$ and $\mathcal{Z}(\widehat{T}_1) \not\approx \mathcal{Z}(\widehat{T}_2)$. By (ii) and Corollary 6.2 each $f \in \mathcal{D}'_{\mathcal{T}}(-R,R)$ has the form

$$f = p\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\Omega_{T_1,T_2}$$

for some polynomial p. Suppose that $\Omega_{T_1,T_2} \in C^\infty(-R, R)$. Then by Theorem 13.16(ii) and (18.27) we have $\zeta_{T_1} \in C^\infty(\mathbb{R}^1)$. This contradicts Proposition 8.20(v) and Corollary 13.2. Thus, there is $m = m(\mathcal{T}) \in \mathbb{Z}_+$ such that $C^m_{\mathcal{T}}(-R, R) = \{0\}$. This completes the proof. □

Theorem 18.8 implies that $C^\infty_{\mathcal{T}}(-R, R) = \{0\}$, provided that $\mathcal{Z}(\mathcal{T}) = \varnothing$, $T_1 \in \mathfrak{N}(\mathbb{R}^1)$, and $R = r_1 + r_2 > r(\mathcal{T})$. We shall now show that the value R in this statement cannot be decreased in general.

Definition 18.1. A set $E \subset \mathbb{C}$ is called a *sparse set* if either $E = \varnothing$ or for each $\varepsilon > 0$, there exists nonzero $T_\varepsilon \in \mathcal{E}'(\mathbb{R}^1)$ such that

$$\operatorname{supp} T_\varepsilon \subset [-\varepsilon, \varepsilon] \quad \text{and} \quad \widehat{T_\varepsilon}(\lambda) = 0 \quad \text{for all } \lambda \in E. \tag{18.30}$$

It is easy to verify that every finite set is sparse. Furthermore, if $E \subset \mathbb{C}$ is sparse and infinite, then E has the form

$$E = \{\lambda_1, \lambda_2, \ldots\}, \quad \text{where } |\lambda_1| \leqslant |\lambda_2| \leqslant \cdots \text{ and } \lim_{n\to\infty} |\lambda_n| = +\infty. \tag{18.31}$$

Proposition 18.1. *Let $E \subset \mathbb{C}$ be sparse. Then the following statements are valid.*

(i) *The sets $-E = \{z \in \mathbb{C} : -z \in E\}$ and $(-E) \cup E$ are sparse.*
(ii) *If $E_1 \subset E$, then E_1 is sparse.*
(iii) *If $c(R, E)$ is the cardinality of the set $\{z \in E : |z| < R\}$, then*

$$\lim_{R\to+\infty} \frac{c(R, E)}{R} = 0.$$

Proof. To prove (i) it is enough to consider the functions $\widehat{T_\varepsilon}(-z)$ and $\widehat{T_\varepsilon}(-z)\widehat{T_\varepsilon}(z)$ where $T_\varepsilon \in \mathcal{E}'(\mathbb{R}^1)$ satisfies (18.30) (see Theorem 6.3). Assertion (ii) is obvious from Definition 18.1. Using now Theorem 6.3 and Proposition 6.1(ii), (iii), we arrive at (iii). □

Let us now present some necessary and sufficient conditions for sparseness of $E \subset \mathbb{C}$.

Proposition 18.2. *Assume that $E \subset \mathbb{C}$ has the form (18.31). Then the following statements are valid.*

(i) *If*

$$\sum_{n=1}^{\infty} (1 + |\lambda_n|)^{-1} < +\infty, \tag{18.32}$$

then E is sparse. Moreover, if $E \subset \{z \in \mathbb{C}\backslash\{0\} : |\arg z - \pi/2| \leqslant \pi/2 - \varepsilon\}$ for some $\varepsilon \in (0, \pi/2)$, then E is sparse if and only if (18.32) holds.
(ii) *If $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ and $\lambda_{n+1} - \lambda_n \to +\infty$ as $n \to \infty$, then E is sparse.*

Proof. Relation (18.32) implies that for all $a, b \in \mathbb{R}^1$, $a < b$, there exists a function $u \in C(\mathbb{R}^1)$ such that $\operatorname{supp} u \subset [a, b]$ and $\widehat{u}(\lambda_n) = 0$ for all n (see [144, Theorem 1.4.3]). This, together with [145, Appendix III, Sect. 1], yields (i). Next, it is not difficult to adapt the argument in the proof of Lemma 5.3.1 in [225] to show (ii). □

Theorem 18.9. *Let $r^*(\mathcal{T}) < R < r_1 + r_2$, $T_1 \in \mathfrak{N}(\mathbb{R}^1)$, and assume that there exists $l > 0$ such that the set*

$$E_l(\mathcal{T}) = \left\{\lambda \in \mathcal{Z}\big(\widehat{T_1}\big) : \big|\widehat{T_2}(\lambda)\big| < (2 + |\lambda|)^{-l}\right\}$$

is sparse. Also let $\{M_q\}_{q=1}^{\infty}$ be a sequence of positive numbers satisfying (18.15). Then there exists a nontrivial function $f \in C^{\infty}_{\mathcal{T}}(-R, R)$ such that

$$\sup_{t \in (-R,R)} \left|f^{(q)}(t)\right| \leqslant M_q \quad \textit{for all } q. \tag{18.33}$$

If, in addition, $T_1, T_2 \in \mathcal{E}'_{\natural}(\mathbb{R}^1)$, then f can be chosen even.

Proof. Let $0 < \varepsilon < (r_1 + r_2 - R)/3$, $l > 0$, and assume that $E_l(\mathcal{T})$ is sparse. Select $s \in \mathbb{N}$ so that $s > n_\lambda(\widehat{T_1})$ for all $\lambda \in \mathcal{Z}(\widehat{T_1})$. Then for each $\delta \in (0, \varepsilon/(2s+1))$, there exists nonzero $u \in \mathcal{E}'(-\delta, \delta)$ such that $\widehat{u}(\lambda) = 0$ for all $\lambda \in E_l(\mathcal{T})$. Take nonzero even $v \in \mathcal{D}(-\delta, \delta)$ and define $\varphi \in \mathcal{D}(-\varepsilon, \varepsilon)$ by the relation $\widehat{\varphi}(z) = (\widehat{u}(z)\widehat{u}(-z))^s \widehat{v}(z)$ (see Theorem 6.3). Then the function φ is even, and $\widehat{\varphi}^{(j)}(\lambda) = 0$ for all $\lambda \in E_l(\mathcal{T})$, $j \in \{0, \ldots, n_\lambda(\widehat{T_1})\}$. Setting

$$f_1 = \zeta_{T_1} * \varphi',$$

we deduce from Proposition 8.20(v) and Corollary 13.2 that $f_1 = 0$ in $(-r_1+\varepsilon, r_1-\varepsilon)$ and $\operatorname{supp} f_1 \neq \varnothing$. Proposition 13.5(iii) implies that

$$c_{\lambda,\eta}(T_1, f_1) = 0$$

for all $\lambda \in E_l(\mathcal{T})$ and $\eta \in \{0, \ldots, m(\lambda, T_1)\}$. In addition, for each $\alpha > 0$,

$$|c_{\lambda,\eta}(T_1, f_1)| \leqslant c_1(2 + |\lambda|)^{-\alpha}, \quad \lambda \in \mathcal{Z}\big(\widehat{T_1}\big),\ \eta \in \{0, \ldots, m(\lambda, T_1)\}, \tag{18.34}$$

where $c_1 > 0$ is independent of λ, η (see (6.34)).

Next, let $\lambda \in \mathcal{Z}(\widehat{T_1}) \backslash E_l(\mathcal{T})$, $\eta \in \{0, \ldots, m(\lambda, T_1)\}$, and let $\gamma_{\lambda,\eta} \in \mathbb{C}$ be defined by

$$\sum_{\eta=\nu}^{m(\lambda,T_1)} \gamma_{\lambda,\eta} \binom{\nu}{\eta} \widehat{T}_2^{(\eta-\nu)}(\lambda) = c_{\lambda,\eta}(T_1, f_1), \quad \nu = 0, \ldots, m(\lambda, T_1). \tag{18.35}$$

Since $\lambda \notin E_l(\mathcal{T})$, it follows from (18.35), (18.34) and (18.22) that for each $\beta > 0$,

$$|\gamma_{\lambda,\eta}| \leqslant c_2(2 + |\lambda|)^{-\beta},$$

where $c_2 > 0$ is independent of λ, η. Using Proposition 8.17(ii), we now define $f_2 \in C^\infty_{T_1}(\mathbb{R}^1)$ by the formula

$$f_2 = \sum_{\lambda \in \mathcal{Z}(\widehat{T}_1) \setminus E_l(\mathcal{T})} \sum_{\eta=0}^{m(\lambda, T_1)} \gamma_{\lambda,\eta} \mathrm{e}^{\lambda,\eta}.$$

In view of (13.5) and (18.35),

$$f_2 * T_2 = f_1.$$

Hence,

$$f_2 \in C^\infty_{\mathcal{T}}(\varepsilon - r_1 - r_2, r_1 + r_2 - \varepsilon).$$

We claim that $[-r_1, r_2] \cap \operatorname{supp} f_2 \neq \varnothing$. Otherwise, Theorem 13.9(ii), together with (18.35), yields

$$c_{\lambda,\eta}(T_1, f_1) = 0$$

for all $\lambda \in \mathcal{Z}(\widehat{T_1})$ and $\eta \in \{0, \ldots, m(\lambda, T_1)\}$. Then by Proposition 13.6(ii) we have $f_1 = 0$, which is a contradiction.

To continue, by Theorems 8.1(ii) and 6.1(i) we can select $h \in \mathcal{D}_\natural(-\varepsilon, \varepsilon)$, $h \geqslant 0$, so that the convolution $f_3 = f_2 * h$ is nontrivial in $(-R, R)$ and

$$\left|h^{(q)}(t)\right| \leqslant M_q$$

for all $t \in (-\varepsilon, \varepsilon)$. Then the function $f = cf_3$ with

$$c = \left(\int_{-R-\varepsilon}^{R+\varepsilon} |f_3(t)| \, \mathrm{d}t\right)^{-1}$$

is in $C^\infty_{\mathcal{T}}(-R, R)$, and (18.33) is fulfilled.

Moreover, if $T_1, T_2 \in \mathcal{E}'_\natural(\mathbb{R}^1)$, then f is even. Theorem 18.9 is thereby established. □

Several remarks are in order here.

Remark 18.1. Theorem 18.9 remains valid if $T_1 \in \mathfrak{N}(\mathbb{R}^1)$ and for some $l > 0$,

$$\sum_{\lambda \in E_l(\mathcal{T})} (1 + |\lambda|)^{-1} < +\infty$$

(see Proposition 18.2(i)).

Remark 18.2. For the case where $\mathcal{Z}(\mathcal{T}) = \varnothing$, Theorem 18.9 fails without assumption (18.15). In fact, if $f \in C^\infty_{\mathcal{T}}(-R, R)$ and (18.10) is valid, then (18.33) implies that $f \in \mathrm{QA}_{\mathcal{T}}(-R, R)$. Now it follows from Theorem 18.3(iv) that $f = 0$.

Remark 18.3. If $\mathcal{Z}(\mathcal{T}) \neq \varnothing$, then Theorem 18.9 is true for each $R > r(\mathcal{T})$ without any additional assumptions on $T_1, T_2 \in \mathcal{E}'(\mathbb{R}^1)$ (see Proposition 13.1(ii)). However, for $\mathcal{Z}(\mathcal{T}) = \varnothing$, Theorem 18.9 fails in general without assumption concerning the sparseness of $E_l(\mathcal{T})$ for some $l > 0$ (see Theorem 18.10 below).

We shall now consider the case where

$$\mathcal{D}'_{\mathcal{T}}(-R, R) = \{0\} \quad \text{for } r^*(\mathcal{T}) < R < r_1 + r_2.$$

Theorem 18.10.

(i) *Let* $\mathcal{Z}(\mathcal{T}) = \varnothing$, $T_1 \in \mathfrak{N}(\mathbb{R}^1)$, *and assume that* $T_1 = \psi_1 * \psi_2$, *where* $\psi_1 \in \mathfrak{N}(\mathbb{R}^1)$, $\psi_2 \in \mathcal{E}'(\mathbb{R}^1)$, *and* $\mathcal{Z}(\widehat{\psi_1}) \cap \mathcal{Z}(\widehat{\psi_2}) = \varnothing$. *Also suppose that for each* $\lambda \in \mathcal{Z}(\widehat{\psi_2})$,

$$\sum_{\eta=0}^{m(\lambda,\psi_2)} \left|\widehat{T}_2^{(\eta)}(\lambda)\right| \leqslant M_q(2+|\lambda|)^{-q}, \quad q = 1, 2, \ldots, \tag{18.36}$$

where the constants $M_q > 0$ *do not depend on* λ *and satisfy* (18.10). *Then* $\mathcal{D}'_{\mathcal{T}}(-R, R) = \{0\}$ *for* $R > \max\{r_1, r_2 + r(\psi_1)\}$. *In addition,* $C^\infty_{\mathcal{T}}(-R, R) = \{0\}$ *if* $R \geqslant r_2 + r(\psi_1)$ *and* $R > r^*(\mathcal{T})$.

(ii) *There exists a family* $\mathcal{T} = \{T_1, T_2\}$ *satisfying all the assumptions in* (i) *such that* T_1 *and* T_2 *are even,* $r(T_2) + r(\psi_1) > r^*(\mathcal{T})$, *and* $\mathcal{D}'_{\mathcal{T}}(-R, R) \neq \{0\}$ *for* $R = r(T_2) + r(\psi_1)$.

(iii) *There exists a family* $\mathcal{T} = \{T_1, T_2\}$ *satisfying all the assumptions in* (i) *such that* $T_1, T_2 \in \mathcal{E}'_\natural(\mathbb{R}^1)$, $r(T_2) + r(\psi_1) > r^*(\mathcal{T})$, *and* $C^\infty_{\mathcal{T}}(-R, R) \neq \{0\}$ *for each* $R \in (r^*(\mathcal{T}), r(T_2) + r(\psi_1))$.

Proof. The proof of (i) can be found in [225, Part III, Sect. 3.1]. To prove (ii) we define $\psi_1, \psi_2 \in (\mathcal{E}'_\natural \cap \mathfrak{N})(\mathbb{R}^1)$ by

$$\widehat{\psi_1}(z) = \cos \pi z, \quad \widehat{\psi_2}(z) = \frac{\sin(\pi z/2)}{z}, \quad z \in \mathbb{C}.$$

The rest of the proof of (ii) now duplicates Theorem 3.1.14 of [225]. Part (iii) can be obtained from (ii) by means of the standard smoothing method (see Theorem 6.1). □

The following result shows that assumption (18.10) in Theorem 18.10 cannot be weakened.

Theorem 18.11. *For each sequence* $\{M_q\}_{q=1}^\infty$ *of positive numbers satisfying* (18.15), *there exist* $T_1, \psi_1, \psi_2 \in (\mathcal{E}'_\natural \cap \mathfrak{N})(\mathbb{R}^1)$, *and* $T_2 \in \mathcal{D}_\natural(\mathbb{R}^1)$ *with the following properties.*

(1) *Estimates* (18.36) *hold for all* $\lambda \in \mathcal{Z}(\widehat{\psi_2})$.
(2) $\mathcal{Z}(\widehat{\psi_1}) \cap \mathcal{Z}(\widehat{\psi_2}) = \varnothing$.
(3) $T_1 = \psi_1 * \psi_2$ *and* $\mathcal{Z}(\widehat{T_1}) \cap \mathcal{Z}(\widehat{T_2}) = \varnothing$.
(4) $C^\infty_{\mathcal{T}}(-R, R) \neq \{0\}$ *for some* $R > \max\{r(T_1), r(T_2) + r(\psi_1)\}$, *where* $\mathcal{T} = \{T_1, T_2\}$.

For a proof see V.V. Volchkov [225], Part III, Theorem 1.16.

To conclude we complement Theorem 18.10(i) by the following result.

Theorem 18.12. *Let* $\mathcal{Z}(\mathcal{T}) = \varnothing$, $T_1 \in (\mathrm{Inv}_+ \cap \mathrm{Inv}_-)(\mathbb{R}^1)$, *and assume that there exists* $\psi \in \mathfrak{M}(\mathbb{R}^1)$ *such that* $r(\psi) > 0$ *and for each* $\lambda \in \mathcal{Z}(\widehat{\psi})$,

$$\sum_{\eta=0}^{m(\lambda,\psi)} \left(\left|\widehat{T}_1^{(\eta)}(\lambda)\right| + \left|\widehat{T}_2^{(\eta)}(\lambda)\right|\right) \leqslant M_q(2+|\lambda|)^{-q}, \quad q = 1, 2, \ldots, \tag{18.37}$$

where $M_q > 0$ *are independent of* λ *and satisfy* (18.10). *Then* $\mathcal{D}'_{\mathcal{T}}(-R, R) = \{0\}$ *for* $R > \max\{r^*(\mathcal{T}), r_1 + r_2 - r(\psi)\}$. *This statement is no longer valid in general if*

$$r^*(\mathcal{T}) < R = r_1 + r_2 - r(\psi).$$

Proof. Let $f \in \mathcal{D}'_{\mathcal{T}}(-R, R)$, $R > \max\{r^*(\mathcal{T}), r_1+r_2-r(\psi)\}$. By Theorem 18.6(i) we have (18.18) with $T = T_1$ and $w \in \mathcal{W}_{\mathcal{T},R}$. Relation (18.17) yields

$$\widehat{T}_1\widehat{w}_{2,2} + \widehat{w}_{1,2} = \widehat{T}_2\widehat{w}. \tag{18.38}$$

Using now (18.37), Proposition 6.10, and (8.6), we obtain

$$\left|\widehat{w}_{1,2}^{(s)}(\lambda)\right| \leqslant M_q(2+|\lambda|)^{c_1-q}, \quad q = 1, 2, \ldots, \tag{18.39}$$

where $\lambda \in \mathcal{Z}(\widehat{\psi})$, $s \in \{0, \ldots, m(\lambda, \psi)\}$, and the constant $c_1 > 0$ is independent of λ, q, s. Now define $g = \zeta_\psi * w_{1,2}$. In view of Proposition 13.5(iii) and Theorem 8.5, one has

$$c_{\lambda,\eta}(\psi, g) = \sum_{\nu=\eta}^{m(\lambda,\psi)} \binom{\nu}{\eta} a^{\lambda,\nu}_{m(\lambda,\psi)}(\widehat{\psi})\widehat{w}_{1,2}^{(\nu-\eta)}(\lambda)$$

for all $\lambda \in \mathcal{Z}(\widehat{\psi})$ and $\eta \in \{0, \ldots, m(\lambda, \psi)\}$. Estimates (8.6), (18.39), and Proposition 6.6(i) imply that

$$|c_{\lambda,\eta}(\psi, g)| \leqslant M_q(2+|\lambda|)^{c_2-q}, \quad q = 1, 2, \ldots,$$

where $c_2 > 0$ is independent of λ, η, q. According to (18.10), Lemma 8.1(i), and Proposition 8.18, $g \in \mathrm{QA}(\mathbb{R}^1)$. Next, by assumption on R and (18.16),

$$\mathrm{supp}\, w_{1,2} \subset \big(-r(\psi), r(\psi)\big). \tag{18.40}$$

Therefore, g vanishes on $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ (see Proposition 8.20(v)). As $g \in \mathrm{QA}(\mathbb{R}^1)$, we conclude from Theorem 8.1(i) that $g = 0$. If $w_{1,2} \neq 0$, this, together with Proposition 8.20(v), (18.40), and Corollary 13.2, yields $\zeta_\psi = 0$, a contradiction. Thus, $w_{1,2}$ must vanish. Since $\mathcal{Z}(\mathcal{T}) = \varnothing$, we see from (18.38) and Theorem 6.4 that

$$w = T_1 * \varphi$$

for some $\varphi \in \mathcal{E}'(\mathbb{R}^1)$. In view of (18.18) with $T = T_1$, one has $f = 0$ and $\mathcal{D}'_{\mathcal{T}}(-R, R) = \{0\}$. In addition, Theorem 18.10(ii) with $\psi_2 = \psi$ shows that this result fails in general if $r^*(\mathcal{T}) < R = r_1 + r_2 - r(\psi)$. □

18.3 The Case of "Small" Intervals. Connections with Division-Type Problems for Entire Functions

Our main purpose in this section is to establish analogues of Theorem 18.8 for system (18.4). To begin with, we introduce some auxiliary notions. Throughout the section we assume that $i_1, i_2 \in \mathcal{I}$ are fixed and $i_1 \neq i_2$.

For $\mathcal{T} \in \mathfrak{T}(\mathbb{R}^1)$ and $R > 0$, let $\mathcal{V}_{\mathcal{T},R}$ be the set of all distributions $v \in \mathcal{E}'(\mathbb{R}^1)$ with the following properties:

(a) $v = 0$ in $\{x \in \mathbb{R}^1 \colon |x| > r(P) + r(Q) - R\}$, where $P = T_{i_1}$ and $Q = T_{i_2}$;
(b) for each $i \in \mathcal{I}\backslash\{i_1, i_2\}$, there exist $v_{1,i}, v_{2,i} \in \mathcal{E}'(\mathbb{R}^1)$ such that

$$v_{\mu,i} = 0 \quad \text{in } \left\{x \in \mathbb{R}^1 \colon |x| > r(T_\mu) + r(T_i) - R\right\},\ \mu \in \{i_1, i_2\}, \tag{18.41}$$

and

$$P * v_{2,i} + Q * v_{1,i} = T_i * v. \tag{18.42}$$

Also we set

$$\mathcal{V}_{\mathcal{T},R,\natural} = \mathcal{V}_{\mathcal{T},R} \cap \mathcal{E}'_\natural(\mathbb{R}^1).$$

Of course, the sets $\mathcal{V}_{\mathcal{T},R}$ and $\mathcal{V}_{\mathcal{T},R,\natural}$ depend on i_1 and i_2. Relations (18.41) and (18.42) and Theorem 6.3 show that the problem of describing of the sets $\mathcal{V}_{\mathcal{T},R}$ and $\mathcal{V}_{\mathcal{T},R,\natural}$ is closely connected with division-type problems for some classes of entire functions (see, e.g., Berenstein and Struppa [19], Sect. 1).

Let us consider some properties of $\mathcal{V}_{\mathcal{T},R}$ and $\mathcal{V}_{\mathcal{T},R,\natural}$. In the sequel $\mathfrak{T}_\natural(\mathbb{R}^1)$ denotes the set of all families $\mathcal{T} \in \mathfrak{T}(\mathbb{R}^1)$ such that $T_i \in \mathcal{E}'_\natural(\mathbb{R}^1)$ for all $i \in \mathcal{I}$.

Proposition 18.3. *Let $\mathcal{T} \in \mathfrak{T}_\natural(\mathbb{R}^1)$, $R > 0$, and let $v \in \mathcal{V}_{\mathcal{T},R}$. Then the following assertions hold.*

(i) *If v is even (respectively, odd), then the distributions $v_{\mu,i}$ in (18.41) and (18.42) can be chosen even (respectively, odd).*
(ii) *There exist $w_1, w_2 \in \mathcal{V}_{\mathcal{T},R,\natural}$ such that $v = w_1 + w_2'$. In particular, $\mathcal{V}_{\mathcal{T},R} = \{0\}$ if and only if $\mathcal{V}_{\mathcal{T},R,\natural} = \{0\}$.*

Proof. To prove (i) it is enough to observe that if v is even (respectively, odd), then, together with $v_{\mu,i}$, the distributions

$$\frac{1}{2}\big(v_{\mu,i}(t) + v_{\mu,i}(-t)\big) \quad \left(\text{respectively, } \frac{1}{2}\big(v_{\mu,i}(t) - v_{\mu,i}(-t)\big)\right)$$

satisfy (18.41) and (18.42).

Turning to (ii), first note that for each odd $u \in \mathcal{E}'(\mathbb{R}^1)$, there exists $w \in \mathcal{E}'_{\natural}(\mathbb{R}^1)$ such that $w' = u$ (see Theorem 6.3). Now define

$$w_1(t) = \frac{1}{2}\big(v(t) + v(-t)\big), \qquad w_2'(t) = \frac{1}{2}\big(v(t) - v(-t)\big).$$

Using (i) and the definition of $\mathcal{V}_{\mathcal{T},R,\natural}$, we arrive at (ii). □

Proposition 18.4. *Let $\mathcal{I} = \mathbb{N}$, $i_1 = 1$, $i_2 = 2$. Then for each $\alpha \in [1, 2]$, there exists $\mathcal{T} = \{T_i\}_{i\in\mathcal{I}} \in \mathfrak{T}_{\natural}(\mathbb{R}^1)$ with the following properties:*

(1) *$T_1 \in \mathfrak{N}(\mathbb{R}^1)$, $\mathcal{Z}(\widehat{T_1}) \not\approx \mathcal{Z}(\widehat{T_2})$, and $r(T_i) = 1$ for all $i \in \mathbb{N}$;*
(2) *$\mathcal{V}_{\mathcal{T},R,\natural} \neq \{0\}$ for $1 < R \leqslant \alpha$;*
(3) *$\mathcal{V}_{\mathcal{T},R} = \{0\}$ for each $R > \alpha$.*

Proof. We define $\mathcal{T} = \{T_1, T_2, \dots\}$ as follows. Let

$$\widehat{T_1}(z) = \frac{\sin z}{z}, \quad \widehat{T_2}(z) = \cos z, \quad z \in \mathbb{C}. \tag{18.43}$$

Then $\mathcal{Z}(\widehat{T_1}) \not\approx \mathcal{Z}(\widehat{T_2})$ and $r(T_1) = r(T_2) = 1$. Next, for the case where $\alpha = 1$, we set

$$T_i(t) = \begin{cases} f(t) & \text{if } |t| \leqslant 1, \\ 0 & \text{if } |t| > 1, \end{cases} \quad i \in \{3, 4, \dots\},$$

where $f \in (\mathrm{RA} \cap L^1)(\mathbb{R}^1)$ is a nonzero even function. Suppose that $v \in \mathcal{V}_{\mathcal{T},R}$ for some $R > 1$. By (18.41)–(18.43) we conclude that

$$T_i * v' = 0$$

in $(1-R, R-1)$ for $i \geqslant 3$. Since $f \in \mathrm{RA}(\mathbb{R}^1)$, this yields $f * v' = 0$ in $\mathbb{R}^1$. Hence, $\widehat{f}\,\widehat{v'} = 0$. As $\widehat{f} \in C(\mathbb{R}^1)$, this gives $v' = 0$ and $\mathcal{V}_{\mathcal{T},R} = \{0\}$.

Assume now that $\alpha \in (1, 2)$. Let us consider $g \in (\mathcal{E}'_{\natural} \cap \mathrm{Inv})(\mathbb{R}^1)$ such that $r(g) = 2 - \alpha$, $\widehat{g}(0) \neq 0$, and all the zeros of $\widehat{g}$ are real and simple. We note that the function $\widehat{g}$ has an infinite number of zeros (see Proposition 6.1(iv) and Theorem 6.3).

Denote by $\{\lambda_1, \lambda_2, \dots\}$ the sequence of all positive zeros of $\widehat{g}$ arranged in ascending order of magnitude. For $i \in \{3, 4, \dots\}$, we put

$$T_i(t) = \begin{cases} \cos(\lambda_{i-2}t) & \text{if } |t| \leqslant 1, \\ 0 & \text{if } |t| > 1. \end{cases} \tag{18.44}$$

Suppose that $v \in \mathcal{V}_{\mathcal{T},R,\natural}$ for some $R > \alpha$. As before, by (18.41)–(18.44) we have

$$\widehat{v'}(\lambda_{i-2}) = 0$$

for each $i \geqslant 3$. Combining this with Corollary 6.3, we obtain $v' = 0$. In view of Proposition 18.3(i), $\mathcal{V}_{\mathcal{T},R} = \{0\}$ for all $R > \alpha$. To continue, for $\mu \in \{1, 2\}$, $i \in \{3, 4, \dots\}$, we define $v_{\mu,i} \in \mathcal{E}'(\mathbb{R}^1)$ with $\operatorname{supp} v_{\mu,i} \subset [\alpha - 2, 2 - \alpha]$ by

$$v'_{1,i}(t) = \big(g' * T_i\big)(t-1) + \big(g' * T_i\big)(t+1), \quad |t| < 1,$$
$$v_{2,i}(t) = \big(g' * T_i\big)(t-1) - \big(g' * T_i\big)(t+1), \quad |t| < 1.$$

Then (18.42) is satisfied with $v = g$. Thus, $\mathcal{V}_{\mathcal{T},R,\natural} \neq \{0\}$ for $1 < R \leqslant \alpha$.

Finally, if $\alpha = 2$, then it is enough to put $T_i = T_1$ for $i \in \{3, 4, \ldots\}$. This completes the proof. □

We now present the following generalization of Theorem 18.8.

Theorem 18.13. *Let $\mathcal{T} \in \mathfrak{T}(\mathbb{R}^1)$, $P = T_{i_1}$, $Q = T_{i_2}$, and assume that $P \in \mathfrak{N}(\mathbb{R}^1)$, $\mathcal{Z}(\widehat{P}) \not\approx \mathcal{Z}(\widehat{Q})$, and*

$$r(T_i) < R \leqslant r(P) + r(T_\mu) \quad \textit{for all } i, \mu \in \mathcal{I},\ \mu \neq i_1.$$

Then the following statements are valid.

(i) *Let $f \in \mathcal{D}'(-R, R)$. Then for f to belong to $\mathcal{D}'_{\mathcal{T}}(-R, R)$, it is necessary and sufficient that*

$$f = \Omega_{P,Q} * v \quad \textit{in } (-R, R) \tag{18.45}$$

for some $v \in \mathcal{V}_{\mathcal{T},R}$.

(ii) $\mathcal{D}'_{\mathcal{T}}(-R, R) = \{0\}$ *if and only if* $\mathcal{V}_{\mathcal{T},R} = \{0\}$.

Proof. To prove (i), first assume that $f \in \mathcal{D}'_{\mathcal{T}}(-R, R)$. Theorem 18.8 shows that (18.45) holds for some $v \in \mathcal{E}'(\mathbb{R}^1)$ with

$$\operatorname{supp} v \subset [R - r(P) - r(Q), r(P) + r(Q) - R].$$

We claim that $v \in \mathcal{V}(\mathcal{T}, R)$. Let $i \in \mathcal{I} \backslash \{i_1, i_2\}$, and let

$$f_i = \Omega_{P,Q} * v * T_i.$$

Then $f_i \in \mathcal{D}'_P(\mathbb{R}^1)$ and $f_i = 0$ in $(r(T_i) - R, R - r(T_i))$. By Theorem 13.4(i) we have $f_i = \zeta_P * v_{1,i}$ for some $v_{1,i} \in \mathcal{E}'(\mathbb{R}^1)$ satisfying (18.41) with $\mu = i_1$. Equality (18.27) now gives

$$\Omega_{P,Q} * (v * T_i - v_{1,i} * Q) = 0 \quad \text{in } \mathbb{R}^1.$$

This, together with (18.26), (13.5), (18.23), and (18.24), implies that

$$\big(\widehat{v}\widehat{T_i} - \widehat{v}_{1,i}\widehat{Q}\big)^{(\eta)}(\lambda) = 0$$

for all $\lambda \in \mathcal{Z}(\widehat{P})$ and $\eta \in \{0, \ldots, m(\lambda, P)\}$. In view of Theorem 6.4, there exists $v_{2,i} \in \mathcal{E}'(\mathbb{R}^1)$ satisfying (18.42). Moreover, by Theorem 6.2 $v_{2,i}$ satisfies (18.41) with $\mu = i_2$. Thus, (18.45) is fulfilled for some $v \in \mathcal{V}_{\mathcal{T},R}$. The converse result follows from (18.45), (18.42), (18.27), and Theorem 13.4(i).

As for (ii), let $\mathcal{D}'_{\mathcal{T}}(-R, R) = \{0\}$. By (i),

$$\Omega_{P,Q} * v = 0$$

for all $v \in \mathcal{V}_{\mathcal{T},R}$. As before, this yields $\widehat{v}^{(\eta)}(\lambda) = 0$ for all $\lambda \in \mathcal{Z}(\widehat{P})$ and $\eta \in \{0, \dots, m(\lambda, P)\}$. By assumption on v and Corollary 6.3 we obtain $v = 0$ and $\mathcal{V}_{\mathcal{T},R} = \{0\}$. The converse statement is obvious from (i). ◻

Corollary 18.2. *Let $\mathcal{T} \in \mathfrak{T}_\natural(\mathbb{R}^1)$, $R > 0$, and assume that $\mathcal{T}$ and R satisfy all the assumptions in Theorem 18.13. Also let $f \in \mathcal{D}'_\natural(-R, R)$. Then $f \in \mathcal{D}'_{\mathcal{T},\natural}(-R, R)$ if and only if*

$$f = \Omega'_{P,Q} * v \quad in (-R, R)$$

for some $v \in \mathcal{V}_{\mathcal{T},R,\natural}$.

Proof. By assumption on $\mathcal{T}$ and (18.26) we see that $\Omega_{P,Q}$ is odd. The desired result now follows from Theorem 18.13 and Proposition 18.3. ◻

As another consequence of Theorem 18.13, we have the following result.

Corollary 18.3. *Let $\mathcal{I} = \mathbb{N}$, $i_1 = 1$, $i_2 = 2$. Then for each $\alpha \in [1, 2]$, there exists $\mathcal{T} = \{T_i\}_{i \in \mathcal{I}} \in \mathfrak{T}_\natural(\mathbb{R}^1)$ with the following properties:*

(1) $T_1 \in \mathfrak{N}(\mathbb{R}^1)$, $\mathcal{Z}(\widehat{T_1}) \not\approx \mathcal{Z}(\widehat{T_2})$, *and* $r(T_i) = 1$ *for all* $i \in \mathbb{N}$;
(2) $\mathcal{D}'_{\mathcal{T},\natural}(-R, R) \neq \{0\}$ *for* $1 < R \leqslant \alpha$;
(3) $\mathcal{D}'_{\mathcal{T}}(-R, R) = \{0\}$ *for* $R > \alpha$.

The proof follows from Theorem 18.13, Proposition 18.4 and Theorem 18.1(i).

We end this section with the following generalization of Theorem 18.12.

Theorem 18.14. *Let $\mathcal{T} \in \mathfrak{T}(\mathbb{R}^1)$ and $T_{i_0} \in (\mathrm{Inv}_+ \cap \mathrm{Inv}_-)(\mathbb{R}^1)$ for some $i_0 \in \mathcal{I}$. Let $\mathcal{Z}(\mathcal{T}) = \varnothing$ and suppose that (18.3) is satisfied. Assume that for each $\nu \in \mathcal{I}\backslash\{i_0\}$, there exists $\psi_\nu \in \mathfrak{M}(\mathbb{R}^1)$ such that $r(\psi_\nu) > 0$ and for each $\lambda \in \mathcal{Z}(\widehat{\psi_\nu})$,*

$$\sum_{\eta=0}^{m(\lambda,\psi_\nu)} \left(\left|\widehat{T}_{i_0}^{(\eta)}(\lambda)\right| + \left|\widehat{T}_\nu^{(\eta)}(\lambda)\right|\right) \leqslant M_{q,\nu}(2 + |\lambda|)^{-q}, \quad q = 1, 2, \dots,$$

where the constants $M_{q,\nu} > 0$ are independent of λ and satisfy (18.14). Then $\mathcal{D}'_{\mathcal{T}}(-R, R) = \{0\}$, provided that $R > r(T_{i_0}) + r(T_\nu) - r(\psi_\nu)$ for all $\nu \in \mathcal{I}\backslash\{i_0\}$.

This theorem is a best possible result, in the sense that $r(T_{i_0}) + r(T_\nu) - r(\psi_\nu)$ cannot be replaced in general by a smaller constant, as, for example, Theorem 18.12 shows. The proof of Theorem 18.14 is an immediate extension of that of Theorem 18.12, and therefore we omit it.

18.4 The Deconvolution Problem. Explicit Reconstruction Formulae

Let $m \in \mathbb{N}$, $m \geqslant 2$, let $T_1, \dots, T_m$ be nonzero distributions in the class $\mathcal{E}'(\mathbb{R}^1)$ such that $\bigcap_{l=1}^m \mathcal{Z}(\widehat{T_l}) = \varnothing$, and let $g_1, \dots, g_m \in \mathcal{D}'(\mathbb{R}^1)$. Assume that a distribution

$f \in \mathcal{D}'(\mathbb{R}^1)$ satisfies the system of convolution equations

$$f * T_l = g_l, \quad l = 1, \dots, m. \tag{18.46}$$

Theorem 18.1(i) shows that f is determined uniquely by (18.46). In this section we consider a construction for recovering the distribution f by $g_1, \dots, g_m$.

Following Sect. 8.2, for $l, k \in \{1, \dots, m\}$ and $\lambda_l \in \mathcal{Z}(\widehat{T_l})$, we define $T_{\lambda_1,\dots,\lambda_m,k} \in \mathcal{E}'(\mathbb{R}^1)$ by the formula

$$\widehat{T_{\lambda_1,\dots,\lambda_m,k}} = \prod_{\substack{l=1\\ l\neq k}}^{m} \widehat{(T_l)_{\lambda_l,0}}.$$

The first our result is as follows.

Theorem 18.15. *Assume that $T_1, \dots, T_m \in \mathfrak{R}(\mathbb{R}^1)$, $\bigcap_{l=1}^m \mathcal{Z}(\widehat{T_l}) = \varnothing$ and that for each $l \in \{1, \dots, m\}$, all the zeros of $\widehat{T_l}$ are simple. If (18.46) is satisfied, then*

$$f = \sum_{\lambda_1 \in \mathcal{Z}(\widehat{T_1})} \cdots \sum_{\lambda_m \in \mathcal{Z}(\widehat{T_m})} f_{\lambda_1,\dots,\lambda_m}, \tag{18.47}$$

where the series converges unconditionally in $\mathcal{D}'(\mathbb{R}^1)$,

$$f_{\lambda_1,\dots,\lambda_m} = \sum_{k=1}^{m} \frac{c_k(\lambda_1,\dots,\lambda_m)}{\widehat{T}_k'(\lambda_k)} (g_k * T_{\lambda_1,\dots,\lambda_m,k}), \tag{18.48}$$

and $(c_1(\lambda_1, \dots, \lambda_m), \dots, c_m(\lambda_1, \dots, \lambda_m)) \in \mathbb{C}^m$ is an arbitrary solution of the system

$$\begin{cases} \lambda_1 c_1 + \cdots + \lambda_m c_m = -1, \\ c_1 + \cdots + c_m = 0. \end{cases} \tag{18.49}$$

In addition, the right-hand side of (18.48) is independent of our choice of the constants $c_1(\lambda_1, \dots, \lambda_m), \dots, c_m(\lambda_1, \dots, \lambda_m) \in \mathbb{C}$ satisfying (18.49).

Proof. By assumption on $T_1, \dots, T_m$ one has

$$\delta_0 = \sum_{\lambda_1 \in \mathcal{Z}(\widehat{T_1})} \cdots \sum_{\lambda_m \in \mathcal{Z}(\widehat{T_m})} (T_1)_{\lambda_1,0} * \cdots * (T_m)_{\lambda_m,0},$$

where the series converges unconditionally in $\mathcal{D}'(\mathbb{R}^1)$ (see the proof of Proposition 8.9). Convolving with f, we arrive at (18.47) with

$$f_{\lambda_1,\dots,\lambda_m} = f * (T_1)_{\lambda_1,0} * \cdots * (T_m)_{\lambda_m,0}.$$

This, together with Proposition 8.15, gives the desired result. □

Remark 18.4. Assume that $T_1, \dots, T_m \in \mathfrak{M}(\mathbb{R}^1)$. Because of Proposition 8.3, for each $l \in \{1, \dots, m\}$, there exists a polynomial p_l such that $p_l(\frac{\mathrm{d}}{\mathrm{d}t})T_l \in \mathfrak{R}(\mathbb{R}^1)$ and

$\bigcap_{l=1}^{m} \mathcal{Z}(q_l) = \varnothing$, where $q_l(z) = p_l(\mathrm{i}z)$. If (18.46) is fulfilled, then

$$f * p_l\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)T_l = p_l\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)g_l, \quad l = 1, \dots, m.$$

Using now Theorem 18.15, we can reconstruct f by $g_1, \dots, g_m$ for the case where $T_1, \dots, T_m \in \mathfrak{M}(\mathbb{R}^1)$ and all the zeroes of $\widehat{T_1} \dots \widehat{T_m}$ are simple.

We shall now investigate the case $m = 2$ in greater detail. In this case relation (18.48) can be rewritten as

$$f_{\lambda_1,\lambda_2} = \frac{1}{\lambda_2 - \lambda_1}\left(\frac{(T_2)_{\lambda_2,0} * g_1}{\widehat{T}_1'(\lambda_1)} - \frac{(T_1)_{\lambda_1,0} * g_2}{\widehat{T}_2'(\lambda_2)}\right)$$

(see (18.49)). For the next step, one omits the assumption that all the zeroes of $\widehat{T_1}$ and $\widehat{T_2}$ are simple.

Let $l \in \{1, 2\}$ and $\lambda_l \in \mathcal{Z}(\widehat{T_l})$. We set

$$\nu = \nu(\lambda_1, \lambda_2) = m(\lambda_1, T_1) + m(\lambda_2, T_2) + 2,$$

$$U_{1,\lambda_1,\lambda_2} = \frac{1}{(\lambda_2 - \lambda_1)^{2\nu}} \sum_{q=0}^{\nu} (-1)^{\nu-q} \binom{2\nu}{\nu+q}$$
$$\times \sum_{j=0}^{m(\lambda_1,T_1)} a_j^{\lambda_1,0}(\widehat{T_1})\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_1\right)^{q+j+m(\lambda_2,T_2)+1}$$
$$\times \left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_2\right)^{\nu-q} (T_2)_{\lambda_2,0},$$

$$U_{2,\lambda_1,\lambda_2} = \frac{1}{(\lambda_2 - \lambda_1)^{2\nu}} \sum_{q=1}^{\nu} (-1)^{\nu+q} \binom{2\nu}{\nu-q}$$
$$\times \sum_{j=0}^{m(\lambda_2,T_2)} a_j^{\lambda_2,0}(\widehat{T_2})\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_2\right)^{q+j+m(\lambda_1,T_1)+1}$$
$$\times \left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_1\right)^{\nu-q} (T_1)_{\lambda_1,0}.$$

Theorem 18.16. *Let $T_1, T_2 \in \mathfrak{R}(\mathbb{R}^1)$, $\mathcal{Z}(\widehat{T_1}) \cap \mathcal{Z}(\widehat{T_2}) = \varnothing$, $g_1, g_2 \in \mathcal{D}'(\mathbb{R}^1)$, and suppose that $f \in \mathcal{D}'(\mathbb{R}^1)$ satisfies (18.46) with $m = 2$. Then*

$$f = \sum_{\lambda_1 \in \mathcal{Z}(\widehat{T_1})} \sum_{\lambda_2 \in \mathcal{Z}(\widehat{T_2})} F_{\lambda_1,\lambda_2},$$

where the series converges unconditionally in $\mathcal{D}'(\mathbb{R}^1)$, and

$$F_{\lambda_1,\lambda_2} = g_1 * U_{1,\lambda_1,\lambda_2} + g_2 * U_{2,\lambda_1,\lambda_2}.$$

Proof. This is just a repetition of the proof of Theorem 18.15 with the reference to Proposition 8.15 replaced by a reference to Proposition 8.16. □

To conclude we point out that there exist multidimensional analogues of Theorems 18.15 and 18.16 for $T_1, \ldots, T_m \in \mathfrak{R}(\mathbb{R}^n)$. Moreover, it is easy to obtain similar constructions for recovering a distribution by its convolutions on various homogeneous spaces. In order to arrive at the correspondent results, it is enough to repeat the argument in the proof of Theorem 18.15 with attention to Propositions 9.16, 9.17, 10.19, and 10.20. We leave for the reader to examine the details.

Chapter 19
Recent Developments in the Spectral Analysis Problem for Higher Dimensions

We have seen in Chap. 9 that the transmutation operators $\mathfrak{A}_{k,j}$ relate convolution equations on $\mathbb{R}^1$ to convolution equations on $\mathbb{R}^n (n > 1)$. This property and the results of Chap. 13 are further used in Chap. 14 to study mean periodic functions on $\mathbb{R}^n$. We have more one-dimensional results at our disposal now, so we can study mean periodic functions on $\mathbb{R}^n$ more systematically.

The entire chapter centers around local analogues of a well-known result of Brown, Schreiber, and Taylor [42] which is as follows: for any collection $\{T_i\}_{i\in\mathcal{I}}$ of radial distributions of compact support on $\mathbb{R}^n$, the system of convolution equations in $\mathcal{E}(\mathbb{R}^n)$

$$f * T_i = 0, \quad i \in \mathcal{I}, \tag{19.1}$$

has only the trivial solution $f = 0$ if and only if the Fourier transforms $\widehat{T_i}$ have no common zeroes. In [14], Berenstein and Gay proved a local version of the Brown–Schreiber–Taylor theorem. More precisely, if there exists $i_0 \in \mathcal{I}$ such that T_{i_0} is invertible and

$$\sup_{\lambda\in\mathcal{Z}(\widetilde{T}_{i_0})} \frac{|\operatorname{Im}\lambda|}{\log(2+|\lambda|)} < +\infty,$$

then

$$\left\{f \in \mathcal{D}'(B_R) : f * T_i = 0 \ \forall i \in \mathcal{I}\right\} = \{0\},$$

provided that $R > r(T_{i_0}) + \sup_{i\in\mathcal{I}} r(T_i)$ and $\bigcap_{i\in\mathcal{I}} \mathcal{Z}(\widetilde{T}_i) = \varnothing$. In this connection, Berenstein and Gay posed the following problem: generalize their result to arbitrary radial distributions T_i. In addition, it is naturally to present the complete picture of the corresponding phenomenon when $R \leqslant r(T_{i_0}) + \sup_{i\in\mathcal{I}} r(T_i)$. These questions are discussed in Sects. 19.1 and 19.3. In Sect. 19.2 we describe various spaces of solutions of (19.1) on domains in $\mathbb{R}^n$. Answers are provided in terms of expansions associated with cylindrical functions. In Sect. 19.4 we consider applications to the Pompeiu problem. One of our most intriguing results asserts that the global Pompeiu property always implies the local Pompeiu property (Theorem 19.16).

V.V. Volchkov, V.V. Volchkov, *Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group,* Springer Monographs in Mathematics,

19.1 Solution of the Berenstein–Gay Problem. Generalizations

Throughout in this chapter we assume that $n \geqslant 2$. In this section we generalize the above mentioned Berenstein–Gay result to arbitrary families of radial distributions.

Let $\mathcal{T} = \{T_i\}_{i \in \mathcal{I}}$ be a given family of nonzero compactly supported distributions on $\mathbb{R}^n$. Suppose that $\mathcal{O}$ is an open subset of $\mathbb{R}^n$ such that the set

$$\mathcal{O}_{T_i} = \left\{x \in \mathbb{R}^n : \mathring{B}_{r(T_i)}(x) \subset \mathcal{O}\right\}$$

is nonempty for all $i \in \mathcal{I}$. Then for each $f \in \mathcal{D}'(\mathcal{O})$, the convolution $f * T_i$ is well defined as distribution in $\mathcal{D}'(\mathcal{O}_{T_i})$.

Let us regard the system of convolution equations

$$(f * T_i)(x) = 0, \quad x \in \mathcal{O}_{T_i},\ i \in \mathcal{I}, \tag{19.2}$$

with unknown $f \in \mathcal{D}'(\mathcal{O})$. By analogy with Sect. 18.1 denote by $\mathcal{D}'_{\mathcal{T}}(\mathcal{O})$ the set of all distributions $f \in \mathcal{D}'(\mathcal{O})$ satisfying (19.2). For $m \in \mathbb{Z}_+$ or $m = \infty$, we set

$$C^m_{\mathcal{T}}(\mathcal{O}) = \left(\mathcal{D}'_{\mathcal{T}} \cap C^m\right)(\mathcal{O}).$$

If the set $\mathcal{O}$ is $\mathrm{O}(n)$-invariant, we put

$$C^m_{\mathcal{T},\natural}(\mathcal{O}) = \left(\mathcal{D}'_{\mathcal{T},\natural} \cap C^m\right)(\mathcal{O}),$$

where

$$\mathcal{D}'_{\mathcal{T},\natural}(\mathcal{O}) = \left(\mathcal{D}'_{\mathcal{T}} \cap \mathcal{D}'_{\natural}\right)(\mathcal{O}).$$

Let $\mathfrak{T}_{\natural}(\mathbb{R}^n)$ be the set of all families $\mathcal{T} = \{T_i\}_{i \in \mathcal{I}}$ such that

$$T_i \in \mathcal{E}'_{\natural}\left(\mathbb{R}^n\right) \quad \text{and} \quad T_i \neq 0 \quad \text{for each } i \in \mathcal{I}.$$

For $\mathcal{T} \in \mathfrak{T}_{\natural}(\mathbb{R}^n)$, define the family $\Lambda(\mathcal{T}) = \{\Lambda(T_i)\}_{i \in \mathcal{I}}$ of distributions in the class $\mathcal{E}'_{\natural}(\mathbb{R}^1)$ by

$$\widehat{\Lambda(T_i)} = \widetilde{T}_i, \quad i \in \mathcal{I}.$$

It follows by (9.50) that $r(T_i) = r(\Lambda(T_i))$ for all $i \in \mathcal{I}$.

We now consider the problem of existence of a nonzero solution of system (19.2).

If $\mathcal{O}$ is a ball and $\mathcal{T} \in \mathfrak{T}_{\natural}(\mathbb{R}^n)$, then this problem reduces to the one-dimensional case by means of the following result.

Theorem 19.1. *Let $\mathcal{T} \in \mathfrak{T}_{\natural}(\mathbb{R}^n)$, and let*

$$r(T_i) < R \leqslant +\infty \quad \textit{for each } i \in \mathcal{I}. \tag{19.3}$$

Assume that $k \in \mathbb{Z}_+$ and $j \in \{1, \ldots, d(n,k)\}$ are fixed. Then the following statements are equivalent.

(i) $\mathcal{D}'_{\mathcal{T}}(B_R) = \{0\}$.

(ii) $(\mathcal{D}'_{\mathcal{T}} \cap \mathcal{D}'_{k,j})(B_R) = \{0\}$.
(iii) $\mathcal{D}'_{\Lambda(\mathcal{T}),\natural}(-R, R) = \{0\}$.

The same remains valid, provided that $\mathcal{D}'_{\mathcal{T}}(B_R)$, $(\mathcal{D}'_{\mathcal{T}} \cap \mathcal{D}'_{k,j})(B_R)$, *and* $\mathcal{D}'_{\Lambda(\mathcal{T}),\natural}(-R, R)$ *are replaced by* $C^\infty_{\mathcal{T}}(B_R)$, $(C^\infty_{\mathcal{T}} \cap \mathcal{D}'_{k,j})(B_R)$, *and* $C^\infty_{\Lambda(\mathcal{T}),\natural}(-R, R)$, *respectively.*

Proof. The implication (i)→(ii) is obvious. Next, it follows by Theorem 9.3(ii) and Proposition 14.7(ii) that (ii) implies (iii). Now let (iii) hold and suppose that $f \in \mathcal{D}'_{\mathcal{T}}(B_R)$. By Propositions 14.5 and 14.7(ii) we deduce that

$$f^{k,j} \in \mathcal{D}'_{\mathcal{T}}(B_R) \quad \text{and} \quad \mathfrak{A}_{k,j}\big(f^{k,j}\big) \in \mathcal{D}'_{\Lambda(\mathcal{T}),\natural}(-R, R)$$

for all $k \in \mathbb{Z}$ and $j = \{1, \dots, d(n,k)\}$. In combination with (iii) and Theorem 9.3(ii), this gives $f^{k,j} = 0$ for all k, j. Therefore, $f = 0$, and (i) is a consequence of (iii). To prove the same result for the classes $C^\infty_{\mathcal{T}}(B_R)$, $(C^\infty_{\mathcal{T}} \cap \mathcal{D}'_{k,j})(B_R)$, and $C^\infty_{\Lambda(\mathcal{T}),\natural}(-R, R)$, it is sufficient to repeat the above arguments using Proposition 9.1(iii), (vi) and Theorem 9.3(iv). □

Remark 19.1. Using Theorem 9.3(i) and Corollary 9.2, we conclude that if $C^m_{\mathcal{T},\natural}(B_R) = \{0\}$ for some $m \in \mathbb{Z}_+$, then $C^m_{\Lambda(\mathcal{T}),\natural}(-R, R) = \{0\}$. However, the converse result fails in general (see Theorems 18.8(iii) and 19.7).

For $\mathcal{T} \in \mathfrak{T}_\natural(\mathbb{R}^n)$, we set

$$\mathcal{Z}_{\mathcal{T}} = \bigcap_{i \in \mathcal{I}} \mathcal{Z}_{T_i},$$

$$r_*(\mathcal{T}) = \inf_{i \in \mathcal{I}} r(T_i), \quad r^*(\mathcal{T}) = \sup_{i \in \mathcal{I}} r(T_i), \quad R_{\mathcal{T}} = r_*(\mathcal{T}) + r^*(\mathcal{T}).$$

Let $\mathfrak{T}_1(\mathbb{R}^n)$ denote the set of all families $\mathcal{T} \in \mathfrak{T}_\natural(\mathbb{R}^n)$ such that

$$0 < r_*(\mathcal{T}) = r(T_{i_1}) \quad \text{and} \quad r^*(\mathcal{T}) = r(T_{i_2})$$

for some $i_1, i_2 \in \mathcal{I}$.

Notice that if $\mathcal{T} \in \mathfrak{T}_\natural(\mathbb{R}^n)$ and $\mathcal{Z}_{\mathcal{T}} \neq \varnothing$, then

$$\Phi_{\lambda,0,0,1} * T_i = 0$$

in $\mathbb{R}^n$ for all $\lambda \in \mathcal{Z}_{\mathcal{T}}$, $i \in \mathcal{I}$ (see Proposition 14.2(ii)). In particular, $C^\infty_{\mathcal{T}}(\mathbb{R}^n) \neq \{0\}$. So we shall regard system (19.2) for the case where $\mathcal{Z}_{\mathcal{T}} = \varnothing$.

We now use Theorem 19.1 in order to establish the following basic result.

Theorem 19.2. *Let* $\mathcal{T} \in \mathfrak{T}_\natural(\mathbb{R}^n)$, $\mathcal{Z}_{\mathcal{T}} = \varnothing$, *and suppose that* (19.3) *is satisfied. Let* $\mathcal{O} \in \mathfrak{S}(R, r(T_\nu))$ *for some* $\nu \in \mathcal{I}$, *and let* $f \in \mathcal{D}'_{\mathcal{T}}(\mathcal{O})$ *(see the definition of* $\mathfrak{S}(R, r)$ *in Sect. 1.2). Then the following statements are valid.*

(i) *If*

$$r_*(\mathcal{T}) + r(T_i) < R \leqslant +\infty \quad \textit{for all } i \in \mathcal{I}, \tag{19.4}$$

then $f = 0$.

(ii) *If* $\mathcal{T} \in \mathfrak{T}_1(\mathbb{R}^n)$, $R = R_{\mathcal{T}}$, $B_R \subset \mathcal{O}$, *and* $f \in C^\infty(\dot{B}_R)$, *then* $f = 0$.

Proof. We can assume, without loss of generality, that $\mathcal{O} = B_R$ (see Theorem 14.2 and the definition of $\mathfrak{S}(R, r)$). Assertion (i) now follows from Theorems 18.1 and 19.1. To prove (ii) it is enough to repeat the arguments in the proof of Theorem 19.1 using Theorem 18.1 and Remark 9.1. □

Some analogues of Theorem 19.2 can be obtained as a consequence of Theorem 19.1 (see Remark 19.1). We now show that the value R in Theorem 19.2 cannot be decreased in the general case (see also Theorems 19.12 and 19.14 below).

Theorem 19.3. *For each* $\eta > 0$, *there exists* $T_\eta \in \mathfrak{N}(\mathbb{R}^n)$ *such that the following assertions hold.*

(i) $r(T_\eta) = 1$ *and* $\mathcal{Z}(\widetilde{T}_\eta) \subset \mathbb{R}^1$ *for all* η.

(ii) $\mathcal{Z}(\widetilde{T}_{\eta_1}) \cap \mathcal{Z}(\widetilde{T}_{\eta_2}) = \varnothing$ *for* $\eta_1 \neq \eta_2$.

(iii) *For each* $m \in \mathbb{Z}_+$ *there exists nonzero* $f \in C^m(B_2)$ *such that* $f * T_\eta = 0$ *in* B_1 *for all* η.

We point out that this result is no longer valid with $m = \infty$ (see, for instance, Theorem 19.13(i)) . However, by regularization we may conclude from (iii) that for any $\varepsilon \in (0, 1)$, there exists nonzero $f_\varepsilon \in C^\infty(B_{2-\varepsilon})$ such that $f_\varepsilon * T_\eta = 0$ in $B_{1-\varepsilon}$ for all $\eta > 0$.

Proof of Theorem 19.3. By analogy with the proof of Theorem 18.2, we define $T_\eta \in \mathfrak{N}(\mathbb{R}^n)$ and $T \in \mathfrak{N}(\mathbb{R}^n)$ by the formulae

$$\widetilde{T}_\eta(z) = 1 - e^\eta \cos z, \quad \widetilde{T}(z) = \cos z, \quad z \in \mathbb{C}. \tag{19.5}$$

Then (i) and (ii) are valid. For $k \in \mathbb{Z}_+$ and $j \in \{1, \dots, d(n, k)\}$, we set

$$f = \zeta_{T,k,j}.$$

Let $m \in \mathbb{Z}_+$ be fixed. If k is large enough, then by Theorem 14.7(v) we infer that $f \in C^m(\mathbb{R}^n)$. In addition,

$$f * T_\eta = \zeta_{T,k,j}$$

in $\mathbb{R}^n$ for all η (see (14.15), Proposition 14.2(ii) and (19.5)). In view of Theorem 14.7(ii), f satisfies (iii), proving the theorem. □

Remark 19.2. The previous result shows that the assumption on R in Theorem 19.2(i) cannot be weakened in general. However, for some families $\mathcal{T} \in \mathfrak{T}_\natural(\mathbb{R}^n)$, Theorem 19.2 remains valid with $R > r^*(\mathcal{T})$ (see Corollary 19.1 below).

We shall now prove another uniqueness theorem for system (19.2).

Theorem 19.4. *Let $\mathcal{T} \in \mathfrak{T}_\natural(\mathbb{R}^n)$, $\mathcal{Z}_\mathcal{T} = \varnothing$, and let $\mathcal{K}$ be a compact subset of $\mathbb{R}^n$. Assume that for some $\nu \in \mathcal{I}$, the set $\mathcal{O} = \mathbb{R}^n \backslash \mathcal{K}$ is a ζ domain with $\zeta = r(T_\nu)$. Then $\mathcal{D}'_\mathcal{T}(\mathcal{O}) = \{0\}$.*

Notice that for $r^*(\mathcal{T}) < +\infty$, this result is obvious from Theorem 19.2(i).

Proof of Theorem 19.4. First, assume that $\mathcal{O} = B_{r,\infty}$ for some $r \geqslant 0$. Let $f \in \mathcal{D}'_\mathcal{T}(\mathcal{O})$ and $T \in \mathcal{T}$. Using Proposition 14.13, we obtain

$$a_{\lambda,\eta,k,j}(T, f) = b_{\lambda,\eta,k,j}(T, f) = 0$$

for all $\lambda \in \mathcal{Z}_T$, $\eta \in \{0, \ldots, n(\lambda, T)\}$, $k \in \mathbb{Z}_+$, and $j \in \{1, \ldots, d(n,k)\}$ (see the proof of Theorem 18.1(i) for the case $R = +\infty$). Now Proposition 14.11(ii) yields $f = 0$. This proves the desired result for the case $\mathcal{O} = B_{r,\infty}$. In view of Theorem 14.2(i) and Definition 1.1, the general case reduces to this one. □

For the rest of the section, we assume that $i_0 \in \mathcal{I}$ is fixed.

Theorem 19.5. *Let $\mathcal{T} \in \mathfrak{T}_\natural(\mathbb{R}^n)$, $\mathcal{Z}_\mathcal{T} = \varnothing$, $T = T_{i_0} \in \mathrm{Inv}_+(\mathbb{R}^n)$, and suppose that (19.3) holds. Assume that for each $\nu \in \mathcal{I}\backslash\{i_0\}$, there exists $\psi_\nu \in \mathfrak{M}(\mathbb{R}^n)$ such that $r(\psi_\nu) > 0$ and for each $\lambda \in \mathcal{Z}_{\psi_\nu}$,*

$$\sum_{\eta=0}^{n(\lambda,\psi_\nu)} \left(\left|\widetilde{T}^{(\eta)}(\lambda)\right| + \left|\widetilde{T}_\nu^{(\eta)}(\lambda)\right|\right) \leqslant M_{q,\nu}(2 + |\lambda|)^{-q}, \quad q = 1, 2, \ldots,$$

where the constants $M_{q,\nu} > 0$ are independent of λ and satisfy (18.14). *Let $R > r(T) + r(T_\nu) - r(\psi_\nu)$ for all $\nu \in \mathcal{I}\backslash\{i_0\}$, and let $\mathcal{O} \in \mathfrak{S}(R, r(T_i))$ for some $i \in \mathcal{I}$. Then $\mathcal{D}'_\mathcal{T}(\mathcal{O}) = \{0\}$. In particular, this is true if $\psi_\nu = T \in \mathfrak{M}(\mathbb{R}^n)$ for all $\nu \in \mathcal{I}\backslash\{i_0\}$.*

Proof. As before, without loss of generality we assume that $\mathcal{O} = B_R$ (see Theorem 14.2(i) and the definition of $\mathfrak{S}(R, r)$ in Sect. 1.2). Now the desired result follows at once from Theorems 19.1 and 18.14. □

Next, let $\mathcal{T} \in \mathfrak{T}_\natural(\mathbb{R}^n)$ and $R \in (0, +\infty]$. Denote by $\mathcal{W}(\mathcal{T}, R)$ the set of all distributions $w \in \mathcal{E}'_\natural(\mathbb{R}^n)$ with the following properties:

(a) $\operatorname{supp} w \subset \dot{B}_{r(T)}$, where $T = T_{i_0}$;
(b) for each $\nu \in \mathcal{I}\backslash\{i_0\}$, there exist $w_{1,\nu}, w_{2,\nu} \in \mathcal{E}'_\natural(\mathbb{R}^n)$ such that

$$w_{1,\nu} = 0 \quad \text{in } \left\{x \in \mathbb{R}^n : |x| > r(T) + r(T_\nu) - R\right\}$$

and

$$T * w_{2,\nu} + w_{1,\nu} = T_\nu * w.$$

The set $\mathcal{W}(\mathcal{T}, R)$ depends on i_0. It is easy to see that

$$w \in \mathcal{W}(\mathcal{T}, R) \quad \text{if and only if} \quad \Lambda(w) \in \mathcal{W}_{\Lambda(\mathcal{T}),R,\natural}$$

(see the proof of Proposition 18.3).

Theorem 19.6. *Let $\mathcal{T} \in \mathfrak{T}_\natural(\mathbb{R}^n)$, $T = T_{i_0} \in \mathrm{Inv}_+(\mathbb{R}^n)$, and assume that (19.3) is satisfied. Then the following assertions hold.*

(i) *If $f \in \mathcal{D}'(B_R)$, then for f to belong to $\mathcal{D}'_{\mathcal{T}}(B_R)$, it is necessary and sufficient that for all $k \in \mathbb{Z}_+$ and $j \in \{1, \dots, d(n,k)\}$,*

$$f^{k,j} = \zeta_{T,k,j} * w \quad \textit{in } B_R$$

for some $w \in \mathcal{W}(\mathcal{T}, R)$ depending on k, j.

(ii) *$\mathcal{D}'_{\mathcal{T}}(B_R) = \{0\}$ if and only if for each $w \in \mathcal{W}(\mathcal{T}, R)$, there exists $\psi \in \mathcal{E}'_\natural(\mathbb{R}^n)$ such that $w = T * \psi$.*

Proof. First, assume that $f \in \mathcal{D}'(B_R)$. In view of Propositions 14.7(ii) and 14.5, Theorem 18.6, and Remark 13.1, for f to belong to $\mathcal{D}'_{\mathcal{T}}(B_R)$, it is necessary and sufficient that for all $k \in \mathbb{Z}_+$ and $j \in \{1, \dots, d(n,k)\}$,

$$\mathfrak{A}_{k,j}\big(f^{k,j}\big) = \zeta'_{\Lambda(T)} * v \quad \text{in } (-R, R)$$

for some $v \in \mathcal{W}_{\Lambda(\mathcal{T}),R,\natural}$. Bearing (14.14) in mind and using Theorem 9.3(i), (ii), we arrive at (i).

Part (ii) is obvious from (i) and Theorem 14.7(iii). □

Next, for the remainder of this section, we suppose that $i_1, i_2 \in \mathcal{I}$ are fixed and $i_1 \neq i_2$. For $\mathcal{T} \in \mathfrak{T}_\natural(\mathbb{R}^n)$ and $R > 0$, let $\mathcal{V}(\mathcal{T}, R)$ denote the set of all distributions $v \in \mathcal{E}'_\natural(\mathbb{R}^n)$ with the following properties:

(a) $v = 0$ in $\{x \in \mathbb{R}^n : |x| > r(P) + r(Q) - R\}$, where $P = T_{i_1}$ and $Q = T_{i_2}$;

(b) for each $i \in \mathcal{I} \backslash \{i_1, i_2\}$, there exist $v_{1,i}, v_{2,i} \in \mathcal{E}'_\natural(\mathbb{R}^n)$ such that

$$v_{\mu,i} = 0 \quad \text{in } \big\{x \in \mathbb{R}^n : |x| > r(T_\mu) + r(T_i) - R\big\}, \quad \mu \in \{i_1, i_2\},$$

and

$$P * v_{2,i} + Q * v_{1,i} = T_i * v.$$

Assume now that $U_1 \in \mathfrak{N}(\mathbb{R}^n)$, $U_2 \in \mathcal{E}'_\natural(\mathbb{R}^n)$, $U_2 \neq 0$, and $\mathcal{Z}(\widetilde{U}_1) \not\subset \mathcal{Z}(\widetilde{U}_2)$. Since $\widehat{\Lambda(U_\nu)} = \widetilde{U}_\nu$, $\nu = 1, 2$, one sees, by the definition of $\Omega_{\Lambda(U_1),\Lambda(U_2)}$, that the distribution $\Omega'_{\Lambda(U_1),\Lambda(U_2)}$ is even (see Sect. 18.1). For $k \in \mathbb{Z}_+$ and $j \in \{1, \dots, d(n,k)\}$, we define $\Omega_{U_1,U_2,k,j} \in \mathcal{D}'(\mathbb{R}^n)$ by

$$\Omega_{U_1,U_2,k,j} = -\mathfrak{A}^{-1}_{k,j}\big(\Omega'_{\Lambda(U_1),\Lambda(U_2)}\big). \tag{19.6}$$

Lemma 19.1.

(i) $\Omega_{U_1,U_2,k,j} \in \mathcal{D}'_{U_1}(\mathbb{R}^n)$ *and*

$$\Omega_{U_1,U_2,k,j} * U_2 = \zeta_{U_1,k,j}.$$

(ii) *If $R > 0$, then for each $s \in \mathbb{N}$, there exists a constant $c > 0$ such that $\Omega_{U_1,U_2,k,j} \in C^s(B_R)$ for all $k > c$ and $j \in \{1, \dots, d(n,k)\}$.*

Proof. Part (i) is a consequence of (19.6), (18.27), (14.14), (9.80), and Theorem 9.5(i). Next, for each $R > 0$, there exists $u \in C^2_\natural(-R, R)$ such that $\Omega'_{\Lambda(U_1),\Lambda(U_2)} = p(\frac{d^2}{dt^2})u$ in $(-R, R)$ for some polynomial p. Assertion (ii) now follows from (19.6), (9.80), and Theorem 9.5(i), (iii). □

We conclude this section with the following result.

Theorem 19.7. *Let $\mathcal{T} \in \mathfrak{T}_\natural(\mathbb{R}^n)$, $P = T_{i_1}$, $Q = T_{i_2}$, and assume that $P \in \mathfrak{N}(\mathbb{R}^n)$, $\mathcal{Z}(\widetilde{P}) \not\asymp \mathcal{Z}(\widetilde{Q})$, and*

$$r(T_i) < R \leqslant r(P) + r(T_\mu) \quad \text{for all } i, \mu \in \mathcal{I},\ \mu \neq i_1.$$

Then the following assertions hold.

(i) *If $f \in \mathcal{D}'(B_R)$, then in order that $f \in \mathcal{D}'_\mathcal{T}(B_R)$, it is necessary and sufficient that for all $k \in \mathbb{Z}_+$ and $j \in \{1, \dots, d(n,k)\}$,*

$$f^{k,j} = \Omega_{P,Q,k,j} * v \quad \text{in } B_R \tag{19.7}$$

for some $v \in \mathcal{V}(\mathcal{T}, R)$ depending on k, j.

(ii) *$\mathcal{D}'_\mathcal{T}(B_R) = \{0\}$ if and only if $\mathcal{V}(\mathcal{T}, R) = \{0\}$.*

(iii) *If $\mathcal{D}'_\mathcal{T}(B_R) \neq \{0\}$, then $C^m_\mathcal{T}(B_R) \neq \{0\}$ for each $m \in \mathbb{Z}_+$.*

We point out that assertion (iii) fails in general with $m = \infty$ (see Theorem 19.13(i) below).

Proof of Theorem 19.7. Part (i) is proved in the same way as Theorem 19.6(i) by using Corollary 18.2.

Turning to (ii), first suppose that $\mathcal{D}'_\mathcal{T}(B_R) = \{0\}$. The definition of $\mathcal{V}(\mathcal{T}, R)$ shows that $v \in \mathcal{V}(\mathcal{T}, R)$ if and only if $\Lambda(v) \in \mathcal{V}_{\Lambda(\mathcal{T}),R,\natural}$. Using this fact together with Theorem 19.1 and Corollary 18.2, we obtain

$$\mathcal{V}(\mathcal{T}, R) = \{0\}.$$

The converse result follows from (i) and Proposition 9.1(vi).

Let us prove (iii). If $\mathcal{D}'_\mathcal{T}(B_R) \neq \{0\}$, then there exist $k \in \mathbb{Z}_+$ $j \in \{1, \dots, d(n,k)\}$, and $v \in \mathcal{V}(\mathcal{T}, R)$ such that the convolution $\Omega_{P,Q,k,j} * v$ is nonzero in B_R (see Proposition 9.1(vi) and (19.7)). Thanks to Theorem 9.3(i), (ii) and (19.6), we infer that

$$\Omega_{P,Q,k,j} * v \neq 0 \quad \text{in } B_R$$

for all k, j. Let $m \in \mathbb{Z}_+$. Since $v \in \mathcal{E}'(\mathbb{R}^n)$, one concludes that $\operatorname{ord} v < +\infty$. This, together with Lemma 19.1, shows that $\Omega_{P,Q,k,j} * v \in C^m_\mathcal{T}(B_R)$ if the number k is large enough. This concludes the proof. □

19.2 Expansions Associated with Cylindrical Functions

In the present section we shall investigate various classes of solutions of system (19.2) for some domains $\mathcal{O}$ with spherical symmetry.

For $\mathcal{T} \in \mathfrak{T}_{\natural}(\mathbb{R}^n)$ and $\lambda \in \mathcal{Z}_{\mathcal{T}}$, we denote

$$n(\lambda, \mathcal{T}) = \min_{i \in \mathcal{I}} n(\lambda, T_i).$$

In what follows we write $\mathcal{T} \in \mathfrak{T}_2(\mathbb{R}^n)$ if

$$\mathcal{T} \in \mathfrak{T}_1\big(\mathbb{R}^n\big) \quad \text{and} \quad T_i \in \mathcal{D}\big(\mathbb{R}^n\big) \tag{19.8}$$

for some $i \in \mathcal{I}$ such that $r(T_i) = r_*(\mathcal{T})$. Also, we write $\mathcal{T} \in \mathfrak{T}_3(\mathbb{R}^n)$ if (19.8) holds for all $i \in \mathcal{I}$ such that $r(T_i) = r^*(\mathcal{T})$. As before, we assume that $i_0 \in \mathcal{I}$ is fixed.

The following result characterizes some classes of solutions of (19.2) in a ball.

Theorem 19.8. *Let $\mathcal{T} \in \mathfrak{T}_{\natural}(\mathbb{R}^n)$ and $T = T_{i_0}$. Then the following assertions are true.*

(i) *Assume that (19.4) is satisfied, let $T \in \mathfrak{M}(\mathbb{R}^n)$, and let $f \in \mathcal{D}'(B_R)$. For f to belong to $\mathcal{D}'_{\mathcal{T}}(B_R)$, it is necessary and sufficient that for all $k \in \mathbb{Z}_+$ and $j \in \{1, \dots, d(n,k)\}$,*

$$f^{k,j} = \sum_{\lambda \in \mathcal{Z}_{\mathcal{T}}} \sum_{\eta=0}^{n(\lambda,\mathcal{T})} \gamma_{\lambda,\eta,k,j} \Phi_{\lambda,\eta,k,j}, \tag{19.9}$$

where $\gamma_{\lambda,\eta,k,j} \in \mathbb{C}$, and the series in (19.9) converges in $\mathcal{D}'(B_R)$. If $\mathcal{T} \in (\mathfrak{T}_2 \cup \mathfrak{T}_3)(\mathbb{R}^n)$, this statement remains valid with $R \in [R_{\mathcal{T}}, +\infty]$.

(ii) *Suppose that (19.4) is fulfilled, let $T \in \mathfrak{M}(\mathbb{R}^n)$, and let $f \in C^\infty(B_R)$. Then $f \in C^\infty_{\mathcal{T}}(B_R)$ if and only if for all k, j, relation (19.9) holds with the series converging in $\mathcal{E}(B_R)$. If $\mathcal{T} \in \mathfrak{T}_1(\mathbb{R}^n)$, the same is true for $R \in [R_{\mathcal{T}}, +\infty]$.*

(iii) *Assume that (19.4) holds and let $\alpha > 0$, $T \in \mathfrak{G}_\alpha(\mathbb{R}^n)$, $r(T) > 0$, and $f \in C^\infty(B_R) \cap G^\alpha(\dot{B}_{r(T)})$. Then $f \in C^\infty_{\mathcal{T}}(B_R)$ if and only if for all k, j, equality (19.9) holds with the series converging in $\mathcal{E}(B_R)$. If $\mathcal{T} \in \mathfrak{T}_1(\mathbb{R}^n)$, this assertion remains to be true with $R \in [R_{\mathcal{T}}, +\infty]$.*

(iv) *Assume that (19.3) is satisfied, let $T \in \mathfrak{M}(\mathbb{R}^n)$, $r(T) > 0$, and suppose that $f \in C^\infty(B_R) \cap \mathrm{QA}(\dot{B}_{r(T)})$. Then $f \in C^\infty_{\mathcal{T}}(B_R)$ if and only if for all k, j, decomposition (19.9) holds with the series converging in $\mathcal{E}(B_R)$. Moreover, if $f \in C^\infty_{\mathcal{T}}(B_R)$, then $f^{k,j} \in \mathrm{QA}_{\mathcal{T}}(B_R)$ for all k, j.*

(v) *Let $T \in \mathfrak{E}(\mathbb{R}^n)$, $f \in \mathcal{D}'(\mathbb{R}^n)$, and assume that f is of finite order in $\mathbb{R}^n$. Then $f \in \mathcal{D}'_{\mathcal{T}}(\mathbb{R}^n)$ if and only if for all k, j, equality (19.9) is satisfied with the series converging in $\mathcal{E}(\mathbb{R}^n)$. In particular, if $f \in \mathcal{D}'_{\mathcal{T}}(\mathbb{R}^n)$, then $f^{k,j} \in C^\infty_{\mathcal{T}}(\mathbb{R}^n)$ for all k, j.*

Proof. First note that $\mathfrak{M}(\mathbb{R}^n) \cap \mathcal{D}(\mathbb{R}^n) = \varnothing$ (see Theorem 8.5). Using now Theorem 14.19 and Remark 14.2, we see that the desired result is a consequence of Theorems 14.11, 14.17, 14.18, and 14.15 and Proposition 14.13. □

Remark 19.3. In the case where $\mathcal{Z}_{\mathcal{T}} = \varnothing$ assertions (ii)–(iv) of Theorem 19.8 give refinements of Theorem 19.2 for the correspondent classes $\mathcal{T}$ and f. For instance, if (19.3) is satisfied and $T = T_{i_0} \in \mathfrak{M}(\mathbb{R}^n)$, $r(T) > 0$, $\mathcal{Z}_{\mathcal{T}} = \varnothing$, and $f \in C^{\infty}_{\mathcal{T}} \cap \mathrm{QA}(\mathring{B}_{r(T)})$, then $f = 0$ (see (19.9) and Proposition 9.1(vi)).

Let us now establish an analog of the previous result for a spherical annulus.

Theorem 19.9. *Let $\mathcal{T} \in \mathfrak{T}_{\natural}(\mathbb{R}^n)$, $T = T_{i_0}$, and suppose that*

$$0 \leqslant r < r' < R' < R \leqslant +\infty, \quad R' - r' = 2r(T).$$

Then the following assertions hold.

(i) *Let $T \in \mathfrak{I}(\mathbb{R}^n)$, assume that*

$$r + 2\big(r_*(\mathcal{T}) + r(T_i)\big) < R \quad \text{for all } i \in \mathcal{I}, \tag{19.10}$$

and let $f \in \mathcal{D}'(B_{r,R})$. Then $f \in \mathcal{D}'_{\mathcal{T}}(B_{r,R})$ if and only if for all k, j,

$$f^{k,j} = \sum_{\lambda \in \mathcal{Z}_{\mathcal{T}}} \sum_{\eta=0}^{n(\lambda,\mathcal{T})} \alpha_{\lambda,\eta,k,j} \Phi_{\lambda,\eta,k,j} + \beta_{\lambda,\eta,k,j} \Psi_{\lambda,\eta,k,j}, \tag{19.11}$$

where $\alpha_{\lambda,\eta,k,j}, \beta_{\lambda,\eta,k,j} \in \mathbb{C}$, and the series in (19.11) converges in $\mathcal{D}'(B_{r,R})$. If $\mathcal{T} \in (\mathfrak{T}_2 \cup \mathfrak{T}_3)(\mathbb{R}^n)$, this statement remains valid with $R \in [r + 2R_{\mathcal{T}}, +\infty]$.

(ii) *Assume that (19.10) is satisfied, let $T \in \mathfrak{I}(\mathbb{R}^n)$, and let $f \in C^{\infty}(B_{r,R})$. Then $f \in C^{\infty}_{\mathcal{T}}(B_{r,R})$ if and only if for all k, j, relation (19.11) holds with the series converging in $\mathcal{E}(B_{r,R})$. If $\mathcal{T} \in \mathfrak{T}_1(\mathbb{R}^n)$, the same is true for $R \in [r+2R_{\mathcal{T}}, +\infty]$.*

(iii) *Suppose that (19.10) holds true, and let $\alpha > 0$, $T \in \mathfrak{I}_{\alpha}(\mathbb{R}^n)$, and $f \in C^{\infty}(B_{r,R}) \cap G^{\alpha}(\mathring{B}_{r',R'})$. Then $f \in C^{\infty}_{\mathcal{T}}(B_{r,R})$ if and only if for all k, j, relation (19.11) holds with the series converging in $\mathcal{E}(B_{r,R})$. If $\mathcal{T} \in \mathfrak{T}_1(\mathbb{R}^n)$, this assertion remains to be true with $R \in [r + 2R_{\mathcal{T}}, +\infty]$.*

(iv) *Let $T \in \mathfrak{I}(\mathbb{R}^n)$, let*

$$r + 2r(T_i) < R \leqslant +\infty \quad \text{for all } i \in \mathcal{I}, \tag{19.12}$$

and suppose that $f \in C^{\infty}(B_{r,R}) \cap \mathrm{QA}(\mathring{B}_{r',R'})$. Then $f \in C^{\infty}_{\mathcal{T}}(B_{r,R})$ if and only if for all k, j, decomposition (19.11) holds with the series converging in $\mathcal{E}(B_{r,R})$. Moreover, if $f \in C^{\infty}_{\mathcal{T}}(B_{r,R})$, then $f^{k,j} \in \mathrm{QA}_{\mathcal{T}}(B_{r,R})$ for all k, j.

Proof. Theorem 14.25 and Remark 14.3 show that the required result follows from Theorems 14.11, 14.24, and 14.15 and Proposition 14.13. □

We point out that the assumptions on R in Theorems 19.8(i)–(ii) and 19.9(i)–(ii) cannot be relaxed either (see Theorem 19.14 and Remark 19.3). Let us regard the case where these assumptions can be weakened.

Theorem 19.10. *Let $\mathcal{T} \in \mathfrak{T}_{\natural}(\mathbb{R}^n)$, $T = T_{i_0}$, and assume that for all $\lambda \in \mathcal{Z}_T$ and $\nu \in \mathcal{I}\backslash\{i_0\}$, the following estimates hold:*

$$\sum_{\eta=0}^{n(\lambda,T)} \left|\widetilde{T}_{\nu}^{(\eta)}(\lambda)\right| \leqslant M_{q,\nu}(1+|\lambda|)^{-2q}, \quad q = 1, 2, \dots, \tag{19.13}$$

where the constants $M_{q,\nu} > 0$ do not depend on λ, and

$$\sum_{p=1}^{\infty} \frac{1}{\inf_{q\geqslant p} M_{q,\nu}^{1/2q}} = +\infty. \tag{19.14}$$

Then assertions (i) *and* (ii) *of Theorem 19.8 remain valid provided that* (19.4) *is replaced by* (19.3)*. In addition, assertions* (i) *and* (ii) *of Theorem 19.9 are true if* (19.10) *is replaced by* (19.12).

A similar result for the one-dimensional case was obtained in Sect. 18.1 (see Theorem 18.4). The proof of Theorem 19.10 can be carried out along the same lines (see Proposition 14.2, Theorems 14.17, 14.24, and 14.15, and Propositions 9.18 and 9.21).

Corollary 19.1. *Assume that $\mathcal{T}$ satisfies all the assumptions in Theorem 19.10 and suppose that $T \in \mathfrak{M}(\mathbb{R}^n)$ and* (19.3) *is fulfilled. Then $\mathcal{D}'_{\mathcal{T}}(B_R) = \{0\}$ if $\mathcal{Z}_{\mathcal{T}} = \varnothing$.*

The proof follows from (19.9) and Proposition 9.1(vi).

Thanks to the following result, we see that assumption (19.14) in Theorem 19.10 cannot be relaxed in the general case.

Theorem 19.11. *Let $\mathcal{I}\backslash\{i_0\} \neq \varnothing$. Then for each sequence $\{M_q\}_{q=1}^{\infty}$ of positive numbers satisfying*

$$\sum_{p=1}^{\infty} \frac{1}{\inf_{q\geqslant p} M_{q}^{1/2q}} < +\infty, \tag{19.15}$$

there exist $T_{i_0} \in \mathfrak{N}(\mathbb{R}^n)$ and $T_\nu \in \mathcal{E}'_{\natural}(\mathbb{R}^n)$, $T_\nu \neq 0$, $\nu \in \mathcal{I}\backslash\{i_0\}$, such that the family $\mathcal{T} = \{T_i\}_{i\in\mathcal{I}}$ possesses the following properties:

(1) *for each $\nu \in \mathcal{I}\backslash\{i_0\}$, estimate* (19.13) *is satisfied with $M_{q,\nu} = M_q$;*

(2) *$r^*(\mathcal{T}) < +\infty$, and assertions* (i) *and* (ii) *of Theorem 19.8 fail for some $R > r^*(\mathcal{T})$.*

The proof of this theorem can be found in [225, Part III, Sect. 4.1].

19.3 More on the Berenstein–Gay Problem: The Case $R \leqslant r_1 + r_2$

Throughout in this section we assume that $\mathcal{T} = \{T_1, T_2\}$ is a family of nonzero distributions in $\mathcal{E}'_{\natural}(\mathbb{R}^n)$. As usual, we set

$$\mathcal{Z}_{\mathcal{T}} = \mathcal{Z}_{T_1} \cap \mathcal{Z}_{T_2}$$

and

$$r^*(\mathcal{T}) = \max\{r_1, r_2\} \quad \text{where } r_\nu = r(T_\nu),\ \nu \in \{1, 2\}.$$

In this section we shall obtain some analogues of the Berenstein–Gay result for the system

$$(f * T_\nu)(x) = 0, \quad |x| + r_\nu < R,\ \nu = 1, 2, \tag{19.16}$$

where $r^*(\mathcal{T}) < R \leqslant +\infty$. By analogy with the previous sections we write $\mathcal{D}'_{\mathcal{T}}(B_R)$ for the set of all $f \in \mathcal{D}'(B_R)$ satisfying (19.16). Similarly, we set

$$\mathcal{D}'_{\mathcal{T},\natural}(B_R) = \big(\mathcal{D}'_{\mathcal{T}} \cap \mathcal{D}'_{\natural}\big)(B_R), \qquad C^m_{\mathcal{T}}(B_R) = \big(\mathcal{D}'_{\mathcal{T}} \cap C^m\big)(B_R)$$

and

$$C^m_{\mathcal{T},\natural}(B_R) = \big(\mathcal{D}'_{\mathcal{T},\natural} \cap C^m\big)(B_R)$$

for $m \in \mathbb{Z}_+$ or $m = \infty$.

As a consequence of Theorem 19.2(i), we obtain

$$\mathcal{D}'_{\mathcal{T}}(B_R) = \{0\}, \quad \text{provided that } R > r_1 + r_2 \text{ and } \mathcal{Z}_{\mathcal{T}} = \varnothing.$$

On the other hand, if $\mathcal{Z}_{\mathcal{T}} \neq \varnothing$, then $(\mathcal{D}'_{\mathcal{T}} \cap \mathrm{RA})(\mathbb{R}^n) \neq \{0\}$ (see Proposition 14.2). Therefore, we shall consider system (19.16) for the case where

$$\mathcal{Z}_{\mathcal{T}} = \varnothing \quad \text{and} \quad r^*(\mathcal{T}) < R \leqslant r_1 + r_2.$$

Theorem 19.12. *Let $\mathcal{Z}_{\mathcal{T}} = \varnothing$. Then the following assertions hold.*

(i) *Let $R = r_1 + r_2 > r^*(\mathcal{T})$, $T_1 \in \mathrm{Inv}_+(\mathbb{R}^n)$, and assume that there exists a sequence $\zeta_1, \zeta_2, \ldots$ of complex numbers such that*

$$(2 + |\zeta_m|)^\alpha \Big(\big|\widetilde{T}_1(\zeta_m)\big| + \big|\widetilde{T}_2(\zeta_m)\big|\Big) \to 0 \quad \text{as } m \to \infty$$

for each $\alpha > 0$ and

$$|\mathrm{Im}\,\zeta_m| \leqslant c \log(2 + |\zeta_m|),$$

where the constant $c > 0$ is independent of m. Then $\mathcal{D}'_{\mathcal{T}}(B_R) = \{0\}$. In particular, this is true if $T_1 \in \mathfrak{M}(\mathbb{R}^n)$ and $\mathcal{Z}(\widetilde{T}_1) \approx \mathcal{Z}(\widetilde{T}_2)$. This occurs, for example, if $T_2 \in \mathcal{D}(\mathbb{R}^n)$.

(ii) *Assume that $T_1 \in \mathfrak{N}(\mathbb{R}^n)$, $r^*(\mathcal{T}) < R \leqslant r_1 + r_2$, $\mathcal{Z}(\widetilde{T}_1) \not\approx \mathcal{Z}(\widetilde{T}_2)$, and let $f \in \mathcal{D}'(B_R)$. Then $f \in \mathcal{D}'_{\mathcal{T}}(B_R)$ if and only if that for all $k \in \mathbb{Z}_+$ and $j \in \{1, \ldots, d(n,k)\}$,*

$$f^{k,j} = \Omega_{T_1,T_2,k,j} * v \quad \text{in } B_R$$

for some $v \in \mathcal{E}'_\natural(\mathbb{R}^n)$ *depending on* k, j *such that* $\operatorname{supp} v \subset \dot{B}_{r_1+r_2-R}$. *In particular,* $C^\infty_{\mathcal{T}}(B_R) \neq \{0\}$ *for* $R < r_1 + r_2$.

(iii) *If* $T_1 \in \mathfrak{N}(\mathbb{R}^n)$, $\mathcal{Z}(\widetilde{T}_1) \not\approx \mathcal{Z}(\widetilde{T}_2)$, *and* $r^*(\mathcal{T}) < R = r_1+r_2$, *then* $C^m_{\mathcal{T}}(B_R) \neq \{0\}$ *for each* $m \in \mathbb{Z}_+$.

We note that assertion (iii) fails with $m = \infty$ (see Theorem 19.13(i) below). Next, Theorem 19.13(ii) shows that for all $s, m \in \mathbb{Z}_+$ there exists $\mathcal{T} = \{T_1, T_2\}$ such that

$$T_1 \in \big(\mathfrak{N} \cap C^s\big)(\mathbb{R}^n), \quad T_2 \in \big(\mathcal{E}'_\natural \cap C^s\big)(\mathbb{R}^n), \quad \mathcal{Z}_{\mathcal{T}} = \varnothing,$$

and $C^m_{\mathcal{T}}(B_R) \neq \{0\}$ for $R = r(T_1) + r(T_2)$.

Proof of Theorem 19.12. Assertion (i) is a consequence of Theorems 19.1 and 18.7. Assertions (ii) and (iii) follow from Theorem 19.7. □

Important information concerning the class $C^\infty_{\mathcal{T}}(B_R)$ is contained in the following theorem.

Theorem 19.13.

(i) *Let* $\mathcal{Z}_{\mathcal{T}} = \varnothing$, $T_1 \in \mathfrak{M}(\mathbb{R}^n)$, $R = r_1 + r_2 > r^*(\mathcal{T})$, *and assume that* $f \in \mathcal{D}'_{\mathcal{T}}(B_R) \cap C^\infty(B_{r_1+\varepsilon})$ *for some* $\varepsilon \in (0, R - r_1)$. *Then* $f = 0$ *in* B_R.

(ii) *Let* $T_1 \in \mathfrak{N}(\mathbb{R}^n)$ *and* $r(T_1) > 0$. *Then there exists* $T_2 \in \mathcal{E}'_\natural(\mathbb{R}^n)$ *such that* $r(T_2) = r(T_1)$ *and the family* $\mathcal{T} = \{T_1, T_2\}$ *possesses the following properties:*

(1) $\mathcal{Z}_{\mathcal{T}} = \varnothing$;

(2) *If* $R = r(T_1) + r(T_2)$, *then for each* $m \in \mathbb{Z}_+$, *there exists a nontrivial function* $f \in C^m_{\mathcal{T}}(B_R)$ *such that* $f = 0$ *in* $B_{r(T_1)}$.

Proof. To prove (i) it is enough to show that

$$\big(\mathcal{D}'_{\mathcal{T}} \cap \mathcal{D}'_\natural\big)(B_R) \cap C^\infty(B_{r_1+\varepsilon}) = \{0\}$$

(see the proof of Theorem 19.1). However,

$$\big(\mathcal{D}'_{\mathcal{T}} \cap \mathcal{D}'_\natural\big)(B_R) \cap C^\infty(B_{r_1+\varepsilon}) = \big(\mathcal{D}'_{\mathcal{T}} \cap C^\infty_\natural\big)(B_R)$$

in view of Theorem 14.19. Using now Theorems 19.1 and 18.8(i), (iii), we arrive at (i).

Turning to (ii), let $T_1 \in \mathfrak{N}(\mathbb{R}^n)$, $r(T_1) > 0$, and $T_2 = T_1 + \delta_0$, where δ_0 is the Dirac measure supported at the origin in $\mathbb{R}^n$. Then $T_2 \in \mathcal{E}'_\natural(\mathbb{R}^n)$, $r(T_2) = r(T_1)$, and $\widetilde{T}_2(\lambda) = 1$ for each $\lambda \in \mathcal{Z}_{T_1}$. Thus, $\mathcal{Z}_{\mathcal{T}} = \varnothing$, where $\mathcal{T} = \{T_1, T_2\}$. Suppose that $m \in \mathbb{Z}_+$ and $R = r(T_1) + r(T_2)$. By Theorem 14.7(ii) and Lemma 19.1(i) we deduce that the distribution $f = \Omega_{T_1,T_2,k,j}$ is in $\mathcal{D}'_{\mathcal{T}}(B_R)$ for all $k \in \mathbb{Z}_+$ and $j \in \{1, \dots, d(n,k)\}$. In addition, by the definition of T_2 one has

$$f = f + f * T_1 = \Omega_{T_1,T_2,k,j} * T_2 = \zeta_{T_1,k,j}.$$

If k is large enough, then $f \in C^m_{\mathcal{T}}(B_R)$ for an arbitrary $j \in \{1, \dots, d(n,k)\}$ (see Lemma 19.1(ii)). Because of Theorem 14.7(ii), this proves (ii). □

Thus, by Theorem 19.13, $C^\infty_{\mathcal{T}}(B_R) = \{0\}$, provided that $\mathcal{Z}_{\mathcal{T}} = \varnothing$, $T_1 \in \mathfrak{M}(\mathbb{R}^n)$, and $R = r_1 + r_2 > r(\mathcal{T})$. The following result ensures us that the value R in this statement cannot be decreased in the general case.

Theorem 19.14. *Assume that $T_1 \in \mathfrak{N}(\mathbb{R}^n)$ and there exists $l > 0$ such that the set*

$$E_l(\mathcal{T}) = \left\{\lambda \in \mathcal{Z}_{T_1} : \left|\widetilde{T}_2(\lambda)\right| < (2 + |\lambda|)^{-l}\right\}$$

is sparse (see Definition 18.1). Also let $\{M_q\}_{q=1}^{\infty}$ be a sequence of positive numbers satisfying estimate (19.15). Then the following statements are valid.

(i) *If $r^*(\mathcal{T}) < R < r_1 + r_2$, then there exists a nontrivial function $f \in C^\infty_{\mathcal{T},\natural}(B_R)$ such that*

$$\sup_{x \in B_R} \left|\left(\Delta^q f\right)(x)\right| \leqslant M_q \quad \textit{for all } q.$$

(ii) *If $R > r > 0$ and $2r^*(\mathcal{T}) < R - r < 2(r_1 + r_2)$, then there is a nontrivial function $f \in C^\infty_\natural(B_{r,R})$ such that*

$$(f * T_\nu)(x) = 0, \quad r + r_\nu < |x| < R - r_\nu, \ \nu = 1, 2,$$

and

$$\sup_{x \in B_{r,R}} \left|\left(\Delta^q f\right)(x)\right| \leqslant M_q \quad \textit{for all } q.$$

Notice that condition (19.15) in this theorem cannot be weakened in general (see Remark 19.3). In addition, Theorem 19.14 fails without the assumption about the sparseness of $E_l(\mathcal{T})$ for some $l > 0$ (see Theorem 19.15 below). This assumption is satisfied for a broad class of families $\mathcal{T}$ (see Propositions 18.1 and 18.2).

The proof of Theorem 19.14 requires some preparation.

Lemma 19.2. *Let $\varepsilon > 0$ and assume that $\{M_q\}_{q=1}^{\infty}$ is a sequence of positive numbers satisfying (19.15). Then there exist nonzero functions $u_1, u_2 \in \mathcal{D}_\natural(-\varepsilon, \varepsilon)$ such that $\mathcal{Z}(\widehat{u}_1) \cap \mathcal{Z}(\widehat{u}_2) = \varnothing$ and*

$$|\widehat{u}_1(z)| + |\widehat{u}_2(z)| \leqslant M_q (2 + |z|)^{-2q} \mathrm{e}^{\varepsilon |\mathrm{Im}\, z|}$$

for all $z \in \mathbb{C}$, $q \in \mathbb{N}$.

Proof. Thanks to Theorem 8.1(ii), for each $c > 0$, there exists a nonzero function $u \in \mathcal{D}_\natural(-\varepsilon/4, \varepsilon/4)$ such that $|u^{(q)}(t)| \leqslant c^{-1-q} M_q^{1/q}$ for all $t \in \mathbb{R}^1$, $q \in \mathbb{N}$. This, together with (6.33), yields

$$|\widehat{u}(z)| \leqslant \varepsilon c^{-1-q} M_q^{1/q} |z|^{-q} \mathrm{e}^{\varepsilon |\mathrm{Im}\, z|/4}, \quad |z| \geqslant 1, \ q \in \mathbb{N}.$$

Hence, for some $\zeta \in (1/2, 1)$, the functions $u_1 = u * u$ and $u_2(t) = u_1(\zeta t)$ satisfy the requirements of the lemma if c is large enough. □

Corollary 19.2. *If $\varepsilon > 0$ and $\{M_q\}_{q=1}^{\infty}$ is a sequence of positive numbers satisfying (19.15), then there exist nonzero functions $v_1, v_2 \in \mathcal{D}_\natural(B_\varepsilon)$ such that $\mathcal{Z}(\widetilde{v}_1) \cap \mathcal{Z}(\widetilde{v}_2) = \varnothing$ and*

$$\left|\left(\Delta^q v_1\right)(x)\right| + \left|\left(\Delta^q v_2\right)(x)\right| \leqslant M_q$$

for all $x \in \mathbb{R}^n$ and $q \in \mathbb{N}$.

The proof follows from Lemmas 19.2 and 8.1, Theorem 6.3, and (6.38).

Proof of Theorem 19.14. To prove (i) first observe that for each $\varepsilon \in (0, (r_1 + r_2 - R)/3)$, there is a nontrivial function $f_1 \in C^\infty_{\mathcal{T},\natural}(B_{R+2\varepsilon})$ (see Theorems 19.1 and 18.9). By Corollary 19.2, Theorem 19.2(i), and Theorem 6.1 we can choose $u \in \mathcal{D}_\natural(B_\varepsilon)$ so that the convolution $f_2 = f_1 * u$ is nontrivial in B_R and

$$\sup_{x \in B_\varepsilon} \left|\left(\Delta^q u\right)(x)\right| \leqslant M_q \quad \text{for all } q.$$

Now define

$$f = cf_2 \quad \text{with } c = \left(\int_{B_{R+\varepsilon}} |f_2(x)| \mathrm{d}x\right)^{-1}.$$

Then the function f satisfies all the requirements of assertion (i).

The proof of (ii) is similar to that of (i), the only change being that, instead of Theorems 18.9 and 19.1, we now use [225, Part III, Theorem 4.9]. □

To conclude we shall establish multidimensional analogues of Theorems 18.10 and 18.12.

Theorem 19.15.

(i) *Let $\mathcal{Z}_{\mathcal{T}} = \varnothing$, $T_1 \in \mathrm{Inv}_+(\mathbb{R}^n)$, and assume that there exists $\psi \in \mathfrak{M}(\mathbb{R}^n)$ such that $r(\psi) > 0$ and for each $\lambda \in \mathcal{Z}_\psi$,*

$$\sum_{\eta=0}^{n(\lambda,\psi)} \left(\left|\widetilde{T}_1^{(\eta)}(\lambda)\right| + \left|\widetilde{T}_2^{(\eta)}(\lambda)\right|\right) \leqslant M_q(2 + |\lambda|)^{-q}, \quad q = 1, 2, \ldots$$

where the constants $M_q > 0$ are independent of λ and satisfy (18.10). *Then $\mathcal{D}'_{\mathcal{T}}(B_R) = \{0\}$, provided that $R > \max\{r^*(\mathcal{T}), r_1 + r_2 - r(\psi)\}$. This statement is no longer valid in general if*

$$r^*(\mathcal{T}) < R = r_1 + r_2 - r(\psi).$$

(ii) *Let $\mathcal{Z}_{\mathcal{T}} = \varnothing$, $T_1 \in \mathfrak{N}(\mathbb{R}^n)$, and assume that $T_1 = \psi_1 * \psi_2$, where $\psi_1 \in \mathfrak{N}(\mathbb{R}^n)$, $\psi_2 \in \mathcal{E}'_\natural(\mathbb{R}^n)$, and $\mathcal{Z}(\widetilde{\psi}_1) \cap \mathcal{Z}(\widetilde{\psi}_2) = \varnothing$. Also suppose that for each $\lambda \in \mathcal{Z}_{\psi_2}$, the estimates*

$$\sum_{\eta=0}^{n(\lambda,\psi_2)} \left|\widetilde{T}_2^{(\eta)}(\lambda)\right| \leqslant M_q(1 + |\lambda|)^{-q}, \quad q = 1, 2, \ldots,$$

hold, where the constants $M_q > 0$ *do not depend on* λ *and satisfy* (18.10). *Then* $\mathcal{D}'_{\mathcal{T}}(B_R) = \{0\}$, *provided that* $R > \max\{r_1, r_2 + r(\psi_1)\}$. *In addition,* $C^{\infty}_{\mathcal{T}}(B_R) = \{0\}$ *if* $R \geqslant r_2 + r(\psi_1)$ *and* $R > r^*(\mathcal{T})$.

(iii) *There exists a family* $\mathcal{T} = \{T_1, T_2\}$ *satisfying all the assumptions of assertion* (ii) *such that* $r(T_2) + r(\psi_1) > r^*(\mathcal{T})$ *and* $C^m_{\mathcal{T}}(B_R) \neq \{0\}$ *for each* $m \in \mathbb{N}$, *provided that* $R = r(T_2) + r(\psi_1)$.

(iv) *There exists a family* $\mathcal{T} = \{T_1, T_2\}$ *satisfying all the assumptions of* (ii) *such that* $r(T_2) + r(\psi_1) > r^*(\mathcal{T})$ *and* $C^{\infty}_{\mathcal{T}}(B_R) \neq \{0\}$ *for each* $R \in (r^*(\mathcal{T}), r(T_2) + r(\psi_1))$.

Proof. Assertions (i), (ii), and (iv) are consequences of Theorems 18.12, 18.10, and 19.1. The proof of (iii) can be found in [225, Part III, Theorem 4.10]. □

Using Theorems 19.1 and 18.11, one can also obtain the multidimensional analog of Theorem 18.11. This shows that assumption (18.10) in Theorem 19.15 cannot be relaxed either.

19.4 The Equivalence of the Local and the Global Pompeiu Properties

Let $\mathrm{M}(n)$ be the group of Euclidean motions in $\mathbb{R}^n$, and let $\psi \in \mathcal{E}'(\mathbb{R}^n)$. For fixed $g \in \mathrm{M}(n)$, we define the action of the distribution $g\psi \in \mathcal{E}'(\mathbb{R}^n)$ by

$$\langle g\psi, f(x)\rangle = \big\langle \psi, f\big(g^{-1}x\big)\big\rangle, \quad f \in \mathcal{E}\big(\mathbb{R}^n\big).$$

For $R \in (0, +\infty]$, we set $M_{\psi,R} = \{g \in \mathrm{M}(n) : \operatorname{supp} g\psi \subset B_R\}$.

Let $\Psi = \{\psi_i\}_{i\in\mathcal{I}}$ be a family of distributions in the class $\mathcal{E}'(\mathbb{R}^n)$ such that $r(\psi_i) < R \leqslant +\infty$ for all $i \in \mathcal{I}$. The family Ψ is said to have the *Pompeiu property* in B_R if there is no nontrivial function $f \in C^{\infty}(B_R)$ such that

$$\langle g\psi_i, f\rangle = 0 \quad \text{for all } i \in \mathcal{I},\ g \in M_{\psi_i,R}. \tag{19.17}$$

The most interesting case is that of a single distribution $\psi \in \mathcal{E}'(\mathbb{R}^n)$ which is the characteristic function of some compact set E in $\mathbb{R}^n$. In this case, the set E is said to have the Pompeiu property in B_R if the same is true for $\Psi = \{\psi\}$. Denote by $\mathrm{Pomp}(B_R)$ the family of all compact subsets of $\mathbb{R}^n$ having the Pompeiu property in B_R.

We now present the central result of this section.

Theorem 19.16. *Let* $\Psi = \{\psi_i\}_{i\in\mathcal{I}}$ *be a family of nonzero distributions in the class* $\mathcal{E}'(\mathbb{R}^n)$, *and let* $r(\psi_\nu) + \inf_{i\in\mathcal{I}} r(\psi_i) < R \leqslant +\infty$ *for all* $\nu \in \mathcal{I}$. *Then the following assertions are equivalent.*

(i) *The family* Ψ *has the Pompeiu property in* $\mathbb{R}^n$.

(ii) *The family* Ψ *has the Pompeiu property in the ball* B_R.

Proof. It is enough to prove the implication (i)→(ii). Without loss of generality, we can assume that $R < +\infty$ and $\operatorname{supp}\psi_i \subset \mathring{B}_{r(\psi_i)}$ for all $i \in \mathcal{I}$. Let (i) hold, and let $\varepsilon_\nu = R - r(\psi_\nu) - \inf_{i\in\mathcal{I}} r(\psi_i)$. For $h \in B_{\varepsilon_i/3}$, define $T_{i,h} \in \mathcal{E}'_\natural(\mathbb{R}^n)$ by the relation $T_{i,h} = (\tau_h\psi_i)^{0,1}$, where $\tau_h\psi_i = \psi_i(\cdot - h)$ (see (9.8) and (9.9)). Then $\operatorname{supp} T_{i,h} \subset B_{r(\psi_i)+\varepsilon_i/3}$. In addition, the set

$$\mathcal{A}_i = \{h \in B_{\varepsilon_i/3} : T_{i,h} \neq 0\}$$

is dense in $B_{\varepsilon_i/3}$. In fact, if $\psi_i \in (\mathcal{E}' \cap L^1)(\mathbb{R}^n)$, this follows from Theorem 1.1 in [225, Part V]. The general case reduces to this one by means of the standard smoothing trick (see Proposition 9.1(iv) and Theorem 6.1). Assume that there exists $\lambda \in \mathbb{C}$ such that $\widetilde{T_{i,h}}(\lambda) = 0$ for each i and all $h \in \mathcal{A}_i$. Then the function $u = \Phi_{\lambda,0,0,1}$ satisfies $u * T_{i,h} = 0$ for all i and for all $h \in \mathcal{A}_i$ (see (14.4)). Since u is real-analytic, the same is true for all i and all $h \in \mathbb{R}^n$. Bearing in mind that u is radial, we obtain

$$\langle g\psi_i, u\rangle = 0 \quad \text{for all } g \in \mathrm{M}(n),\ i \in \mathcal{I}.$$

This contradicts (i), and hence $\bigcap_{i\in\mathcal{I}}\bigcap_{h\in\mathcal{A}_i} \mathcal{Z}(\widetilde{T_{i,h}}) = \varnothing$. Assume now that $f \in C^\infty(B_R)$ and (19.17) holds. Then $f * T_{i,h} = 0$ for each i and all $h \in \mathcal{A}_i$. Theorem 19.2 gives $f = 0$, and (ii) is proved. □

Theorem 19.16 solves Problem 8.1 in [225, Part IV]. As a consequence, one has the following result.

Corollary 19.3. *Let E be a compact subset of $\mathbb{R}^n$, and let $r_E = r(\chi_E)$, where χ_E is the characteristic function of E. If $E \in \mathrm{Pomp}(\mathbb{R}^n)$ and $R > 2r_E$, then $E \in \mathrm{Pomp}(B_R)$.*

We note that for each $\varepsilon \in (0, 1)$, this corollary fails in general with $R = (2-\varepsilon)r_E$ (see [225, Part IV, Theorem 1.6]).

Using the argument in the proof of Theorem 19.16, we now give a proof of the Brown–Schreiber–Taylor theorem on the Pompeiu property [42].

Theorem 19.17. *Suppose that a family $\Psi = \{\psi_i\}_{i\in\mathcal{I}}$ of nonzero distributions in $\mathcal{E}'(\mathbb{R}^n)$ does not have the Pompeiu property in $\mathbb{R}^n$. Then there exists $\lambda \in \mathbb{C}$ such that for each i, the function $\widehat{\psi_i}$ vanishes identically on the analytic variety*

$$S_\lambda = \left\{(\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n : \zeta_1^2 + \cdots + \zeta_n^2 = \lambda^2\right\}.$$

Proof. If Ψ does not have the Pompeiu property, the proof of Theorem 19.16 shows that there exists $\lambda \in \mathbb{C}$ such that $\langle g\psi_i, \Phi_{\lambda,0,0,1}\rangle = 0$ for each i and all $g \in \mathrm{M}(n)$. This implies easily that $\langle g\psi_i, f\rangle = 0$ for all $f \in C^\infty(\mathbb{R}^n)$ satisfying $\Delta f + \lambda^2 f = 0$ (see the proof of Lemma 1.4 in [225, Part 4]). Setting $f(x) = \mathrm{e}^{-\mathrm{i}(x_1\zeta_1+\cdots+x_n\zeta_n)}$, where $(\zeta_1, \dots, \zeta_n) \in S_\lambda$, we obtain $\widehat{\psi_i} = 0$ on S_λ. □

For the rest of the section, we assume that Ω is a nonempty open bounded set in $\mathbb{R}^n$. Let $\partial\Omega$ be the boundary of Ω, E the closure of Ω, and r_E as defined in

Corollary 19.3. The Pompeiu problem asks under what conditions there exists a nonzero function $f \in C^\infty(\mathbb{R}^n)$ satisfying

$$\int_{g\Omega} f(x)\,\mathrm{d}x = 0 \tag{19.18}$$

for all rigid motions g of $\mathbb{R}^n$? A large amount of research has gone into this problem, but it remains open. We answer this question in the case where f is not real-analytic, the set $\mathbb{R}^n \backslash E$ is connected, and $\partial\Omega$ is locally the graph of a Lipschitz function by showing that these assumptions imply that Ω is a ball.

Theorem 19.18. *Suppose that $\partial\Omega$ is locally the graph of a Lipschitz function and that the set $\mathbb{R}^n \backslash E$ is connected. Let $R > 2r_E$ and assume that f is locally integrable in B_R and that (19.18) is satisfied for each $g \in \mathrm{M}(n)$ such that $gE \subset B_R$. Then either Ω is a ball or there exists a nonzero polynomial p depending only on Ω such that*

$$p(\Delta) f = 0 \quad \text{in } B_R. \tag{19.19}$$

Proof. If the only function f satisfying the requirements of the theorem is $f = 0$, then (19.19) holds for all polynomials p. Suppose that $f \neq 0$. Without loss of generality, we assume that $E \subset \mathring{B}_{r_E}$. Let $\varepsilon = R - 2r_E$ and put

$$A_\varepsilon = \left\{h \in \mathrm{M}(n) : h\mathring{B}_{r_E} \subset B_{r_E+\varepsilon/3}\right\}.$$

For $h \in A_\varepsilon$, define $T^h \in \mathcal{E}'_\natural(\mathbb{R}^n)$ by $T^h = (h\chi_E)^{0,1}$, where χ_E is the characteristic function of E (see (9.9)). Let $\mathcal{Z}_E = \bigcap_{h \in A_\varepsilon} \mathcal{Z}(\widetilde{T^h})$. Since $f \neq 0$, the proof of Theorem 19.16 gives $\mathcal{Z}_E \neq \varnothing$. First, consider the case in which the set $\mathcal{Z}_E$ is finite: $\mathcal{Z}_E = \{\lambda_1, \dots, \lambda_N\}$. For each $l \in \{1, \dots, N\}$, set $m_l = 1 + \min_{h \in A_\varepsilon} n(\lambda_l, T^h)$ and define $U_{h,l} \in \mathcal{E}'(\mathbb{R}^n)$ by

$$\widetilde{U_{h,l}}(z) = \left(z^2 - \lambda_l^2\right)^{-n(\lambda_l, T^h)-1} \widetilde{T^h}(z), \quad z \in \mathbb{C}.$$

Then $\bigcap_{h \in A_\varepsilon} \mathcal{Z}(\widetilde{U_{h,l}}) = \varnothing$ and $p(\Delta) f * U_{h,l} = 0$, where $p(z) = \prod_{l=1}^N (z^2 + \lambda_l^2)^{m_l}$. Thus, (19.19) holds by Theorem 19.2.

It remains to consider the case in which the set $\mathcal{Z}_E$ is infinite. In this case, the proof of Theorem 19.17 shows that $\widehat{\chi_E} = 0$ on S_λ for some infinite set of values λ. In addition, $\partial\Omega$ is real-analytic, owing to Williams [262]. In view of Berenstein and Yang [23], this implies that Ω is a ball. □

As a consequence, we obtain the following result.

Theorem 19.19. *Let Ω be as in Theorem 19.18. Assume that $R > 2r_E$ and that there exists $f \in C(B_R)$ which is not real-analytic such that (19.18) holds for all $g \in \mathrm{M}(n)$ for which $gE \subset B_R$. Then Ω is a ball. Conversely, if Ω is a ball, then there exists $f \in C^\infty(\mathbb{R}^n)$ which is not real-analytic such that (19.18) holds for all $g \in \mathrm{M}(n)$.*

Proof. In view of the ellipticity of the operator $p(\Delta)$, every solution of (19.19) is real-analytic. Once Theorem 19.18 has been established, the set Ω is a ball. The converse result follows from Theorem 14.8(ii). □

When $n = 2$, there is a close connection between the Pompeiu property and the theory of analytic functions. In what follows, each point $(x_1, x_2) \in \mathbb{R}^2$ is identified with the complex number $z = x_1 + ix_2$. Let $\Omega \subset \mathbb{C}$ be a Jordan region with piecewise smooth boundary Γ. We say that Γ has the *Morera property* if each function $f \in C(\mathbb{C})$ which satisfies

$$\int_{g\Gamma} f(z)\,dz = 0 \quad \text{for every } g \in \mathrm{M}(2) \tag{19.20}$$

is entire.

Suppose now that $E = \mathrm{Cl}\Omega \in \mathrm{Pomp}(\mathbb{R}^2)$. If $f \in C^\infty(\mathbb{C})$ satisfies (19.20), it follows from the Green's theorem that

$$\int_{gE} \frac{\partial f}{\partial \overline{z}}\,dx_1\,dx_2 = \frac{1}{2i}\int_{g\Gamma} f(z)\,dz = 0.$$

Hence, $\frac{\partial f}{\partial \overline{z}} = 0$, and f is entire. A standard smoothing shows that it is actually sufficient to assume that $f \in C(\mathbb{C})$. Thus, the Pompeiu property for $E = \mathrm{Cl}\Omega$ implies the Morera property for $\Gamma = \partial\Omega$. Actually, the two properties are equivalent, since for each $h \in C^\infty(\mathbb{C})$, there exists $f \in C^\infty(\mathbb{C})$ such that $\frac{\partial f}{\partial \overline{z}} = h$ (see Hörmander [126], Theorem 4.4.6).

Chapter 20
$\mathcal{E}'_{\natural\natural}(X)$ Spectral Analysis on Domains of Noncompact Symmetric Spaces of Arbitrary Rank

In Chap. 19 we developed a new approach to problems of spectral analysis on domains in $\mathbb{R}^n$. The main advantage of this approach is that it enables the results of the previous chapter to be extended to the case of noncompact symmetric spaces $X = G/K$. We do this in the present chapter (see Sects. 20.1–20.3). In particular, in Sects. 20.1 and 20.2 we give a symmetric space analog of the local version of the Brown–Schreiber–Taylor theorem and investigate the problem of existence of a nontrivial solution for systems of convolution equations in more detail. In the study of similar questions on $\mathbb{R}^n$ ($n > 1$) the class $\mathcal{E}'_\natural(\mathbb{R}^n)$ has played an important role. In the same way the class $\mathcal{E}'_{\natural\natural}(X)$ which we introduced in Part II is essential in Sects. 20.1 and 20.2. As in Chap. 14, natural analogues of the main results fail to be true for the class $\mathcal{E}'_\natural(X)$ if rank $X \geqslant 2$. In Sect. 20.3 we restrict ourselves to the case where rank $X = 1$ and prove some results concerning the exponential representation problem.

The theory given here applies to the special problems arising from integral geometry. Namely, in Sect. 20.4 we present a local version of Zalcman's two-radii theorem for weighted ball means on symmetric spaces X with complex group G.

20.1 Symmetric Space Analogues of the Local Version of the Brown–Schreiber–Taylor Theorem

Throughout this chapter we assume that $X = G/K$ is a symmetric space of noncompact type. Let $\mathcal{T} = \{T_i\}_{i\in\mathcal{I}}$ be a given family of nonzero K-invariant distributions on X with compact supports. Assume that $\mathcal{O}$ is an open subset of X such that for each $i \in \mathcal{I}$, the set

$$G_{\mathcal{O},T_i} = \left\{g \in G\colon g\mathring{B}_{r(T_i)} \subset \mathcal{O}\right\}$$

is nonempty. Then for all $f \in \mathcal{D}'(\mathcal{O})$, the convolution $f \times T_i$ is well defined as distribution in $\mathcal{D}'(\mathcal{O}_{T_i})$, where

V.V. Volchkov, V.V. Volchkov, *Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group,* Springer Monographs in Mathematics,

$$\mathcal{O}_{T_i} = \{x = go \in \mathcal{O}\colon g \in G_{\mathcal{O},T_i}\}, \quad i \in \mathcal{I}.$$

Let us consider the system of convolution equations

$$(f \times T_i)(x) = 0, \quad x \in \mathcal{O}_{T_i},\ i \in \mathcal{I}, \tag{20.1}$$

with unknown $f \in \mathcal{D}'(\mathcal{O})$. Denote by $\mathcal{D}'_{\mathcal{T}}(\mathcal{O})$ the set of all distributions $f \in \mathcal{D}'(\mathcal{O})$ satisfying (20.1). For $m \in \mathbb{Z}_+$ or $m = \infty$, we put

$$C^m_{\mathcal{T}}(\mathcal{O}) = \big(\mathcal{D}'_{\mathcal{T}} \cap C^m\big)(\mathcal{O}).$$

Next, if $\mathcal{O}$ is K-invariant, we set

$$\mathcal{D}'_{\mathcal{T},\natural}(\mathcal{O}) = \big(\mathcal{D}'_{\mathcal{T}} \cap \mathcal{D}'_{\natural}\big)(\mathcal{O}) \quad \text{and} \quad C^m_{\mathcal{T},\natural}(\mathcal{O}) = \big(\mathcal{D}'_{\mathcal{T},\natural} \cap C^m\big)(\mathcal{O}).$$

Let $\Lambda_+(\mathcal{T}) = \{\Lambda_+(T_i)\}_{i\in\mathcal{I}}$ be the family of distributions in the class $\mathcal{E}'(\mathfrak{a})$ such that

$$\widehat{\Lambda_+(T_i)} = \widetilde{T}_i$$

for each $i \in \mathcal{I}$ (see Sect. 10.6). By (10.104) we see that $r(T_i) = r(\Lambda_+(T_i))$ for all i. If

$$r(T_i) < R \leqslant +\infty \quad \text{for each } i \in \mathcal{I}, \tag{20.2}$$

then we set

$$\mathcal{D}'_{\Lambda_+(\mathcal{T}),W}(\mathcal{B}_R) = \bigcap_{i\in\mathcal{I}} \mathcal{D}'_{\Lambda_+(T_i),W}(\mathcal{B}_R),$$

where $\mathcal{D}'_{\Lambda_+(T_i),W}(\mathcal{B}_R)$ is the set of all W-invariant distributions $f \in \mathcal{D}'(\mathcal{B}_R)$ satisfying the convolution equation

$$f * \Lambda_+(T_i) = 0.$$

As usual, for $m \in \mathbb{Z}_+$ or $m = \infty$, we set

$$C^m_{\Lambda_+(T_i),W}(\mathcal{B}_R) = \big(\mathcal{D}'_{\Lambda_+(T_i),W} \cap C^m\big)(\mathcal{B}_R),$$

$$C^m_{\Lambda_+(\mathcal{T}),W}(\mathcal{B}_R) = \bigcap_{i\in\mathcal{I}} C^m_{\Lambda_+(T_i),W}(\mathcal{B}_R).$$

In this section we shall study the problem of existence of a nonzero solution of system (20.1). The following result will be one of our most important tools.

Theorem 20.1. *Assume that* (20.2) *is satisfied. Then following assertions are equivalent.*

(i) $\mathcal{D}'_{\mathcal{T}}(B_R) = \{0\}$.
(ii) $\mathcal{D}'_{\mathcal{T},\natural}(B_R) = \{0\}$.
(iii) $\mathcal{D}'_{\Lambda_+(\mathcal{T}),W}(\mathcal{B}_R) = \{0\}$.

The same is true if $\mathcal{D}'_{\mathcal{T}}(B_R)$ *is replaced by* $C^{\infty}_{\mathcal{T}}(B_R)$*,* $\mathcal{D}'_{\mathcal{T},\natural}(B_R)$ *by* $C^{\infty}_{\mathcal{T},\natural}(B_R)$*, and* $\mathcal{D}'_{\Lambda_+(\mathcal{T}),W}(\mathcal{B}_R)$ *by* $C^{\infty}_{\Lambda_+(\mathcal{T}),W}(\mathcal{B}_R)$*.*

Proof. It is clear that (i) implies (ii). In addition, by Theorems 10.15 and 10.12(ii) we conclude that (iii) is a consequence of (ii). Assume now that (iii) holds. To prove (i), take $f \in \mathcal{D}'_{\mathcal{T}}(B_R)$ and suppose that $\delta \in \widehat{K}_M$. It follows by Proposition 15.1(iii) that $f_\delta \in \mathcal{D}'_{\mathcal{T}}(B_R)$. In view of Proposition 15.1(iv), every matrix entry of $\mathfrak{A}_\delta(f_\delta)$ is in the class $\mathcal{D}'_{\Lambda_+(\mathcal{T}),W}(\mathcal{B}_R)$. By our assumption, $\mathfrak{A}_\delta(f_\delta) = 0$ in $\mathcal{B}_R$. Now Theorem 10.12(iii) implies that $f_\delta = 0$ in B_R. This gives, by Proposition 10.2(iii), that $f = 0$ and $\mathcal{D}'_{\mathcal{T}}(B_R) = \{0\}$.

To prove the theorem for the classes $C^{\infty}_{\mathcal{T}}(B_R)$, $C^{\infty}_{\mathcal{T},\natural}(B_R)$, and $C^{\infty}_{\Lambda_+(\mathcal{T}),W}(\mathcal{B}_R)$, it is sufficient to repeat the above arguments. □

Remark 20.1. Using Theorems 10.15 and 10.12(ii), we infer that if $C^{m}_{\mathcal{T},\natural}(B_R) = \{0\}$ for some $m \in \mathbb{Z}_+$, then $C^{m}_{\Lambda_+(\mathcal{T}),W}(\mathcal{B}_R) = \{0\}$. However, it can be shown that in general the condition $C^{m}_{\Lambda_+(\mathcal{T}),W}(\mathcal{B}_R) = \{0\}$ does not imply that $C^{m}_{\mathcal{T}}(B_R) = \{0\}$ (see Theorem 18.8(iii) and Theorem 20.7 below).

Denote by $\mathfrak{T}_{\natural\natural}(X)$ the set of all families $\mathcal{T} = \{T_i\}_{i\in\mathcal{I}}$ such that

$$T_i \in \mathcal{E}'_{\natural\natural}(X) \quad \text{and} \quad T_i \neq 0 \quad \text{for each } i \in \mathcal{I}.$$

Observe that if rank $X = 1$, then every family $\mathcal{T} = \{T_i\}_{i\in\mathcal{I}}$ of nonzero K-invariant compactly supported distributions on X is in the class $\mathfrak{T}_{\natural\natural}(X)$.

As a consequence of Theorem 20.1, we obtain the following statement.

Corollary 20.1. *Let* $\mathcal{T} \in \mathfrak{T}_{\natural\natural}(X)$*, and let* $\mathcal{G} = \{g_i\}_{i\in\mathcal{I}}$ *be the family of distributions in the class* $\mathcal{E}'_{\natural}(\mathbb{R}^1)$ *such that* $\widehat{g_i} = \overset{\circ}{T}_i$ *for all* i*. Assume that (20.2) is satisfied. Then*

(i) $\mathcal{D}'_{\mathcal{T}}(\mathcal{B}_R) = \{0\}$ *if and only if* $\mathcal{D}'_{\mathcal{G},\natural}(-R, R) = \{0\}$.
(ii) $C^{\infty}_{\mathcal{T}}(B_R) = \{0\}$ *if and only if* $C^{\infty}_{\mathcal{G},\natural}(-R, R) = \{0\}$.

The proof is clear from Theorems 20.1 and 19.1.

For $\mathcal{T} \in \mathfrak{T}_{\natural\natural}(X)$, we define

$$\mathcal{Z}_{\mathcal{T}} = \bigcap_{i\in\mathcal{I}} \mathcal{Z}_{T_i},$$

$$r_*(\mathcal{T}) = \inf_{i\in\mathcal{I}} r(T_i), \quad r^*(\mathcal{T}) = \sup_{i\in\mathcal{I}} r(T_i), \quad R_{\mathcal{T}} = r_*(\mathcal{T}) + r^*(\mathcal{T}).$$

Let $\mathfrak{T}_1(X)$ be the set of families $\mathcal{T} \in \mathfrak{T}_{\natural\natural}(X)$ such that

$$0 < r_*(\mathcal{T}) = r(T_{i_1}) \quad \text{and} \quad r^*(\mathcal{T}) = r(T_{i_2})$$

for some $i_1, i_2 \in \mathcal{I}$.

We write $\mathcal{T} \in \mathfrak{T}_2(X)$ if

$$\mathcal{T} \in \mathfrak{T}_1(X) \quad \text{and} \quad T_i \in \mathcal{D}(X) \tag{20.3}$$

for some $i \in \mathcal{I}$ such that $r(T_i) = r_*(\mathcal{T})$.

Also we write $\mathcal{T} \in \mathfrak{T}_3(X)$ if (20.3) is satisfied for all $i \in \mathcal{I}$ such that $r(T_i) = r^*(\mathcal{T})$.

We point out that if $\mathcal{T} \in \mathfrak{T}_{\natural\natural}(X)$ and $\mathcal{Z}_{\mathcal{T}} \neq \varnothing$, then $C^\infty_{\mathcal{T}}(X) \neq \{0\}$ (see Proposition 15.2(i)). For a broad class of domains $\mathcal{O}$, the problem of existence of a nonzero solution of system (20.1) with $\mathcal{T} \in \mathfrak{T}_{\natural\natural}(X)$, $\mathcal{Z}_{\mathcal{T}} = \varnothing$, solves negatively by means of the following result.

Theorem 20.2. *Let $\mathcal{T} \in \mathfrak{T}_{\natural\natural}(X)$, $\mathcal{Z}_{\mathcal{T}} = \varnothing$, and suppose that (20.2) is fulfilled. Let $\mathcal{O} \in \mathfrak{S}(R, r(T_\nu))$ for some $\nu \in \mathcal{I}$ and assume that at least one of the following assumptions holds:*

(i) *The estimates*

$$r_*(\mathcal{T}) + r(T_i) < R \leqslant +\infty, \quad i \in \mathcal{I}, \tag{20.4}$$

are satisfied, and $f \in \mathcal{D}'_{\mathcal{T}}(\mathcal{O})$;

(ii) *$\mathcal{T} \in (\mathfrak{T}_2 \cup \mathfrak{T}_3)(X)$, $T_\mu \in \mathfrak{M}(X)$ for some $\mu \in \mathcal{I}$, $R \in [R_{\mathcal{T}}, +\infty]$, and $f \in \mathcal{D}'_{\mathcal{T}}(\mathcal{O})$;*

(iii) *$\mathcal{T} \in \mathfrak{T}_1(X)$, $T_\mu \in \mathfrak{M}(X)$ for some $\mu \in \mathcal{I}$, $R \in [R_{\mathcal{T}}, +\infty]$, and $f \in C^\infty_{\mathcal{T}}(\mathcal{O})$;*

(iv) *$\mathcal{T} \in \mathfrak{T}_1(X)$, $R = R_{\mathcal{T}}$, $\mathcal{O} = B_R$, and $f \in C^\infty_{\mathcal{T}}(\mathcal{O}) \cap C^\infty(B_{R+\varepsilon})$ for some $\varepsilon > 0$;*

(v) rank *$X = 1$, $\mathcal{O} = X \backslash \mathcal{K}$ for some compact set $\mathcal{K} \subset X$, and $f \in \mathcal{D}'_{\mathcal{T}}(\mathcal{O})$.*

Then $f = 0$ in $\mathcal{O}$.

Proof. First, suppose that at least one of assumptions (i)–(iii) is fulfilled. Owing to Theorem 15.1(i), we can assume, without loss of generality, that $\mathcal{O} = B_R$. We set $\mathcal{G} = \{g_i\}_{i \in \mathcal{I}}$, where $g_i \in \mathcal{E}'_{\natural}(\mathbb{R}^1)$ is defined by

$$\widehat{g_i} = \overset{\circ}{T}_i, \quad i \in \mathcal{I}.$$

By Theorems 6.3 and 10.7, $r(g_i) = r(T_i)$ for all i. Using now Corollary 20.1 and Theorems 18.1 and 18.3, we see that each of assumptions (i)–(iii) yields $f = 0$ in $\mathcal{O}$. Assume now that (iv) is fulfilled. Repeating the arguments in the proof of Theorem 20.1 and using Theorems 10.12 and 19.2(ii), we arrive at the desired result. Finally, let assumption (v) hold. If $\mathcal{O} = B_{r,\infty}$ for some $r \geqslant 0$, one obtains $f = 0$ in $\mathcal{O}$ because of Propositions 15.9(ii) and 15.11 (see the proof of Theorem 19.4). This, together with Theorem 15.1(i), completes the proof. □

Theorem 20.2 provides symmetric space analogue of the local version of the Brown–Schreiber–Taylor theorem (see Theorem 19.2).

We now show that Theorem 20.2 fails in general without the assumption that $\mathcal{T} \in \mathfrak{T}_{\natural\natural}(X)$.

Theorem 20.3. *Assume that* $\operatorname{rank} X \geqslant 2$. *Then for each* $\eta > 0$, *there exists* $T_\eta \in (\mathcal{E}'_\natural \cap \mathrm{Inv})(X)$ *such that the following statements are valid.*

(i) $r(T_\eta) = 1$ *for any* $\eta > 0$.
(ii) *If* $\eta_1, \eta_2 > 0$ *and* $\eta_1 \neq \eta_2$, *then*

$$\left\{\lambda \in \mathfrak{a}^*_{\mathbb{C}} : \widetilde{T}_{\eta_1}(\lambda) = \widetilde{T}_{\eta_2}(\lambda) = 0\right\} = \varnothing. \tag{20.5}$$

(iii) *If* $R > 1$ *and* $\varepsilon \in (0, R)$, *then there exists nontrivial* $f \in C^\infty_\natural(B_R)$ *such that* $f = 0$ *in* $B_{R-\varepsilon}$ *and* $f \times T_\eta = 0$ *in* B_{R-1} *for all* η.

Proof. Let $\varepsilon \in (0, 1)$. Examining the proof of Theorem 15.3, we see that there exists $T \in \mathcal{D}_\natural(X)$ such that $r(T) = 1$, and for each $R > 1$, it follows that $f \times T = 0$ for some nonzero function $f \in C^\infty_\natural(B_R)$ vanishing in $B_{R-\varepsilon}$. For $\eta \geqslant 0$, we set

$$T_\eta = T + (1 + \eta)\delta_0,$$

where, as usual, δ_0 is the Dirac measure supported at origin. Then $r(T_\eta) = 1$, and it follows by Proposition 10.9(ii) that T_η is invertible. By the definition of T_η it is easy to verify that (20.5) holds with $\eta_1 \neq \eta_2$. Finally, since $\varepsilon \in (0, 1)$ and $f = 0$ in $B_{R-\varepsilon}$, we conclude that $f \times T_\eta = 0$ in B_{R-1} for all $\eta \geqslant 0$. □

The following result shows that the value R in Theorem 20.2 cannot be decreased in the general case (see also Theorems 20.8 and 20.10 below).

Theorem 20.4. *For each* $\eta > 0$, *there exists* $T_\eta \in \mathfrak{N}(X)$ *such that the following assertions hold.*

(i) $r(T_\eta) = 1$ *and* $\mathcal{Z}_{T_\eta} \subset (0, +\infty)$ *for all* η.
(ii) *If* $\eta_1, \eta_2 > 0$ *and* $\eta_1 \neq \eta_2$, *then* (20.5) *is satisfied.*
(iii) *For each* $m \in \mathbb{Z}_+$, *there exists nontrivial* $f \in C^m(B_2)$ *such that* $f \times T_\eta = 0$ *in* B_1 *for all* η.

We note that this theorem fails with $m = \infty$ (see, for instance, Theorem 20.9(i)). However, by regularization we may deduce from (iii) that for any $\varepsilon \in (0, 1)$, there is nontrivial $f_\varepsilon \in C^\infty(B_{2-\varepsilon})$ such that

$$f_\varepsilon \times T_\eta = 0$$

in $B_{1-\varepsilon}$ for all η.

Proof of Theorem 20.4. If $\operatorname{rank} X \geqslant 2$, then Theorem 20.4 is a consequence of Theorems 18.2, 10.12(ii), and 10.14(iv). So we assume that $\operatorname{rank} X = 1$. Let $T_\eta, T \in \mathfrak{N}(X)$ be defined by

$$\overset{\circ}{T}_\eta(z) = 1 - \mathrm{e}^\eta \cos z, \quad \overset{\circ}{T}(z) = \cos z, \quad z \in \mathbb{C}.$$

Then (i) and (ii) are true. Next, given $m \in \mathbb{Z}_+$, we choose $\delta \in \widehat{K}_M$ and $j \in \{1, \ldots, d(\delta)\}$ so that $\zeta_{T,\delta,j} \in C^m(X)$ (see Theorem 15.6(v)). Setting

$$f = \zeta_{T,\delta,j}$$

and applying Proposition 15.2(ii), we see that $f \in C^m(X)$ and $f \times T_\eta = \zeta_{T,\delta,j}$ in X for all η. This, together with Theorem 15.6(ii), implies (iii). Hence the theorem. □

Let us consider the case where the assumption on R in Theorem 20.2 can be weakened.

Theorem 20.5. *Let $\mathcal{T} \in \mathfrak{T}_{\natural\natural}(X)$ and $T_{i_0} \in \mathrm{Inv}_+(X)$ for some $i_0 \in \mathcal{I}$. Let $\mathcal{Z}_\mathcal{T} = \varnothing$ and suppose that* (20.2) *is satisfied. Assume that for each $\nu \in \mathcal{I}\backslash\{i_0\}$, there exists $\psi_\nu \in \mathfrak{M}(X)$ such that $r(\psi_\nu) > 0$ and for each $\lambda \in \mathcal{Z}_{\psi_\nu}$,*

$$\sum_{\eta=0}^{n(\lambda,\psi_\nu)} \left(\left|\overset{\circ}{T}{}^{(\eta)}_{i_0}(\lambda)\right| + \left|\overset{\circ}{T}{}^{(\eta)}_{\nu}(\lambda)\right|\right) \leqslant M_{q,\nu}(2+|\lambda|)^{-q}, \quad q = 1, 2, \ldots, \tag{20.6}$$

where the constants $M_{q,\nu} > 0$ are independent of λ and satisfy (18.14). *Let*

$$R > r(T_{i_0}) + r(T_\nu) - r(\psi_\nu)$$

for all $\nu \in \mathcal{I}\backslash\{i_0\}$, and let $\mathcal{O} \in \mathfrak{S}(R, r(T_i))$ for some $i \in \mathcal{I}$. Then $\mathcal{D}'_\mathcal{T}(\mathcal{O}) = \{0\}$. In particular, the same is valid if $\psi_\nu = T_{i_0} \in \mathfrak{M}(X)$ for all $\nu \in \mathcal{I}\backslash\{i_0\}$.

Proof. The result follows from Theorem 15.1(i), Corollary 20.1(i), and Theorem 18.14. □

The following theorem shows that assumption (18.14) in Theorem 20.5 cannot be relaxed in the general case.

Theorem 20.6. *Let $i_0 \in \mathcal{I}$ and $\mathcal{I}\backslash\{i_0\} \neq \varnothing$. Then for each sequence $\{M_q\}_{q=1}^\infty$ of positive numbers satisfying* (18.15), *there exist $T_{i_0} \in \mathfrak{N}(X)$ and $T_\nu \in \mathcal{E}'_{\natural\natural}(X)$, $T_\nu \neq 0$, $\nu \in \mathcal{I}\backslash\{i_0\}$, such that the family $\mathcal{T} = \{T_i\}_{i\in\mathcal{I}}$ has the following properties:*

(1) *$\mathcal{Z}_\mathcal{T} = \varnothing$ and $r^*(\mathcal{T}) < +\infty$;*
(2) *for each $\nu \in \mathcal{I}\backslash\{i_0\}$, estimate* (20.6) *is satisfied with $\psi_\nu = T_{i_0}$ and $M_{q,\nu} = M_q$;*
(3) *$C^\infty_\mathcal{T}(B_R) \neq \{0\}$ for some $R > r^*(\mathcal{T})$.*

Proof. Combine Corollary 20.1(ii) and [225, Part III, the proof of Theorem 1.10]. □

For the rest of the section, we assume that $i_1, i_2 \in \mathcal{I}$ are fixed and $i_1 \neq i_2$. By analogy with Sect. 19.1, for $\mathcal{T} = \{T_i\}_{i\in\mathcal{I}} \in \mathfrak{T}_{\natural\natural}(X)$ and $R > 0$, denote by $\mathcal{U}(\mathcal{T}, R)$ the set of all distributions $u \in \mathcal{E}'_{\natural\natural}(X)$ with the following properties:

(a) $u = 0$ in $\{x \in X : d(o,x) > r(P) + r(Q) - R\}$, where $P = T_{i_1}$ and $Q = T_{i_2}$;
(b) for each $\nu \in \mathcal{I}\backslash\{i_1, i_2\}$, there exist $u_{1,\nu}, u_{2,\nu} \in \mathcal{E}'_{\natural\natural}(X)$ such that

$$u_{\mu,\nu} = 0 \quad \text{in } \left\{x \in X : d(o,x) > r(T_\mu) + r(T_\nu) - R\right\}, \quad \mu \in \{i_1, i_2\},$$

and

$$P \times u_{2,\nu} + Q \times u_{1,\nu} = T_\nu \times u.$$

Next, for the rank one case, we define $\Omega_{U_1,U_2,\delta,j} \in \mathcal{D}'(X)$ by the formula

$$\Omega_{U_1,U_2,\delta,j} = -\mathfrak{A}_{\delta,j}^{-1}\big({\Omega'}_{\Lambda(U_1),\Lambda(U_2)}\big), \tag{20.7}$$

where $\delta \in \widehat{K}_M$, $j \in \{1,\dots,d(\delta)\}$, $U_1 \in \mathfrak{N}(X)$, $U_2 \in \mathcal{E}'_\natural(X)\backslash\{0\}$, and $\mathcal{Z}(\overset{\circ}{U}_1) \not\approx \mathcal{Z}(\overset{\circ}{U}_2)$ (see Sect. 18.2). Therefore, $\Omega_{U_1,U_2,\delta,j} \in \mathcal{D}'_{U_1}(X)$ and

$$\Omega_{U_1,U_2,\delta,j} \times U_2 = \zeta_{U_1,\delta,j}$$

(see (18.27) and (15.19)).

We now present analogues of Theorem 19.6 and Proposition 18.4 for spaces X of arbitrary rank.

Theorem 20.7. *Let $\mathcal{T} \in \mathfrak{T}_{\natural\natural}(X)$, $P = T_{i_1}$, $Q = T_{i_2}$, and suppose that $P \in \mathfrak{N}(X)$, $\mathcal{Z}(\overset{\circ}{P}) \not\approx \mathcal{Z}(\overset{\circ}{Q})$, and*

$$r(T_i) < R \leqslant r(P) + r(T_\mu) \quad \textit{for all } i, \mu \in \mathcal{I},\ \mu \neq i_1.$$

Then the following statements are valid.

(i) $\mathcal{D}'_{\mathcal{T}}(B_R) = \{0\}$ *if and only if* $\mathcal{U}(\mathcal{T}, R) = \{0\}$.
(ii) *If* $\mathcal{D}'_{\mathcal{T}}(B_R) \neq \{0\}$, *then* $C^m_{\mathcal{T}}(B_R) \neq \{0\}$ *for each* $m \in \mathbb{Z}_+$.
(iii) *If* rank $X = 1$ *and* $f \in \mathcal{D}'(B_R)$, *then for* f *to belong to* $\mathcal{D}'_{\mathcal{T}}(B_R)$, *it is necessary and sufficient that for all* $\delta \in \widehat{K}_M$, $j \in \{1,\dots,d(\delta)\}$,

$$f^{\delta,j} = \Omega_{P,Q,\delta,j} \times u \quad \textit{in } B_R$$

for some $u \in \mathcal{U}(\mathcal{T}, R)$ *depending on* δ, j.

It can be shown that assertion (ii) of this theorem fails with $m = \infty$ (see Theorem 20.9(i) below).

Proof of Theorem 20.7. Part (i) is obvious from Theorem 18.8, Remark 13.1, and Corollary 20.1. Next, if rank $X \geqslant 2$, then (ii) follows from the proof of Theorem 19.7 and Remark 20.1. The proof of (iii) and (ii) for the rank one case proceeds as that of Theorem 19.6, except that we must replace $\mathfrak{A}_{k,j}$ by $\mathfrak{A}_{\delta,j}$ (see Sect. 10.8). □

Proposition 20.1. *Let $\mathcal{I} = \mathbb{N}$, $i_1 = 1$, $i_2 = 2$. Then for each $\alpha \in [1, 2]$, there exists $\mathcal{T} = \{T_i\}_{i\in\mathcal{I}} \in \mathfrak{T}_{\natural\natural}(X)$ with the following properties:*

(1) $T_1 \in \mathfrak{N}(X)$, $\mathcal{Z}(\overset{\circ}{T}_1) \not\approx \mathcal{Z}(\overset{\circ}{T}_2)$, *and* $r(T_\nu) = 1$ *for all* $\nu \in \mathbb{N}$;
(2) $\mathcal{D}'_{\mathcal{T}}(B_R) \neq \{0\}$ *for* $1 < R \leqslant \alpha$;
(3) $\mathcal{D}'_{\mathcal{T}}(B_R) = \{0\}$ *for* $R > \alpha$.

The proof follows from Corollaries 18.3 and 20.1.

20.2 $\mathcal{E}'_{\natural\natural}(X)$ Mean Periodic Functions with Respect to a Couple of Distributions

In this section we deal with the family $\mathcal{T} = \{T_1, T_2\}$, where $T_1, T_2 \in \mathcal{E}'_{\natural\natural}(X)\backslash\{0\}$. For brevity, we set

$$r_\nu = r(T_\nu), \quad \nu \in \{1, 2\},$$

and

$$r^*(\mathcal{T}) = \max\{r_1, r_2\}.$$

Also let

$$\mathcal{Z}_{\mathcal{T}} = \mathcal{Z}_{T_1} \cap \mathcal{Z}_{T_2}.$$

We now consider the problem of existence of a nonzero solution of the system

$$(f \times T_\nu)(x) = 0, \quad x \in B_{R-r_\nu}, \ \nu = 1, 2, \tag{20.8}$$

where $R > r(\mathcal{T})$. As before, we set

$$\mathcal{D}'_{\mathcal{T}}(B_R) = \big(\mathcal{D}'_{T_1} \cap \mathcal{D}'_{T_2}\big)(B_R), \qquad \mathcal{D}'_{\mathcal{T},\natural}(B_R) = \big(\mathcal{D}'_{\mathcal{T}} \cap \mathcal{D}'_{\natural}\big)(B_R),$$

and

$$C^m_{\mathcal{T}}(B_R) = \big(\mathcal{D}'_{\mathcal{T}} \cap C^m\big)(B_R), \qquad C^m_{\mathcal{T},\natural}(B_R) = \big(\mathcal{D}'_{\mathcal{T},\natural} \cap C^m\big)(B_R)$$

for $m \in \mathbb{Z}_+$ or $m = \infty$.

Applying Theorem 20.2, we deduce that $\mathcal{D}'_{\mathcal{T}}(B_R) = \{0\}$ if $R > r_2 + r_2$ and $\mathcal{Z}_{\mathcal{T}} = \varnothing$. Next, if $\mathcal{Z}_{\mathcal{T}} \neq \varnothing$, then system (20.8) has a nontrivial solution $f \in \mathrm{RA}(X)$ (see Proposition 15.2). So we shall regard the case where

$$\mathcal{Z}_{\mathcal{T}} = \varnothing \quad \text{and} \quad r^*(\mathcal{T}) < R \leqslant r_1 + r_2.$$

Theorem 20.8. *Let $\mathcal{Z}_{\mathcal{T}} = \varnothing$. Then the following assertions hold.*

(i) *Let $R = r_1 + r_2 > r^*(\mathcal{T})$, $T_1 \in \mathrm{Inv}_+(X)$, and assume that there exists a sequence $\zeta_1, \zeta_2, \dots$ of complex numbers such that for each $\alpha > 0$,*

$$(2 + |\zeta_m|)^\alpha \big(|\overset{\circ}{T}_1(\zeta_m)| + |\overset{\circ}{T}_2(\zeta_m)|\big) \to 0 \quad \text{as } m \to \infty$$

and $|\mathrm{Im}\,\zeta_m| \leqslant c \log(2 + |\zeta_m|)$, where the constant $c > 0$ is independent of m. Then $\mathcal{D}'_{\mathcal{T}}(B_R) = \{0\}$. In particular, this is true if $T_1 \in \mathfrak{M}(X)$ and $\mathcal{Z}(\overset{\circ}{T}_1) \approx \mathcal{Z}(\overset{\circ}{T}_2)$. This occurs, for example, when $T_2 \in \mathcal{D}(X)$.

(ii) *Let $r^*(\mathcal{T}) < R \leqslant r_1 + r_2$, $T_1 \in \mathfrak{N}(X)$, and $\mathcal{Z}(\overset{\circ}{T}_1) \not\approx \mathcal{Z}(\overset{\circ}{T}_2)$. Then $C^m_{\mathcal{T}}(B_R) \neq \{0\}$ for each $m \in \mathbb{Z}_+$. In particular, $C^\infty_{\mathcal{T}}(B_R) \neq \{0\}$ for $R < r_1 + r_2$.*

(iii) *Assume that* rank $X = 1$, *$r^*(\mathcal{T}) < R \leqslant r_1 + r_2$, $T_1 \in \mathfrak{N}(X)$, $\mathcal{Z}(\overset{\circ}{T}_1) \not\approx \mathcal{Z}(\overset{\circ}{T}_2)$, and $f \in \mathcal{D}'(B_R)$. Then $f \in \mathcal{D}'_{\mathcal{T}}(B_R)$ if and only if for all $\delta \in \widehat{K}_M$ and*

$j \in \{1, \ldots, d(\delta)\}$,

$$f^{\delta,j} = \Omega_{T_1,T_2,\delta,j} \times u \quad \textit{in } B_R$$

for some $u \in \mathcal{E}'_{\natural}(X)$ *depending on* δ, j *such that* $\operatorname{supp} u \subset \dot{B}_{r_1+r_2-R}$.

We notice that assertion (ii) fails with $m = \infty$ and $R = r_1 + r_2$ (see Theorem 20.9(i)). In addition, Theorem 20.9(ii) shows that for all $k, m \in \mathbb{Z}_+$, there exists $\mathcal{T}$ such that

$$T_1 \in \big(\mathfrak{N} \cap C^k\big)(X), \quad T_2 \in \mathcal{E}'_{\natural\natural}(X)\backslash\{0\}, \quad \mathcal{Z}_{\mathcal{T}} = \varnothing,$$

and $C^m_{\mathcal{T}}(B_R) \neq \{0\}$ for $R = r(T_1) + r(T_2)$.

Proof of Theorem 20.8. Assertion (i) is a consequence of Corollary 20.1(i) and Theorem 18.7. Next, applying Theorem 20.7, we arrive at (ii) and (iii). □

Let us now consider an analog of Theorem 19.13.

Theorem 20.9.

(i) *Let* $\mathcal{Z}_{\mathcal{T}} = \varnothing$, $T_1 \in \mathfrak{M}(X)$, $R = r_1 + r_2 > r^*(\mathcal{T})$, *and assume that* $f \in \mathcal{D}'_{\mathcal{T}}(B_R) \cap C^\infty(B_{r_1+\varepsilon})$ *for some* $\varepsilon \in (0, R - r_1)$. *Then* $f = 0$ *in* B_R.

(ii) *For every* $T_1 \in \mathfrak{N}(X)$ *with* $r(T_1) > 0$, *there is* $T_2 \in \mathcal{E}'_{\natural\natural}(X)$ *such that* $r(T_2) = r(T_1)$ *and the family* $\mathcal{T} = \{T_1, T_2\}$ *possesses the following properties:*

  (1) $\mathcal{Z}_{\mathcal{T}} = \varnothing$;

  (2) *For* $R = r(T_1) + r(T_2)$ *and for each* $m \in \mathbb{Z}_+$, *there exists a nontrivial function* $f \in C^m_{\mathcal{T}}(B_R)$ *such that* $f = 0$ *in* $B_{r(T_1)}$.

Proof. To prove (i) suppose that $\delta \in \widehat{K}_M$. Because of Proposition 15.1(iii) and (10.18),

$$f_\delta \in \mathcal{D}'_{\mathcal{T}}(B_R) \quad \text{and} \quad f_\delta \in C^\infty(B_{r_1+\varepsilon}).$$

It follows by Proposition 15.1(iv) that each entry in the matrix $\mathfrak{A}_\delta(f_\delta)$ is in the class $(\mathcal{D}'_{\Lambda_+(T_1),W} \cap \mathcal{D}'_{\Lambda_+(T_2),W})(\mathcal{B}_R)$, and its restriction to $B_{r_1+\varepsilon}$ belongs to $C^\infty(\mathcal{B}_{r_1+\varepsilon})$. For the rank one case, this, together with Theorem 18.8(i), (iii), gives $\mathfrak{A}_\delta(f_\delta) = 0$ in $\mathcal{B}_R$. In addition, using Theorems 19.13 and 20.1, we see that the same is true if rank $X \geqslant 2$. Now Theorem 10.12(iii) ensures us that $f_\delta = 0$ in $\mathcal{B}_R$. In combination with Proposition 10.2(iii), this yields $f = 0$ in B_R, as contended.

Turning to (ii), assume that $T_1 \in \mathfrak{N}(X)$ and $r(T_1) > 0$. Define $T_2 \in \mathcal{E}'_{\natural\natural}(X)$ by the formula $\overset{\circ}{T}_2(z) = 1 + \overset{\circ}{T}_1(z)$, $z \in \mathbb{C}$. Then $r(T_2) = r(T_1)$ and $\mathcal{Z}_{\mathcal{T}} = \varnothing$, where $\mathcal{T} = \{T_1, T_2\}$. Next, for the rank one case, we find

$$\Omega_{T_1,T_2,\delta,j} = \zeta_{T_1,\delta,j} \quad \text{for all } \delta \in \widehat{K}_M,\ j \in \{1, \ldots, d(\delta)\} \tag{20.9}$$

(see (20.7) and (15.20)). Let $m \in \mathbb{Z}_+$ and $R = r(T_1)+r(T_2)$. If $s(\delta)$ is large enough, then the function

$$f = \Omega_{T_1,T_2,\delta,j}$$

is in the class $C^m_{\mathcal{T}}(B_R)$ (see Theorem 15.6(v)). Because of (20.9) and Theorem 15.6(ii), this proves (ii) for the rank one case.

If rank $X \geqslant 2$, then (ii) follows from the proof of Theorem 19.13 and Remark 20.1. This completes the proof. □

For the rest of the section, we shall regard some analogues of Theorems 19.14 and 19.15.

Theorem 20.10. *Let $r^*(\mathcal{T}) < R < r_1 + r_2$, $T_1 \in \mathfrak{N}(X)$, and assume that there exists $l > 0$ such that the set*

$$E_l(\mathcal{T}) = \left\{\lambda \in \mathcal{Z}_{T_1} : \left|\mathring{T}_2(\lambda)\right| < (2 + |\lambda|)^{-l}\right\}$$

is sparse (see Definition 18.1). Then for every sequence $\{M_q\}_{q=1}^{\infty}$ of positive numbers satisfying (19.15), there exists a nontrivial function $f \in C^{\infty}_{\natural}(X) \cap C^{\infty}_{\mathcal{T}}(B_R)$ such that

$$\sup_{x \in B_R} \left|(L^q f)(x)\right| \leqslant M_q \quad \textit{for all } q \in \mathbb{N}.$$

As in the Euclidean case, condition (19.15) in this theorem cannot be relaxed in general (see Remark 20.2 in the following section). Next, Theorem 20.10 fails without the assumption that $E_l(\mathcal{T})$ is sparse (see Theorem 20.11 below). This assumption holds for a broad class of families $\mathcal{T}$, in particular, if rank $X = 1$ and T_1, T_2 are the characteristic functions of balls (see [227]).

To prove the theorem we need the following result.

Lemma 20.1. *If $\varepsilon > 0$ and $\{M_q\}_{q=1}^{\infty}$ is a sequence of positive numbers satisfying (19.15), then there exist nonzero functions $v_1, v_2 \in (\mathcal{E}'_{\natural\natural} \cap \mathcal{D})(X)$ such that $(\text{supp}\, v_1) \cup (\text{supp}\, v_2) \subset B_{\varepsilon}$, $\mathcal{Z}(\mathring{v}_1) \cap \mathcal{Z}(\mathring{v}_2) = \varnothing$ and*

$$\left|\left(L^q v_1\right)(x)\right| + \left|\left(L^q v_2\right)(x)\right| \leqslant M_q$$

for all $x \in X$, $q \in \mathbb{N}$.

The proof follows from Lemmas 19.2, 8.1, Theorem 10.7, and Remark 10.2.

Proof of Theorem 20.10. Let $\varepsilon = (r_1 + r_2 - R)/3$. Using Theorems 18.9 and 19.14, we see from Theorem 20.1 that there is a nontrivial function $f_1 \in C^{\infty}_{\mathcal{T},\natural}(B_{R+2\varepsilon})$. The rest of the proof is similar to that of Theorem 19.14 with the references to Corollary 19.2 and Theorem 19.2(i) replaced by the references to Lemma 20.1 and Theorem 20.2. □

We shall now consider the case where $\mathcal{D}'_{\mathcal{T}}(B_R) = \{0\}$ for $r^*(\mathcal{T}) < R < r_1 + r_2$.

Theorem 20.11.

(i) *Let $\mathcal{Z}_{\mathcal{T}} = \varnothing$, $T_1 \in \text{Inv}_+(X)$, and assume that there exists $\psi \in \mathfrak{M}(X)$ such that $r(\psi) > 0$ and for each $\lambda \in \mathcal{Z}_{\psi}$,*

$$\sum_{\eta=0}^{n(\lambda,\psi)} \left(\left|\overset{\circ}{T}_1^{(\eta)}(\lambda)\right| + \left|\overset{\circ}{T}_2^{(\eta)}(\lambda)\right|\right) \leqslant M_q(2+|\lambda|)^{-q}, \quad q = 1, 2, \ldots,$$

where the constants $M_q > 0$ *do not depend on* λ *and satisfy* (18.10). *Then* $\mathcal{D}'_{\mathcal{T}}(B_R) = \{0\}$ *for* $R > \max\{r^*(\mathcal{T}), r_1 + r_2 - r(\psi)\}$. *This statement is no longer valid in general if* $r^*(\mathcal{T}) < R = r_1 + r_2 - r(\psi)$.

(ii) *Let* $\mathcal{Z}_{\mathcal{T}} = \varnothing$, $T_1 \in \mathfrak{N}(X)$, *and assume that* $T_1 = \psi_1 \times \psi_2$, *where* $\psi_1 \in \mathfrak{N}(X)$, $\psi_2 \in \mathcal{E}'_{\natural\natural}(X)$, *and* $\mathcal{Z}(\overset{\circ}{\psi}_1) \cap \mathcal{Z}(\overset{\circ}{\psi}_2) = \varnothing$. *Also suppose that for each* $\lambda \in \mathcal{Z}_{\psi_2}$, *the estimates*

$$\sum_{\eta=0}^{n(\lambda,\psi_2)} \left|\overset{\circ}{T}_2^{(\eta)}(\lambda)\right| \leqslant M_q(1+|\lambda|)^{-q}, \quad q = 1, 2, \ldots, \tag{20.10}$$

hold, where the constants $M_q > 0$ *do not depend on* λ *and satisfy* (18.10). *Then* $\mathcal{D}'_{\mathcal{T}}(B_R) = \{0\}$ *for* $R > \max\{r_1, r_2 + r(\psi_1)\}$. *In addition,* $C^{\infty}_{\mathcal{T}}(B_R) = \{0\}$ *if* $R \geqslant r_2 + r(\psi_1)$ *and* $R > r^*(\mathcal{T})$.

(iii) *There exists a family* $\mathcal{T} = \{T_1, T_2\}$ *satisfying all the assumptions of assertion* (ii) *such that* $r(T_2)+r(\psi_1) > r^*(\mathcal{T})$ *and* $C^m_{\mathcal{T}}(B_R) \neq \{0\}$ *for* $R = r(T_2)+r(\psi_1)$ *and each* $m \in \mathbb{N}$.

(iv) *There exists a family* $\mathcal{T} = \{T_1, T_2\}$ *satisfying all the assumptions of assertion* (ii) *such that* $r(T_2) + r(\psi_1) > r^*(\mathcal{T})$ *and* $C^{\infty}_{\mathcal{T}}(B_R) \neq \{0\}$ *for each* $R \in (r^*(\mathcal{T}), r(T_2) + r(\psi_1))$.

Proof. Parts (i) and (ii) are consequences of Corollary 20.1 and Theorems 18.12 and 18.10. The proof of (iii) duplicates Theorem 4.10 in [225, Part III] with the reference to Theorem 20.8(ii). Assertion (iv) can be obtained from (iii) by means of the standard regularization. □

To conclude we show that assumption (18.10) in Theorem 20.11 cannot be weakened.

Theorem 20.12. *For every sequence* $\{M_q\}_{q=1}^{\infty}$ *of positive numbers satisfying* (18.15), *there exist* $T_1, \psi_1, \psi_2 \in \mathfrak{N}(X)$, *and* $T_2 \in (\mathcal{E}'_{\natural\natural} \cap \mathcal{D})(X)$ *with the following properties:*

(1) *Estimates* (20.10) *hold for all* $\lambda \in \mathcal{Z}(\overset{\circ}{\psi}_2)$;

(2) $\mathcal{Z}(\overset{\circ}{\psi}_1) \cap \mathcal{Z}(\overset{\circ}{\psi}_2) = \varnothing$;

(3) $T_1 = \psi_1 \times \psi_2$ *and* $\mathcal{Z}(\overset{\circ}{T}_1) \cap \mathcal{Z}(\overset{\circ}{T}_2) = \varnothing$;

(4) $C^{\infty}_{\mathcal{T}}(B_R) \neq \{0\}$ *for some* $R > \max\{r(T_1), r(T_2) + r(\psi_1)\}$, *where* $\mathcal{T} = \{T_1, T_2\}$.

The proof follows from Corollary 20.1(ii) and Theorem 18.11.

20.3 Explicit Representation Theorems

Throughout the section we suppose that rank $X = 1$. Recall that in this case each family $\mathcal{T} = \{T_i\}_{i\in\mathcal{I}}$ of nonzero K-invariant compactly supported distributions on X belongs to $\mathfrak{T}_{\natural\natural}(X)$.

For $\mathcal{T} \in \mathfrak{T}_{\natural\natural}(X)$ and $\lambda \in \mathcal{Z}_{\mathcal{T}}$, we denote

$$n(\lambda, \mathcal{T}) = \min_{i\in\mathcal{I}} n(\lambda, T_i).$$

In the present section we shall characterize various classes of solutions of system (20.1) for some domains $\mathcal{O}$ with spherical symmetry. As before, we assume that $i_0 \in \mathcal{I}$ is fixed. The first our result is as follows.

Theorem 20.13. *Let $\mathcal{T} \in \mathfrak{T}_{\natural\natural}(X)$ and $T = T_{i_0}$. Then the following statements hold.*

(i) *Assume that (20.4) is satisfied, let $T \in \mathfrak{M}(X)$, and let $f \in \mathcal{D}'(B_R)$. For f to belong to $\mathcal{D}'_{\mathcal{T}}(B_R)$, it is necessary and sufficient that for all $\delta \in \widehat{K}_M$ and $j \in \{1, \dots, d(\delta)\}$,*

$$f^{\delta,j} = \sum_{\lambda\in\mathcal{Z}_{\mathcal{T}}} \sum_{\eta=0}^{n(\lambda,\mathcal{T})} \gamma_{\lambda,\eta,\delta,j}\Phi_{\lambda,\eta,\delta,j}, \tag{20.11}$$

where $\gamma_{\lambda,\eta,\delta,j} \in \mathbb{C}$, and the series in (20.11) converges in $\mathcal{D}'(B_R)$. If $\mathcal{T} \in (\mathfrak{T}_2 \cup \mathfrak{T}_3)(X)$, this statement remains valid with $R \in [R_{\mathcal{T}}, +\infty]$.

(ii) *Suppose that (20.4) holds, let $T \in \mathfrak{M}(X)$, and let $f \in C^\infty(B_R)$. Then $f \in C^\infty_{\mathcal{T}}(B_R)$ if and only if for all $\delta \in \widehat{K}_M$ and $j \in \{1, \dots, d(\delta)\}$, relation (20.11) holds with the series converging in $\mathcal{E}(B_R)$. If $\mathcal{T} \in \mathfrak{T}_1(X)$, the same is true for $R \in [R_{\mathcal{T}}, +\infty]$.*

(iii) *Assume that (20.4) holds and let $\alpha > 0$, $T \in \mathfrak{G}_\alpha(X)$, $r(T) > 0$, and $f \in C^\infty(B_R) \cap G^\alpha(\mathring{B}_{r(T)})$. Then $f \in C^\infty_{\mathcal{T}}(B_R)$ if and only if for each $\delta \in \widehat{K}_M$ and $j \in \{1, \dots, d(\delta)\}$, relation (20.11) holds with the series converging in $\mathcal{E}(B_R)$. If $\mathcal{T} \in \mathfrak{T}_1(X)$, this assertion remains true for $R \in [R_{\mathcal{T}}, +\infty]$.*

(iv) *Suppose that (20.2) holds, let $T \in \mathfrak{M}(X)$, $r(T) > 0$, and let $f \in C^\infty(B_R) \cap \mathrm{QA}(\mathring{B}_{r(T)})$. Then $f \in C^\infty_{\mathcal{T}}(B_R)$ if and only if for all $\delta \in \widehat{K}_M$ and $j \in \{1, \dots, d(\delta)\}$, relation (20.11) holds with the series converging in $\mathcal{E}(B_R)$. Moreover, if $f \in C^\infty_{\mathcal{T}}(B_R)$, then $f^{\delta,j} \in \mathrm{QA}_{\mathcal{T}}(B_R)$ for all (δ, j).*

(v) *Let $T \in \mathfrak{E}(X)$, $f \in \mathcal{D}'(X)$, and assume that f is of finite order in X. Then $f \in \mathcal{D}'_{\mathcal{T}}(X)$ if and only if for all $\delta \in \widehat{K}_M$ and $j \in \{1, \dots, d(\delta)\}$, relation (20.11) holds with the series converging in $\mathcal{E}(X)$. In particular, if $f \in \mathcal{D}'_{\mathcal{T}}(X)$, then $f^{\delta,j} \in C^\infty_{\mathcal{T}}(X)$ for all $\delta \in \widehat{K}_M$ and $j \in \{1, \dots, d(\delta)\}$.*

Proof. A similar result for the Euclidean space was obtained in Sect. 19.2 (see Theorem 19.8). The proof of Theorem 20.13 can be carried out along the same lines (see Theorems 15.9, 15.15, 15.17, 15.18, and 15.13 and Proposition 15.11). □

Notice that the assumptions about R in Theorem 20.13 cannot be weakened in the general case (see Theorem 20.10).

Remark 20.2. Using Theorem 20.13(iv), we see that if $\mathcal{T} \in \mathfrak{T}_{\natural\natural}(X)$, $T = T_{i_0} \in \mathfrak{M}(X)$, $r(T) > 0$, and (20.2) is satisfied, then $C^{\infty}_{\mathcal{T}}(B_R) \cap \mathrm{QA}(\mathring{B}_{r(T)}) = \{0\}$ if $\mathcal{Z}_{\mathcal{T}} = \varnothing$. This fact shows that condition (19.15) in Theorem 20.10 cannot be weakened.

We now turn to the case of a spherical annulus in X.

Theorem 20.14. *Let $\mathcal{T} \in \mathfrak{T}_{\natural\natural}(X)$, $T = T_{i_0}$, and assume that $0 \leqslant r < r' < R' < R \leqslant +\infty$ and $R' - r' = 2r(T)$. Then the following assertions are true.*

(i) *Let $T \in \mathfrak{N}(X)$, assume that*

$$r + 2\big(r_*(\mathcal{T}) + r(T_i)\big) < R \quad \text{for all } i \in \mathcal{I}, \tag{20.12}$$

and let $f \in \mathcal{D}'(B_{r,R})$. Then $f \in \mathcal{D}'_{\mathcal{T}}(B_{r,R})$ if and only if for all $\delta \in \widehat{K}_M$ and $j \in \{1, \dots, d(\delta)\}$,

$$f^{\delta,j} = \sum_{\lambda \in \mathcal{Z}_{\mathcal{T}}} \sum_{\eta=0}^{n(\lambda,\mathcal{T})} \alpha_{\lambda,\eta,\delta,j} \Phi_{\lambda,\eta,\delta,j} + \beta_{\lambda,\eta,\delta,j} \Psi_{\lambda,\eta,\delta,j}, \tag{20.13}$$

where $\alpha_{\lambda,\eta,\delta,j}, \beta_{\lambda,\eta,\delta,j} \in \mathbb{C}$, and the series in (20.13) converges in $\mathcal{D}'(B_{r,R})$. If $\mathcal{T} \in (\mathfrak{T}_2 \cup \mathfrak{T}_3)(X)$, this statement remains valid with $R \in [r + 2R_{\mathcal{T}}, +\infty]$.

(ii) *Assume that (20.12) is satisfied, let $T \in \mathfrak{N}(X)$, and let $f \in C^{\infty}(B_{r,R})$. Then $f \in C^{\infty}_{\mathcal{T}}(B_{r,R})$ if and only if for all (δ, j), relation (20.13) holds with the series converging in $\mathcal{E}(B_{r,R})$. If $\mathcal{T} \in \mathfrak{T}_1(X)$, the same is true for $R \in [r+2R_{\mathcal{T}}, +\infty]$.*

(iii) *Assume that (20.12) holds, let $\alpha > 0$, $T \in \mathfrak{D}_{\alpha}(X)$, and $f \in C^{\infty}(B_{r,R}) \cap G^{\alpha}(\mathring{B}_{r',R'})$. Then $f \in C^{\infty}_{\mathcal{T}}(B_{r,R})$ if and only if for all (δ, j), relation (20.13) holds with the series converging in $\mathcal{E}(B_{r,R})$. If $\mathcal{T} \in \mathfrak{T}_1(X)$, this assertion remains true for $R \in [r + 2R_{\mathcal{T}}, +\infty]$*

(iv) *Let $T \in \mathfrak{N}(X)$, let*

$$r + 2r(T_i) < R \leqslant +\infty \quad \text{for all } i \in \mathcal{I}, \tag{20.14}$$

and suppose that $f \in C^{\infty}(B_{r,R}) \cap \mathrm{QA}(\mathring{B}_{r',R'})$. Then $f \in C^{\infty}_{\mathcal{T}}(B_{r,R})$ if and only if for all (δ, j), relation (20.13) holds with the series converging in $\mathcal{E}(B_{r,R})$. Moreover, if $f \in C^{\infty}_{\mathcal{T}}(B_{r,R})$, then $f^{\delta,j} \in \mathrm{QA}_{\mathcal{T}}(B_{r,R})$ for all (δ, j).

The proof follows from Theorems 15.9, 15.23, 15.24, and 15.13 and Proposition 15.11.

We now establish analogues of Theorems 19.10 and 19.11.

Theorem 20.15. *Let $\mathcal{T} \in \mathfrak{T}_{\natural\natural}(X)$, $T = T_{i_0} \in \mathfrak{N}(X)$, and assume that for all $\lambda \in \mathcal{Z}_T$ and $\nu \in \mathcal{I}\backslash\{i_0\}$, the following estimates hold*

$$\sum_{\eta=0}^{n(\lambda,T)} \left|\overset{\circ}{T}{}_{\nu}^{(\eta)}(\lambda)\right| \leqslant M_{q,\nu}(1+|\lambda|)^{-2q}, \quad q=1,2,\ldots, \tag{20.15}$$

where the constants $M_{q,\nu} > 0$ do not depend on λ and satisfy (19.14). *Then assertions* (i) *and* (ii) *of Theorem 20.13 remain valid with* (20.4) *replaced by* (20.2). *In addition, parts* (i) *and* (ii) *of Theorem 20.14 are true if* (20.12) *is replaced by* (20.14).

To prove this result it is enough to use the arguments from the proof of Theorem 18.4 along with applying Theorems 15.15, 15.23, and 15.13.

The following theorem shows that assumption (19.14) in Theorem 20.15 cannot be relaxed in the general case.

Theorem 20.16. *Let $\mathcal{I}\backslash\{i_0\} \neq \varnothing$. Then for each sequence $\{M_q\}_{q=1}^{\infty}$ of positive numbers satisfying* (19.15)*, there exist $T_{i_0} \in \mathfrak{N}(X)$ and $T_\nu \in \mathcal{E}'_{\natural}(X)$, $T_\nu \neq 0$, $\nu \in \mathcal{I}\backslash\{i_0\}$, such that the family $\mathcal{T} = \{T_i\}_{i\in\mathcal{I}}$ has the following properties:*

(1) *for each $\nu \in \mathcal{I}\backslash\{i_0\}$, estimate* (20.15) *is fulfilled with $M_{q,\nu} = M_q$;*
(2) *$r^*(\mathcal{T}) < +\infty$ and parts* (i) *and* (ii) *of Theorem 20.13 fail for some $R > r(\mathcal{T})$.*

Proof. Let $\varepsilon \in (0, 1/2)$. It follows by Lemmas 20.1, 8.1, and (10.75) that there is nonzero $\varphi \in \mathcal{D}_{\natural}(X)$ such that $\operatorname{supp}\varphi \subset B_{\varepsilon/4}$ and

$$\left|\overset{\circ}{\varphi}(t)\right| \leqslant M_q(2+|t|)^{-2q} \quad \text{for all } t \in \mathbb{R}^1,\ q \in \mathbb{N}.$$

Assume now that $w \in \mathcal{D}_{\natural}(X)$, $w \neq 0$, $\operatorname{supp} w \subset B_{1+\varepsilon/4,1+3\varepsilon/4}$, and $|\overset{\circ}{w}(t)| \leqslant 1$ for each $t \in \mathbb{R}^1$. Then the convolution $u = \varphi \times w$ satisfies the estimate

$$\left|\overset{\circ}{u}(t)\right| \leqslant M_q(2+|t|)^{-2q} \quad \text{for all } t \in \mathbb{R}^1,\ q \in \mathbb{N}. \tag{20.16}$$

In addition, $u \neq 0$ and $\operatorname{supp} u \subset B_{1,1+\varepsilon}$. We set

$$T_\nu = u, \quad \nu \in \mathcal{I}\backslash\{i_0\}.$$

Also let T_{i_0} be the characteristic function of the ball $B_{1+\alpha}$, where $\alpha \in (0,\varepsilon)$ is chosen so that $\mathcal{Z}(\overset{\circ}{T}_{i_0}) \cap \mathcal{Z}(\overset{\circ}{u}) = \varnothing$ (see (15.24)). Let us prove that the family $\mathcal{T} = \{T_i\}_{i\in\mathcal{I}}$ possesses (1) and (2). Bearing (20.16) in mind, we see that (1) holds. In addition, $\mathcal{Z}_{\mathcal{T}} = \varnothing$, $r(\mathcal{T}) < 1+\varepsilon$, and for each $R \in (1+\varepsilon, 2-\varepsilon)$, we have $C^{\infty}_{\mathcal{T},\natural}(B_R) \neq \{0\}$. In fact, it is enough to choose nonzero $f \in C^{\infty}_{\natural}(B_R)$ so that $f = 0$ in $B_{\varepsilon,R}$ and

$$\int_{B_\varepsilon} f(x)\,\mathrm{d}x = 0.$$

This means that $f \in C^{\infty}_{\mathcal{T},\natural}(B_R)$ and for some $\delta \in \widehat{K}_M$ and $j \in \{1,\ldots,d(\delta)\}$, the function $f^{\delta,j}$ cannot be presented in B_R as series (20.11). Hence the theorem. □

20.4 Zalcman-Type Two-Radii Problems on Domains of G/K (G Complex)

One well-known result in integral geometry is Zalcman's two-radii theorem: each function with vanishing integrals over all balls in $\mathbb{R}^n$ with radii in a fixed set $\{r_1, r_2\}$ vanishes if r_1/r_2 is not a ratio of two zeros of the Bessel function $J_{n/2}$. Examples show that this condition on r_1/r_2 is also necessary (see Zalcman [267–269, 271, 272]).

In this section, we consider the following related problem. Let $r_1, r_2 > 0$, $R > \max\{r_1, r_2\}$, and $f \in L^{1,\mathrm{loc}}(B_R)$. Assume that

$$\int_{B_{r_i}} f(gx)\,\mathrm{d}\mu(x) = 0, \quad g \in G,\ d(o, go) < R - r_i,\ i = 1, 2, \tag{20.17}$$

where

$$\mathrm{d}\mu(x) = \left(J\left(\mathrm{Exp}^{-1} x\right)\right)^{-1/2} \mathrm{d}x$$

(see (10.39) and (10.40)). Under what conditions does it follow that $f = 0$? We answer this question for all symmetric spaces $X = G/K$ of noncompact type with a complex group G.

Let $l = (\dim X)/2$. Denote by E_X the set of all numbers of the form α/β, where $\alpha, \beta > 0$ and $J_l(\alpha) = J_l(\beta) = 0$ (here, as before, J_l is the Bessel function of order l). It is easy to see that E_X is countable and everywhere dense in $(0, +\infty)$ (see (7.9)).

We say that a number $\tau > 0$ is *well approximated* by elements of E_X if for each $m > 0$, there exist positive numbers α, β such that $J_l(\alpha) = J_l(\beta) = 0$ and $|\tau - \alpha/\beta| < (2 + \beta)^{-m}$. Let WA_X be the set of all points well approximated by elements of E_X. We point out the following properties of the set WA_X (see V.V. Volchkov [225], Part II, Lemmas 1.13 and 1.16).

(a) $\tau \in \mathrm{WA}_X$ if and only if $\tau^{-1} \in \mathrm{WA}_X$.
(b) WA_X has zero Lebesgue measure in $(0, +\infty)$.
(c) The intersection of WA_X with any interval $(a, b) \subset (0, +\infty)$ is uncountable.
(d) $\tau \in \mathrm{WA}_X$ if and only if for each $m > 0$, there exists $\gamma > 0$ such that $J_l(\gamma) = 0$ and $|J_l(\tau\gamma)| < (2 + \gamma)^{-m}$.

Let $V_{r_1,r_2}(B_R)$ denote the set of all functions $f \in L^{1,\mathrm{loc}}(B_R)$ satisfying (20.17).

Theorem 20.17. *Let $X = G/K$ be a symmetric space of noncompact type with a complex group G and assume that $r_1, r_2 > 0$, $R > \max\{r_1, r_2\}$.*

(i) *If $f \in V_{r_1,r_2}(B_R)$, $r_1 + r_2 < R$, and $r_1/r_2 \notin E_X$, then $f = 0$.*
(ii) *If $f \in (V_{r_1,r_2} \cap C^\infty)(B_R)$, $r_1 + r_2 = R$, and $r_1/r_2 \notin E_X$, then $f = 0$.*
(iii) *If $f \in V_{r_1,r_2}(B_R)$, $r_1 + r_2 = R$, and $r_1/r_2 \in \mathrm{WA}_X \backslash E_X$, then $f = 0$.*
(iv) *If $r_1 + r_2 = R$ and $r_1/r_2 \notin \mathrm{WA}_X$, then for each integer $m \geqslant 0$, there exists a nontrivial function $f \in (V_{r_1,r_2} \cap C^m)(B_R)$.*
(v) *If $r_1 + r_2 > R$, then there exists a nontrivial function $f \in (V_{r_1,r_2} \cap C^\infty)(B_R)$.*

(vi) *If* $r_1/r_2 \in E_X$, *then there exists a nontrivial real analytic function* $f \in V_{r_1,r_2}(X)$.

We note that the situations described in assertions (i)–(vi) actually occur for suitable r_1, r_2 (see properties (a), (b), and (c) of the set WA_X).

Proof of Theorem 20.17. Let $g \in G$, and let χ_i be the characteristic function of the ball B_{r_i}, $i = 1, 2$. Then

$$\chi_i\big(g^{-1}o\big) = \chi_i(go). \tag{20.18}$$

Using (10.33) and (10.38), one infers that

$$J^{-1/2}\big(\mathrm{Exp}^{-1}(go)\big)\psi_\lambda\big(\mathrm{Exp}^{-1}(go)\big) = J^{-1/2}\big(\mathrm{Exp}^{-1}\big(g^{-1}o\big)\big)\psi_{-\lambda}\big(\mathrm{Exp}^{-1}\big(g^{-1}o\big)\big),$$

where

$$\psi_\lambda(P) = \int_K \mathrm{e}^{\mathrm{i}\langle A_\lambda, \mathrm{Ad}(k)P\rangle}\,\mathrm{d}k, \qquad P \in \mathfrak{p}.$$

Hence,

$$J^{-1/2}\big(\mathrm{Exp}^{-1}(go)\big) = J^{-1/2}\big(\mathrm{Exp}^{-1}\big(g^{-1}o\big)\big). \tag{20.19}$$

In view of (20.18), (20.19), and (10.10), relation (20.17) can be written as

$$f \times T_i = 0 \quad \text{in } B_{R-r_i},\ i = 1, 2,$$

where $T_i(x) = J^{-1/2}(\mathrm{Exp}^{-1}x)\chi_i(x)$, $x \in X$. The proof of Theorem 10.17 shows that $T_i \in \mathcal{E}'_{\natural\natural}(X)$ and

$$\widetilde{T}_i(\lambda) = c_i \mathbf{I}_{l/2}\big(r_i\sqrt{\langle\lambda,\lambda\rangle}\big), \qquad \lambda \in \mathfrak{a}^*_{\mathbb{C}},$$

where $\mathbf{I}_{l/2}(z) = J_{l/2}(z)z^{-l/2}$, and the constant c_i is independent of λ (see (14.18)). Now assertions (i) and (ii) are consequences of Theorem 20.2. Next, recall that all the zeros of $\mathbf{I}_{l/2}$ are real and simple. In addition, $|\mathbf{I}'_{l/2}(\zeta)| > c|\zeta|^{-(l+1)/2}$, where $\zeta \in \mathcal{Z}(\mathbf{I}_{l/2})$, and $c > 0$ is independent of ζ (see (14.22)). Thus, $T_1, T_2 \in \mathfrak{N}(X)$. Using Theorem 20.8(i), (ii) and property (d) of WA_X, we obtain assertions (iii)–(v).

Finally, assume that $r_1/r_2 \in E_X$. Then there exists $\lambda \in \{\mu \in \mathfrak{a}^*_{\mathbb{C}} : \widetilde{T}_1(\mu) = \widetilde{T}_2(\mu) = 0\}$, and the spherical function φ_λ is real-analytic in X and is in the class $V_{r_1,r_2}(X)$ (see the proof of Theorem 10.16). This implies assertion (vi) and completes the proof of Theorem 20.17. □

We now discuss the case $r_1/r_2 \in E_X$ in greater detail. In this case there is $c > 0$ such that $c = \lambda_1/r_1 = \lambda_2/r_2$ for some $\lambda_1, \lambda_2 \in \mathcal{Z}(\mathbf{I}_{l/2})$. Examining the above proof, we see that if $f \in L^{1,\mathrm{loc}}(X)$ and

$$\big(L + |\rho|^2 + c^2\big)f = 0, \tag{20.20}$$

then $f \in V_{r_1,r_2}(X)$. Is the converse true? This question remains open. However, we are able to prove the following weaker result.

Theorem 20.18. *Let* $\{\lambda_q\}_{q=1}^{\infty}$ *be the sequence of all positive zeros of* $\mathbf{I}_{l/2}$*, and let* $c > 0$*. Assume that* $f \in L^{1,\mathrm{loc}}(X)$ *and* $r_q = \lambda_q/c$*,* $q = 1, 2, \ldots$*. Then the following items are equivalent.*

(i) f *satisfies* (20.20).
(ii) $\int_{B_{r_q}} f(gx)\,\mathrm{d}\mu(x) = 0$ *for all* $g \in G$ *and* $q \in \mathbb{N}$.

Proof. We have already observed that (i)→(ii). Thus, we need to prove that (ii)→(i). Define $T_q \in \mathcal{E}'_{\natural\natural}(X)$ by the relation

$$\widetilde{T}_q(\lambda) = \big(\langle\lambda,\lambda\rangle - c^2\big)^{-1}\mathbf{I}_{l/2}\big(r_q\sqrt{\langle\lambda,\lambda\rangle}\,\big), \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}},\ q \in \mathbb{N}.$$

One therefore has $\bigcap_{q=1}^{\infty} \mathcal{Z}(\overset{\circ}{T}_q) = \varnothing$ (see [225], Part II, Lemma 1.29). In addition, equality (10.73) and the proof of Theorem 20.17 show that (ii) can be brought to the form

$$\big((L + |\rho|^2 + c^2)f\big) \times T_q = 0 \quad \text{for all } q.$$

Appealing now to Theorem 20.2, we arrive at the desired statement. □

Chapter 21
Spherical Spectral Analysis on Subsets of Compact Symmetric Spaces

This chapter is devoted to so-called "freak theorems" for systems of convolution equations on compact symmetric spaces and their local versions.

If $\mathcal{X}$ is a compact rank one symmetric space (see Chap. 11), $f \in L^1(\mathcal{X})$, and

$$\int_B f(x)\,\mathrm{d}x = 0 \tag{21.1}$$

for all geodesic balls $B \subset \mathcal{X}$ of (fixed) radii $r_1, \ldots, r_l \in (0, \pi/2)$, then $f = 0$ so long as the equations $R_N^{(\alpha_{\mathcal{X}}+1, \beta_{\mathcal{X}}+1)}(\cos 2r_i) = 0$, $i = 1, \ldots, l$, have no common solution for $N = 1, 2, \ldots$. Unlike the noncompact case (see Theorems 15.15 and 20.2), here l is not necessarily $\geqslant 2$. Similar result is also valid for spherical means. These "freak theorems" have been obtained for $\mathcal{X} = \mathbb{S}^2$ and ball means by Ungar [219], for $\mathcal{X} = \mathbb{S}^2$ and spherical means, in principle already by Radon [180]. For $\mathcal{X} = \mathbb{S}^n$, they have been proved by Schneider [187], and in the general case they follow from results of Berenstein and Zalcman [26]. In Sect. 21.1 we will extend the results we have just described to convolution equations on an arbitrary compact symmetric space.

The situation becomes much more complicated if we want to obtain local analogues of "freak theorems." Here f is defined on a domain $\mathcal{O} \subset \mathcal{X}$, and (21.1) is required to hold only when $\mathrm{Cl}\, B \subset \mathcal{O}$. As before, the object is to determine conditions under which (21.1) implies that f vanishes almost everywhere on $\mathcal{O}$. These questions and their generalizations to systems of convolution equations are discussed in Sect. 21.2.

21.1 The Case of the Whole Space. The Contrast Between the Noncompact Type and the Compact Type

Let Y be a Riemannian globally symmetric space of rank one. Take a distribution $T \in \mathcal{E}'_\natural(Y)$ for which $r(T) > 0$. Here, as usual, $r(T)$ is the radius of the smallest

V.V. Volchkov, V.V. Volchkov, *Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group,* Springer Monographs in Mathematics,

closed ball centered at 0 containing the support of T. Consider the set

$$\mathcal{D}'_T(Y) = \{f \in \mathcal{D}'(Y) \colon f \times T = 0 \text{ on } Y\}.$$

If Y is noncompact, then the spherical transform $\widetilde{T}$ has infinitely many zeroes. Therefore, the spherical function φ_λ belongs to $\mathcal{D}'_T(Y)$ for $\lambda \in \mathcal{Z}(\widetilde{T})$ (see Proposition 15.2). In particular, $\mathcal{D}'_T(Y) \neq \{0\}$.

Suppose Y is compact. Then the set of spherical functions on Y is countable, and the spherical transform

$$\widetilde{T}(j) = \langle T, \varphi_j \rangle, \quad j \in \mathbb{Z}_+,$$

does not necessarily has zeroes (see (11.4) and Badertscher [7]). In this case, $\varphi_j \notin \mathcal{D}'_T(Y)$ for every $j \in \mathbb{Z}_+$. Moreover, we have the following general fact.

Theorem 21.1. *Let U/K be a compact symmetric space of arbitrary rank, $\{T_i\}_{i\in\mathcal{I}}$ be a given family of K-invariant distributions on U/K. Then the system*

$$(f \times T_i)(p) = 0, \quad p \in U/K,\ i \in \mathcal{I}, \tag{21.2}$$

has a nontrivial solution $f \in \mathcal{D}'(U/K)$ if and only if there exists a spherical function φ on U/K such that

$$\langle T_i, \check{\varphi} \rangle = 0 \quad \textit{for all } i \in \mathcal{I}. \tag{21.3}$$

Proof. If condition (21.3) holds, then, thanks to (1.60),

$$\varphi \times T_i = \langle T_i, \check{\varphi} \rangle \varphi = 0, \quad i \in \mathcal{I},$$

i.e., φ satisfies (21.2). Conversely, let $f \in \mathcal{D}'(U/K)$ be a nontrivial solution of the system (21.2). By regularization, we can assume, without loss of generality, that $f \in \mathcal{E}(U/K)$. Denote by Φ the set of all spherical functions on U/K. For $\varphi \in \Phi$, we put

$$V_\varphi = \{g \in \mathcal{E}(U/K) \colon \mathrm{D}g = \lambda_\varphi(\mathrm{D})g\ \forall \mathrm{D} \in \mathbf{D}(U/K)\},$$

where λ_φ is the homomorphism of the algebra $\mathbf{D}(U/K)$ into $\mathbb{C}$ defined by the relation

$$\mathrm{D}\varphi = \lambda_\varphi(\mathrm{D})\varphi, \quad \mathrm{D} \in \mathbf{D}(U/K).$$

Since $L^2(U/K)$ is the countable orthogonal direct sum of the spaces $\{V_\varphi\}_{\varphi\in\Phi}$ (see Helgason [122], Chap. 5, Theorem 4.3), we can write

$$f = \sum_{\varphi\in\Phi} f_\varphi, \quad f_\varphi \in V_\varphi. \tag{21.4}$$

The series in (21.4) converges in $\mathcal{E}(U/K)$, because $f \in \mathcal{E}(U/K)$. Hence,

$$\sum_{\varphi\in\Phi} \langle T_i, \check{\varphi} \rangle f_\varphi = f \times T_i = 0, \quad i \in \mathcal{I}.$$

Bearing in mind that $f \neq 0$, we obtain $\langle T_i, \check{\varphi}\rangle = 0$ $(i \in \mathcal{I})$ for some $\varphi \in \Phi$, as required. Thus, Theorem 21.1 is proved. □

So, Theorem 21.1 gives an answer in the problem of existence of a nonzero solution of the system of convolution equations on the whole space U/K. In particular, we see that if Y is compact, then the space $\mathcal{D}'_{\mathcal{T}}(Y)$ can contain only the zero distribution. In the next section we shall obtain local analogues of Theorem 21.1 for the rank one case.

21.2 Freak-Like Theorems on Subsets of Compact Symmetric Spaces of Rank One

Let $\mathcal{X}$ be a rank one symmetric space of the compact type (see Sect. 11.1), $\mathcal{T} = \{T_i\}_{i\in\mathcal{I}}$ be a family of nonzero distributions in $\mathcal{E}'_{\natural}(\mathfrak{X})$, and let

$$\Lambda(\mathcal{T}) = \{\Lambda(T_i)\}_{i\in\mathcal{I}},$$

where $\Lambda(T_i)$ is given by (11.92) with $T = T_i$. Put $r_i = r(T_i)$, $i \in \mathcal{I}$. If $R > r_i$ for all $i \in \mathcal{I}$, we define

$$\mathcal{D}'_{\mathcal{T}}(B_R) = \bigcap_{i\in\mathcal{I}} \mathcal{D}'_{T_i}(B_R),$$
$$C^s_{\mathcal{T}}(B_R) = \big(\mathcal{D}'_{\mathcal{T}} \cap C^s\big)(B_R), \quad s \in \mathbb{Z}_+ \cup \{\infty\}.$$

Theorem 21.2. *Suppose that $r_i < R \leqslant \pi/2$, $i \in \mathcal{I}$. Then for fixed $k \in \mathbb{Z}_+$, $m \in \{0, \dots, M_{\mathcal{X}}(k)\}$, and $j \in \{1, \dots, d^{k,m}_{\mathcal{X}}\}$, the following assertions are equivalent.*

(i) $\mathcal{D}'_{\mathcal{T}}(B_R) = \{0\}$.
(ii) $(\mathcal{D}'_{\mathcal{T}} \cap \mathcal{D}'_{k,m,j})(B_R) = \{0\}$.
(iii) $\mathcal{D}'_{\Lambda(\mathcal{T}),\natural}(-R, R) = \{0\}$.

The same holds true if we proceed from the classes $\mathcal{D}'_{\mathcal{T}}(B_R)$, $(\mathcal{D}'_{\mathcal{T}} \cap \mathcal{D}'_{k,m,j})(B_R)$, and $\mathcal{D}'_{\Lambda(\mathcal{T}),\natural}(-R, R)$ to the classes $C^\infty_{\mathcal{T}}(B_R)$, $(C^\infty_{\mathcal{T}} \cap \mathcal{D}'_{k,m,j})(B_R)$, and $C^\infty_{\Lambda(\mathcal{T}),\natural}(-R, R)$, respectively.

Proof. This follows from Theorem 11.3 and Proposition 16.3. □

Using Theorem 21.2 and the results in Chap. 18, one can establish analogues of the main statements of Chap. 20 for $\mathcal{X}$. As before, we set

$$r^*(\mathcal{T}) = \sup_{i\in\mathcal{I}} r_i, \quad r_*(\mathcal{T}) = \inf_{i\in\mathcal{I}} r_i, \quad \mathcal{Z}_{\mathcal{T}} = \bigcap_{i\in\mathcal{I}} \mathcal{Z}_{T_i}.$$

Theorem 21.3. *Let $\mathcal{Z}_{\mathcal{T}} = \varnothing$ and assume that $r_*(\mathcal{T}) + r_i < R \leqslant \pi/2$ for all $i \in \mathcal{I}$. Then $\mathcal{D}'_{\mathcal{T}}(B_R) = \{0\}$.*

The assertion of Theorem 21.3 is an immediate consequence of Theorems 18.1(i) and 21.2.

Next, in the same way as in the proof of Theorems 19.12(i), 19.13(i), and 19.15(i), (ii), we obtain the following:

Theorem 21.4. *Let $\mathcal{T} = \{T_1, T_2\}$, $\mathcal{Z}_{\mathcal{T}} = \varnothing$, and $R \in (r^*(\mathcal{T}), \pi/2]$. Then the following are true.*

(i) *If $R = r_1 + r_2$ and $T_1 \in \mathfrak{M}(\mathcal{X})$, then $\mathcal{D}'_{\mathcal{T}}(B_R) \cap C^\infty(B_{r_1+\varepsilon}) = \{0\}$ for each $\varepsilon \in (0, R - r_1)$.*

(ii) *Let $R = r_1 + r_2$ and $T_1 \in \mathrm{Inv}_+(\mathcal{X})$. Suppose that there is a sequence $\{\lambda_i\}_{i=1}^\infty$ of complex numbers such that for any $\alpha > 0$,*

$$(2 + |\lambda_i|)^\alpha \big(|\widetilde{T}_1(\lambda_i)| + |\widetilde{T}_2(\lambda_i)|\big) \to 0 \quad \text{as } i \to \infty$$

and

$$|\mathrm{Im}\,\lambda_i| \leqslant c \log(2 + |\lambda_i|),$$

where the constant $c > 0$ does not depend on i. Then $\mathcal{D}'_{\mathcal{T}}(B_R) = \{0\}$. In particular, if $R = r_1 + r_2$, $T_1 \in \mathfrak{M}(\mathcal{X})$, and $\mathcal{Z}(\widetilde{T}_1) \approx \mathcal{Z}(\widetilde{T}_2)$, then $\mathcal{D}'_{\mathcal{T}}(B_R) = \{0\}$.

(iii) *Let $T_1 \in \mathrm{Inv}_+(\mathcal{X})$. Assume that there is $\psi \in \mathfrak{M}(\mathcal{X})$ such that $r(\psi) > 0$ and for each $\lambda \in \mathcal{Z}_\psi$,*

$$\sum_{\eta=0}^{n(\lambda,\psi)} \big(|\widetilde{T}_1^{(\eta)}(\lambda)| + |\widetilde{T}_2^{(\eta)}(\lambda)|\big) \leqslant M_q (2 + |\lambda|)^{-q}, \quad q = 1, 2, \ldots,$$

where the constants $M_q > 0$ do not depend on λ and satisfy (18.10). *Then $\mathcal{D}'_{\mathcal{T}}(B_R) = \{0\}$ if $R > r_1 + r_2 - r(\psi)$.*

(iv) *Let $T_1 = \psi_1 \times \psi_2 \in \mathfrak{N}(\mathcal{X})$, where $\psi_1 \in \mathfrak{N}(\mathcal{X})$, $\psi_2 \in \mathcal{E}'_\natural(\mathfrak{X})$. Assume that $\mathcal{Z}(\widetilde{\psi}_1) \cap \mathcal{Z}(\widetilde{\psi}_2) = \varnothing$ and*

$$\sum_{\eta=0}^{n(\lambda,\psi_2)} |\widetilde{T}_2^{(\eta)}(\lambda)| \leqslant M_q (1 + |\lambda|)^{-q}, \quad q = 1, 2, \ldots, \tag{21.5}$$

for all $\lambda \in \mathcal{Z}_{\psi_2}$, where the constants $M_q > 0$ do not depend on λ and satisfy (18.10). *Then $\mathcal{D}'_{\mathcal{T}}(B_R) = \{0\}$ if $R > r_2 + r(\psi_1)$. Furthermore, $C^\infty_{\mathcal{T}}(B_R) = \{0\}$ when $R \geqslant r_2 + r(\psi_1)$.*

The following result shows that the assumptions in Theorems 21.3 and 21.4 are precise.

Theorem 21.5.

(i) *If $\mathcal{Z}_{\mathcal{T}} \neq \varnothing$, then $(\mathcal{D}'_{\mathcal{T}} \cap \mathrm{RA}_\natural)(\mathfrak{X}) \neq \{0\}$.*

(ii) *If $T_1 \in \mathfrak{N}(\mathcal{X})$ and*

$$0 < r_1 \leqslant \pi/4, \tag{21.6}$$

then there is $T_2 \in \mathcal{E}'_\natural(\mathfrak{X})$ *with the following properties:*

(1) $r_1 = r_2$;
(2) $\mathcal{Z}_\mathcal{T} = \varnothing$ *with* $\mathcal{T} = \{T_1, T_2\}$;
(3) *for* $R = r_1 + r_2$ *and for each* $l \in \mathbb{Z}_+$, *the class* $C^l_\mathcal{T}(B_R)$ *contains a nontrivial function vanishing in* B_{r_1}.

(iii) *Let* $\mathcal{T} = \{T_1, T_2\}$, *where* $T_1 \in \mathfrak{N}(\mathcal{X})$ *and* $\mathcal{Z}(\widetilde{T}_1) \not\approx \mathcal{Z}(\widetilde{T}_2)$. *Suppose that* $r^*(\mathcal{T}) < R \leqslant \pi/2$ *and* $R \leqslant r_1 + r_2$. *Then* $C^s_\mathcal{T}(B_R) \neq \{0\}$ *for each* $s \in \mathbb{Z}_+$. *In addition, a distribution* $f \in \mathcal{D}'(B_R)$ *belongs to* $\mathcal{D}'_\mathcal{T}(B_R)$ *if and only if for all* $k \in \mathbb{Z}_+$, $m \in \{0, \dots, M_\mathcal{X}(k)\}$, *and* $j \in \{1, \dots, d^{k,m}_\mathcal{X}\}$,

$$f^{k,m,j} = \mathfrak{A}^{-1}_{k,m,j}\big(\big(\Omega'_{\Lambda(T_1),\Lambda(T_2)} * u\big)\big|_{(-R,R)}\big),$$

where u *is a distribution in* $\mathcal{E}'_\natural(\mathbb{R}^1)$ *depending on* k, m, j *such that* $\operatorname{supp} u \subset [R - r_1 - r_2, r_1 + r_2 - R]$. *In particular,* $C^\infty_\mathcal{T}(B_R) \neq \{0\}$ *when* $R < r_1 + r_2$.

(iv) *Let* $\mathcal{T} = \{T_1, T_2\}$, $T_1 \in \mathfrak{N}(\mathcal{X})$, $R \in (r^*(\mathcal{T}), \pi/2]$, *and* $R < r_1 + r_2$. *Assume that for some* $l > 0$, *the set*

$$\big\{\lambda \in \mathcal{Z}_{T_1} : \big|\widetilde{T}_2(\lambda)\big| < (2 + |\lambda|)^{-l}\big\}$$

is sparse. Then for an arbitrary sequence $\{M_q\}_{q=1}^\infty$ *of positive numbers satisfying* (19.14), *there is a nontrivial function* $f \in C^\infty_\mathcal{T}(B_R)$ *such that*

$$\sup_{p \in B_R} \big|\big(L^q f\big)(p)\big| \leqslant M_q \quad \text{for all } q \in \mathbb{N}.$$

(v) *There exists a family* $\mathcal{T} = \{T_1, T_2\}$ *satisfying all the assumptions of Theorem 21.4*(iv) *such that* $r^*(\mathcal{T}) < r_2 + r(\psi_1) \leqslant \pi/2$ *and* $C^s_\mathcal{T}(B_R) \neq \{0\}$ *for all* $s \in \mathbb{Z}_+$ *and* $R = r_2 + r(\psi_1)$.

(vi) *There exists a family* $\mathcal{T} = \{T_1, T_2\}$ *satisfying all the assumptions of Theorem 21.4*(iv) *such that* $r^*(\mathcal{T}) < r_2 + r(\psi_1) \leqslant \pi/2$ *and* $C^\infty_\mathcal{T}(B_R) \neq \{0\}$ *for every* $R \in (r^*(\mathcal{T}), r_2 + r(\psi_1))$.

(vii) *For an arbitrary sequence* $\{M_q\}_{q=1}^\infty$ *of positive numbers such that the series in* (18.10) *is convergent, there exist* $T_1, \psi_1, \psi_2 \in \mathfrak{N}(\mathcal{X})$, *and* $T_2 \in \mathcal{D}_\natural(\mathfrak{X})$ *with the following properties:*

(1) *for each* $\lambda \in \mathcal{Z}(\widetilde{\psi}_2)$, *estimates* (21.5) *are true;*
(2) $\mathcal{Z}(\widetilde{\psi}_1) \cap \mathcal{Z}(\widetilde{\psi}_2) = \varnothing$;
(3) $T_1 = \psi_1 \times \psi_2$ *and* $\mathcal{Z}(\widetilde{T}_1) \cap \mathcal{Z}(\widetilde{T}_2) = \varnothing$;
(4) $r_2 + r(\psi_1) \in (r^*(\mathcal{T}), \pi/2)$ *and* $C^\infty_\mathcal{T}(B_R) \neq \{0\}$ *for some* $R \in (r_2 + r(\psi_1), \pi/2)$, *where* $\mathcal{T} = \{T_1, T_2\}$.

Proof. Part (i) is clear from Proposition 11.10. Parts (ii)–(vi) are proved similarly to Theorems 19.12(ii), 19.13(ii), 19.14(i), and 19.15(iii), (iv). Finally, using Theorems 18.11 and 21.2, we arrive at (vii). □

It is relevant to remark that the value $\pi/4$ in estimate (21.6) cannot be increased. Otherwise, for $r_1 > \pi/4$, we have $r_2 < \pi/4$. Then by Theorem 16.2(i),

$$\left\{f \in \mathcal{D}'_{T_2}(B_{r_1+r_2})\colon\ f = 0 \text{ in } B_{r_1}\right\} = \{0\},$$

contrary to the property (3) in the statement (ii).

To end we note that Theorems 20.7, 20.13, and 20.14 also have natural analogues for $\mathcal{X}$. We leave them for the reader to state and prove.

Bibliography

1. Adams, J.F.: Lectures on Lie Groups. Benjamin, New York (1969)
2. Agmon, S.: Functions of exponential type in an angle and singularities of Taylor series. Trans. Am. Math. Soc. **70**, 492–508 (1951)
3. Agranovsky, M.L., Narayanan, E.K.: L^p-integrability, supports of Fourier transforms and uniqueness for convolution equations. J. Fourier Anal. Appl. **10**, 13–27 (2004)
4. Agranovsky, M.L., Narayanan, E.K.: A local two radii theorem for the twisted spherical means on $\mathbb{C}^n$. Contemp. Math. **382**, 13–27 (2005)
5. Ahlfors, L.V.: Möbius Transformations in Several Dimensions. Lect. Notes Math. University of Minnesota Press, Minnesota (1981)
6. Askey, R., Fitch, J.: Integral representations for Jacobi polynomials and some applications. J. Math. Anal. Appl. **26**, 411–437 (1969)
7. Badertscher, E.: The Pompeiu problem on locally symmetric spaces. J. Anal. Math. **57**, 250–281 (1991)
8. Bagchi, S.M., Sitaram, A.: Spherical mean-periodic functions on semi-simple Lie groups. Pac. J. Math. **84**, 241–250 (1979)
9. Ben Natan, Y., Weit, Y.: Integrable harmonic functions on symmetric spaces of rank one. J. Funct. Anal. **160**, 141–149 (1998)
10. Besse, A.L.: Manifolds All of Whose Geodesics Are Closed. Springer, New York (1978)
11. Berenstein, C.A.: Spectral synthesis on symmetric spaces. Contemp. Math. **63**, 1–25 (1987)
12. Berenstein, C.A., Dostal, M.A.: Some remarks on convolution equations. Ann. Inst. Fourier **23**, 55–74 (1973)
13. Berenstein, C.A., Dostal, M.A.: On convolution equations I. In: Lect. Notes Math., vol. 336, pp. 79–94. Springer, Berlin (1973)
14. Berenstein, C.A., Gay, R.: A local version of the two-circles theorem. Isr. J. Math. **55**, 267–288 (1986)
15. Berenstein, C.A., Gay, R.: Sur la synthèse spectrale dans les espaces symétriques. J. Math. Pures Appl. **65**, 323–333 (1986)

V.V. Volchkov, V.V. Volchkov, *Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group,* Springer Monographs in Mathematics,

16. Berenstein, C.A., Gay, R.: Le problème de Pompeiu locale. J. Anal. Math. **52**, 133–166 (1989)
17. Berenstein, C.A., Gay, R.: Complex Analysis and Special Topics in Harmonic Analysis. Springer, New York (1995)
18. Berenstein, C.A., Struppa, D.C.: Solutions of convolution equations in convex sets. Am. J. Math. **109**, 521–544 (1987)
19. Berenstein, C.A., Struppa, D.C.: Complex analysis and convolution equations. In: Henkin, G.M. (ed.) Several Complex Variables V. Encycl. Math. Sci., vol. 54, pp. 1–108. Springer, New York (1993)
20. Berenstein, C.A., Taylor, B.A.: A new look at interpolation theory for entire functions of one variable. Adv. Math. **33**, 109–143 (1979)
21. Berenstein, C.A., Taylor, B.A.: Mean-periodic functions. Int. J. Math. Sci. **3**, 199–236 (1980)
22. Berenstein, C.A., Taylor, B.A.: Interpolation problems in $\mathbb{C}^n$ with applications to harmonic analysis. J. Anal. Math. **38**, 188–254 (1980)
23. Berenstein, C.A., Yang, P.C.: An inverse Neumann problem. J. Reine Angew. Math. **382**, 1–21 (1987)
24. Berenstein, C.A., Yger, A.: Le problème de la déconvolution. J. Funct. Anal. **54**, 113–160 (1983)
25. Berenstein, C.A., Yger, A.: Effective Bezout identities in $Q[z_1, \dots, z_n]$. Acta Math. **166**, 69–120 (1991)
26. Berenstein, C.A., Zalcman, L.: Pompeiu's problem on symmetric spaces. Comment. Math. Helv. **55**, 593–621 (1980)
27. Berenstein, C.A., Taylor, B.A., Yger, A.: Sur quelques formules explicites de déconvolution. J. Opt. Soc. Am. **14**, 75–82 (1983)
28. Berenstein, C.A., Gay, R., Yger, A.: Inversion of the local Pompeiu transform. J. Anal. Math. **54**, 259–287 (1990)
29. Berenstein, C.A., Chang, D.C., Tie, J.: Laguerre Calculus and Its Applications on the Heisenberg Group. Am. Math. Soc., Providence (2001)
30. Berenstein, C.A., Chang, D.C., Eby, W., Zalcman, L.: Moment versions of the Morera problem in $\mathbf{C^n}$ and $\mathbf{H^n}$. Adv. Appl. Math. **31**, 273–300 (2003)
31. Berenstein, C.A., Chang, D.C., Eby, W.: L^p results for the Pompeiu problem with moments on the Heisenberg group. J. Fourier Anal. Appl. **10**, 545–571 (2004)
32. Berezin, F.A., Gel'fand, I.M.: Some remarks on the theory of spherical functions on symmetric Riemannian manifolds. Tr. Mosk. Mat. Obshch. **5**, 311–356 (1956)
33. Berkani, M., El Harchaoui, M., Gay, R.: Inversion de la transformation de Pompéiu locale dans l'espace hyperbolique quaternionique—cas des deux boules. Complex Var. Theory Appl. **43**, 29–57 (2000)
34. Berndtsson, B., Passare, M.: Integral formulas and an explicit version of the Fundamental Principle. J. Funct. Anal. **84**, 358–372 (1989)
35. Betancor, J.J., Betancor, J.D., Méndez, J.M.R.: Hypoelliptic Jacobi convolution operators on Schwartz distributions. Positivity **8**, 407–422 (2004)

36. Bhanumurthy, T.S.: Plancherel's measure for the factor space $\mathrm{SL}(n, \mathrm{R})/\mathrm{SO}(n, \mathrm{R})$. Dokl. Akad. Nauk SSSR **133**, 503–506 (1960). English transl.: Sov. Math. Dokl. **1**, 860–862 (1960)
37. Bhanumurthy, T.S.: The asymptotic behavior of zonal spherical functions on the Siegel upper half-plane. Dokl. Akad. Nauk SSSR **135**, 1027–1030 (1960). English transl.: Sov. Math. Dokl. **1**, 1325–1329 (1960)
38. Bishop, R.L., Crittenden, R.J.: Geometry of Manifolds. Academic Press, New York (1964)
39. Borel, A.: Le plan projectif des octaves et les sphéres comme espaces homogènes. C.R. Acad. Sci. **230**, 1378–1380 (1950)
40. Bredon, G.E.: Introduction to Compact Transformation Groups. Academic Press, New York (1972)
41. Bremermann, H.: Distributions, Complex Variables, and Fourier Transforms. Addison-Wesley, Reading (1965)
42. Brown, L., Schreiber, B.M., Taylor, B.A.: Spectral synthesis and the Pompeiu problem. Ann. Inst. Fourier **23**, 125–154 (1973)
43. Bruhat, F.: Sur les representations induites des groupes de Lie. Bull. Soc. Math. France **84**, 97–205 (1956)
44. Busemann, H.: The Geometry of Geodesics. Academic Press, New York (1955)
45. Cantor, I.L., Solodovnikov, A.S.: Hypercomplex Numbers. Nauka, Moscow (1973)
46. Cartan, É., Les groupes réels simples finis et continus. Ann. École Norm. **31**, 263–355 (1914)
47. Cartan, É., Sur certaines formes riemanniennes remarquables des géométries a groupe fondamental simple. Ann. Sci. École Norm. Sup. **44**, 345–467 (1927)
48. Cartan, É., Sur la detérmination d'un système orthogonal complet dans un espace de Riemann symétrique clos. Rend. Circ. Mat. Palermo. **53**, 217–252 (1929)
49. Casey, S.D., Walnut, D.F.: Systems of convolution equations, deconvolution, Shannon sampling, and the wavelet and Gabor transforms. SIAM Rev. **36**, 537–577 (1994)
50. Chèbli, H.: Sur un théorème de Paley–Wiener associé à la décomposition spectrale d'un opérateur de Sturm–Liouville sur $(0, \infty)$. J. Funct. Anal. **17**, 447–461 (1974)
51. Chevalley, C., Schafer, R.: The exceptional simple Lie algebras F_4 and E_6. Proc. Nat. Acad. Sci. U.S.A. **36**, 137–141 (1950)
52. Chu, C.-H., Leung, C.-W.: Harmonic functions on homogeneous spaces. Monatsh. Math. **128**, 227–235 (1999)
53. Collingwood, E.F., Lohwater, A.J.: The Theory of Cluster Sets. Cambridge University Press, Cambridge (1966)
54. Delsarte, J.: Les fonctions moyenne-périodiques. J. Math. Pures Appl. **14**, 403–453 (1935)
55. Delsarte, J.: Theorie des fonctions moyenne-périodiques de deux variables. Ann. Math. **72**, 121–178 (1960)

56. Delsarte, J.: Lectures on Topics in Mean Periodic Functions and the Two-Radius Theorem. Tata Institute, Bombay (1961)
57. Dickson, D.G.: Analytic mean periodic functions. Trans. Am. Math. Soc. **110**, 361–374 (1964)
58. Dubrovin, B.A., Novikov, S.P., Fomenko, A.T.: Modern Geometry. Nauka, Moscow (1979)
59. Dunford, N., Schwartz, J.T.: Linear Operators, vol. II. Interscience, New York (1963)
60. Edwards, R.E.: Fourier Series a Modern Introduction, vols. I, II. Springer, New York (1982)
61. Eguchi, M., Hashizume, M., Okamoto, K.: The Paley–Wiener theorem for distributions on symmetric spaces. Hiroshima Math. J. **3**, 109–120 (1973)
62. Ehrenpreis, L.: Solutions of some problems of division I. Am. J. Math. **76**, 883–903 (1954)
63. Ehrenpreis, L.: Solutions of some problems of division II. Am. J. Math. **77**, 286–292 (1955)
64. Ehrenpreis, L.: Solutions of some problems of division III. Am. J. Math. **78**, 685–715 (1956)
65. Ehrenpreis, L.: Solutions of some problems of division IV. Am. J. Math. **82**, 522–588 (1960)
66. Ehrenpreis, L.: Solutions of some problems of division V. Am. J. Math. **84**, 324–348 (1962)
67. Ehrenpreis, L.: Mean periodic functions I. Am. J. Math. **77**, 293–328, 731–733 (1955)
68. Ehrenpreis, L.: The Fundamental Principle for linear constant coefficients partial differential equations. In: Proc. Intern. Symp. Linear Spaces, pp. 161–174. Pergamon, Oxford (1961)
69. Ehrenpreis, L.: Fourier Analysis in Several Complex Variables. Wiley-Interscience, New York (1970)
70. Ehrenpreis, L., Malliavin, P.: Invertible operators and interpolation in AU-spaces. J. Math. Pures Appl. **53**, 165–182 (1974)
71. Ehrenpreis, L., Mautner, F.: Some properties of the Fourier transform on semi-simple Lie groups II. Trans. Am. Math. Soc. **84**, 1–55 (1957)
72. El Harchaoui, M.: Inversion de la transformation de Pompeiu locale dans les espaces hyperboliques réel et complexe (Cas de deux boules). J. Anal. Math. **67**, 1–37 (1995)
73. Erdélyi, A., Magnus, W., Oberhettinger, F., Tricomi, F.G.: Higher Transcendental Functions (Bateman Manuscript Project), vols. I, II. McGraw-Hill, New York (1953)
74. Evgrafov, M.A.: Asymptotic Estimates and Entire Functions, 3rd edn. Nauka, Moscow (1979). English transl. of 1st ed.: Gordon and Breach, New York (1961)
75. Faulkner, J.: Octonion Planes Defined by Quadratic Jordan Algebras. Memoir of the Am. Math. Soc., vol. 104. Am. Math. Soc., Providence (1970)
76. Flensted-Jensen, M.: Paley–Wiener type theorems for a differential operator connected with symmetric spaces. Ark. Mat. **10**, 143–162 (1972)

77. Flensted-Jensen, M., Ragozin, D.L.: Spherical functions are Fourier transforms of L^1-functions. Ann. Sci. Ecole Norm. Sup. **6**, 457–458 (1973)
78. Folland, G.: Harmonic Analysis in Phase Space. Ann. Math. Stud., vol. 122. Princeton University Press, Princeton (1989)
79. Freudenthal, H.: Oktaven, Ausnahmegruppen and Oktavengeometrie. Math. Inst. der Rijksuniversiteit, Utrecht (1951)
80. Gay, R.: Division des fonctionnelles analytiques. Application aux fonctions entières de type exponentiel moyenne-periodiques. In: Lect. Notes Math., vol. 822, pp. 77–89. Springer, Berlin (1980)
81. Gel'fand, I.M.: Spherical functions in symmetric Riemannian spaces. Dokl. Akad. Nauk SSSR **70**, 5–8 (1950)
82. Gel'fond, A.O.: Calculus of Finite Differences. Nauka, Moscow (1967)
83. Gindikin, S.G.: Unitary representations of groups of automorphisms of Riemannian symmetric spaces of null curvature. Funkc. Anal. Ego Prilozh. **1**, 32–37 (1967)
84. Gindikin, S.G., Karpelevič, F.I.: Plancherel measure of Riemannian symmetric spaces of non-positive curvature. Dokl. Akad. Nauk SSSR **145**, 252–255 (1962)
85. Globevnik, J.: Analyticity on rotation invariant families of curves. Trans. Am. Math. Soc. **280**, 247–254 (1983)
86. Globevnik, J.: Integrals over circles passing through the origin and a characterizations of analytic functions. J. Anal. Math. **52**, 199–209 (1989)
87. Globevnik, J.: Zero integrals on circles and characterizations of harmonic and analytic functions. Trans. Am. Math. Soc. **317**, 313–330 (1990)
88. Globevnik, J.: Local support theorems for the k-plane transform on $\mathbb{R}^n$. J. Math. Anal. Appl. **181**, 455–461 (1994)
89. Globevnik, J.: Holomorphic extensions from open families of circles. Trans. Am. Math. Soc. **355**, 1921–1931 (2003)
90. Globevnik, J.: Analyticity on families of circles. Isr. J. Math. **142**, 29–45 (2004)
91. Globevnik, J.: A decomposition of functions with zero means on circles. Ark. Mat. **43**, 383–393 (2005)
92. Globevnik, J.: Analyticity on translates of a Jordan curve. Trans. Am. Math. Soc. **359**, 5555–5565 (2007)
93. Globevnik, J., Quinto, E.T.: Morera theorems via microlocal analysis. J. Geom. Anal. **6**, 20–30 (1996)
94. Globevnik, J., Rudin, W.: A characterization of harmonic functions. Indag. Math. **50**, 419–426 (1988)
95. Globevnik, J., Stout, E.L.: Boundary Morera theorems for holomorphic functions of several complex variables. Duke Math. J. **64**, 571–615 (1991)
96. Godement, R.: A theory of spherical functions I. Trans. Am. Math. Soc. **73**, 496–556 (1952)
97. Golovin, V.D.: On a generalization of the notion of periodic continuation. Dokl. Akad. Nauk SSSR **149**(3), 502–504 (1963)
98. Gonzalez, F., Quinto, E.T.: Support theorems for Radon transforms on higher rank symmetric spaces. Proc. Am. Math. Soc. **122**, 1045–1052 (1994)

99. Gorbacevič, V.V., Oniščik, A.L.: Lie transformation groups. In: Itogi Nauki i Tekhniki, Sovrem. Problemy Matematiki, Fundamentalnye Napravleniya, vol. 20, pp. 103–244. VINITI, Moscow (1988)
100. Grinberg, E., Quinto, E.T.: Morera theorems for complex manifolds. J. Funct. Anal. **178**, 1–22 (2000)
101. Gromoll, D., Klingenberg, W., Meyer, W.: Riemennsche Geometrie im Großen. Springer, New York (1968)
102. Gurevich, D.I.: Counterexamples to a problem of L. Schwartz. Funkc. Anal. Ego Prilozh. **9**(2), 29–35 (1975). English transl.: Funct. Anal. Appl. **9**, 116–120 (1975)
103. Hansen, S.: Localizable analytically uniform spaces and the Fundamental Principle. Trans. Am. Math. Soc. **264**, 235–250 (1981)
104. Hansen, S.: On the Fundamental Principle of L. Ehrenpreis. Preprint, Paderborn University (1982)
105. Harish-Chandra: Spherical functions on a semi-simple Lie group I. Am. J. Math. **80**, 241–310 (1958)
106. Harish-Chandra: Spherical functions on a semi-simple Lie group II. Am. J. Math. **80**, 553–613 (1958)
107. Harish-Chandra: Some results on differential equations and their applications. Proc. Nat. Acad. Sci. U.S.A. **45**, 1763–1764 (1959)
108. Harish-Chandra: Discrete series for semisimple Lie groups II. Acta Math. **116**, 1–111 (1966)
109. Harish-Chandra, Z.: Differential equations and semisimple Lie groups. In: Varadarajan, V.S. (ed.) Harish-Chandra, Collected Papers, vol. III, pp. 6–56. Springer, New York (1984).
110. Harris, L.A.: Banach algebras with involution and Möbius transformations. J. Funct. Anal. **11**, 1–16 (1972)
111. Harris, L.A.: Bounded symmetric homogeneous domains in infinite dimensional spaces. In: Lect. Notes Math., vol. 364, pp. 13–40. Springer, Berlin (1973)
112. Harris, L.A.: Operator Siegel domains. Proc. R. Soc. Edinb., Sect. A **79**, 137–156 (1977)
113. Hayden, T.L., Suffridge, T.J.: Biholomorphic maps in Hilbert space have a fixed point. Pac. J. Math. **38**, 419–422 (1971)
114. Heal, K.M., Hansen, M.L., Rickard, K.M.: Maple V Learning Guide. Springer, New York (1996)
115. Helgason, S.: Differential Geometry and Symmetric Spaces. Academic Press, New York (1962)
116. Helgason, S.: Radon–Fourier transforms on symmetric spaces and related group representations. Bull. Am. Math. Soc. **71**, 757–763 (1965)
117. Helgason, S.: A duality for symmetric spaces with applications to group representations. Adv. Math. **2**, 313–319 (1970)
118. Helgason, S.: Analysis on Lie Groups and Homogeneous Spaces. Conf. Board Math. Sci. Series, vol. 14. Am. Math. Soc., Providence (1972)
119. Helgason, S.: The surjectivity of invariant differential operators on symmetric spaces. Ann. Math. **98**, 451–480 (1973)

120. Helgason, S.: Eigenspaces of the Laplacian; integral representations and irreducibility. J. Funct. Anal. **17**, 328–353 (1974)
121. Helgason, S.: Differential Geometry, Lie Groups, and Symmetric Spaces. Academic Press, New York (1978)
122. Helgason, S.: Groups and Geometric Analysis. Academic Press, New York (1984)
123. Helgason, S.: Geometric Analysis on Symmetric Spaces. Am. Math. Soc., Providence (1994)
124. Hörmander, L.: Linear Partial Differential Operators. Springer, New York (1963)
125. Hörmander, L.: Generators for some rings of analytic functions. Bull. Am. Math. Soc. **73**, 943–949 (1967)
126. Hörmander, L.: The Analysis of Linear Partial Differential Operators, vols. I, II. Springer, New York (1983)
127. Jacobson, N.: Lie Algebras. Wiley-Interscience, New York (1962)
128. John, F.: Abhängigkeiten zwischen den Flächenintegralen einer stetigen Funktion. Math. Ann. **111**, 541–559 (1935)
129. John, F.: Plane Waves and Spherical Means, Applied to Partial Differential Equations. Dover, New York (1971)
130. Johnson, K.D.: Composition series and intertwining operators for the spherical principal series II. Trans. Am. Math. Soc. **215**, 269–283 (1976)
131. Johnson, K., Wallach, N.R.: Composition series and intertwining operators for the spherical principal series I. Trans. Am. Math. Soc. **229**, 137–173 (1977)
132. Kahane, J.P.: Sur quelques problèmes d'unicité et de prolongement relatifs aux fonctions approchables par des sommes d'exponentielle. Ann. Inst. Fourier **5**, 39–130 (1955)
133. Kahane, J.P.: Sur les fonctions moyenne–périodiques bornées. Ann. Inst. Fourier **7**, 293–314 (1957)
134. Kahane, J.P.: Lectures on Mean Periodic Functions. Tata Institute of Fundamental Research, Bombay (1957)
135. Kaneko, A.: Fundamental principle and extension of solutions of linear differential equations with constant coefficients. In: Lect. Notes Math., vol. 287, pp. 122–134. Springer, Berlin (1973)
136. Kargaev, P.P.: Zeroes of mean-periodic functions. Mat. Zametki. **37**(3), 322–325 (1985). English transl.: Math. Notes **37**(3), 181–183 (1985)
137. Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry, vols. I, II. Interscience, New York (1963, 1969)
138. Koornwinder, T.: A new proof of a Paley–Wiener type theorem for the Jacobi transform. Ark. Mat. **13**, 145–159 (1975)
139. Koornwinder, T.H.: Jacobi functions and analysis on noncompact semisimple Lie groups. In: Askey, R.A., et al. (eds.) Special Functions: Group Theoretical Aspects and Applications, pp. 1–85. Reidel, Dordrecht (1984)
140. Korányi, A.: A survey of harmonic functions on symmetric spaces. In: Harmonic Analysis in Euclidean Spaces. Proc. Symp. Pure Math., vol. 35, Part 1. Am. Math. Soc., Providence (1979),

141. Kostant, B.: On the existence and irreducibility of certain series of representations. Bull. Am. Math. Soc. **75**, 627–642 (1969)
142. Köthe, G.: Topological Vector Spaces. Springer, New York (1969)
143. Kuchment, P.A.: Functions on symmetric spaces which are periodic in the mean. Teor. Funkc. Funkc. Anal. Ih Prilozh. **16**, 68–69 (1982)
144. Leont'ev, A.F.: Sequences of Polynomials in Exponents. Nauka, Moscow (1980)
145. Levin, B.Ya.: Distribution of Zeros of Entire Functions. Gostekhizdat, Moscow (1956). English transl.: Am. Math. Soc., Providence (1964)
146. Levin, B.Ya. (in collaboration with Lyubarskiĭ, Yu., Sodin, M., Tkachenko, V.): Lectures on Entire Functions. Transl. Math. Monogr., vol. 150. Am. Math. Soc., Providence (1996).
147. Lyubich, Yu.I.: On a class of integral equations. Mat. Sb. **38**(2), 183–202 (1956)
148. Lyubich, Yu.I.: On a uniqueness theorem for mean periodic functions. J. Sov. Math. **28**(3) (1985)
149. Lyubich, Yu.I., Tkachenko, V.A.: A new proof of the fundamental theorem on functions periodic in the mean. Teor. Funkc. Funkc. Anal. Ih Prilozh. **4**, 162–170 (1967)
150. Malgrange, B.: Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution. Ann. Inst. Fourier **6**, 271–355 (1955–1956)
151. Malgrange, B.: Convolution equations. Rend. Semin. Mat. (Torino) **19**, 19–27 (1959–1960)
152. Meril, A.: Analytic functionals with unbounded carriers and mean periodic functions. Trans. Am. Math. Soc. **278**, 115–136 (1983)
153. Meril, A., Struppa, D.C.: Convolutors in spaces of holomorphic functions. In: Lect. Notes Math., vol. 1286, pp. 253–275. Springer, Berlin (1987)
154. Miščenko, A.S., Fomenko, A.T.: Course of Differential Geometry and Topology. Moscow University, Moscow (1980)
155. Mohanty, P., Ray, S., Sarkar, R., Sitaram, A.: The Helgason Fourier transform for symmetric spaces II. J. Lie Theory **14**(1), 227–242 (2004)
156. Montgomery, D., Samelson, H.: Transformation groups on spheres. Ann. Math. **44**(3), 454–470 (1943)
157. Mostow, G.D.: Strong Rigidity of Locally Symmetric Spaces. Princeton University Press, Princeton (1973)
158. Mourou, M., Trimèche, K.: Transmutation operators and Paley–Wiener theorem associated with a singular differential-difference operator on the real line. Anal. Appl. **1**(1), 43–70 (2003)
159. Napalkov, V.V.: Convolution Equations in Multidimensional Spaces. Nauka, Moscow (1982)
160. Narayanan, E.K., Thangavelu, S.: A spectral Paley–Wiener theorem for the Heisenberg group and a support theorem for the twisted spherical means on $\mathbb{C}^n$. Ann. Inst. Fourier **56**, 459–473 (2006)
161. Nikiforov, A.F., Uvarov, V.B.: Special Functions of Mathematical Physics. Nauka, Moscow (1978)

162. Ochakovskaya, O.A.: Majorants of functions with zero integrals over balls of fixed radius. Dokl. Akad. Nauk **420**(5), 598–600 (2008). English transl.: Dokl. Math. **77**(3), 1–3 (2008)
163. Ochakovskaya, O.A.: Liouville-type theorems for functions with zero integrals over balls of fixed radius. Dokl. Akad. Nauk **415**(2), 171–173 (2007). English transl.: Dokl. Math. **76**(1), 530–532 (2007)
164. Ólafsson, G., Schlichtkrull, H.: A local Paley–Wiener theorem for compact symmetric spaces. Preprint, arxiv: 0/05.0256VI [math.R. I] (2007)
165. Palamodov, V.P.: Linear Differential Operators with Constant Coefficients. Nauka, Moscow (1967). English transl.: Springer, New York (1970)
166. Poincaré, H.: Les fonctions analytiques de deux variables et la représentation conforme. Rend. Circ. Mat. Palermo **23**, 185–220 (1907)
167. Postnikov, M.M.: Lectures on Geometry. Lie Groups and Algebras. Nauka, Moscow (1982)
168. Postnikov, M.M.: Lectures on Geometry. Smooth Manifolds. Nauka, Moscow (1987). English transl.: Mir, Moscow (1989)
169. Prudnikov, A.P., Brychkov, Yu.A., Marichev, O.I.: Integrals and Series. Supplementary Chapters. Nauka, Moscow (1986)
170. Quinto, E.T.: The dependence of the generalized Radon transform on defining measures. Trans. Am. Math. Soc. **257**, 331–346 (1980)
171. Quinto, E.T.: The invertibility of rotation invariant Radon transforms. J. Math. Anal. Appl. **94**, 602–603 (1983)
172. Quinto, E.T.: Pompeiu transforms on geodesic spheres in real analytic manifolds. Isr. J. Math. **84**, 353–363 (1993)
173. Quinto, E.T.: Real analytic Radon transforms on rank one symmetric spaces. Proc. Am. Math. Soc. **117**, 179–186 (1993)
174. Quinto, E.T.: Radon transforms on curves in the plane. In: Quinto, E.T., Cheney, M., Kuchment, P. (eds.) Tomography, Impedance Imaging, and Integral Geometry, South Hadley, MA, 1993. Lect. Appl. Math., vol. 30, pp. 231–244. Am. Math. Soc., Providence (1994)
175. Quinto, E.T.: Radon transforms, differential equations, and microlocal analysis. Contemp. Math. **278**, 57–68 (2001)
176. Quinto, E.T.: Mean value extension theorems and microlocal analysis. Proc. Am. Math. Soc. **131**, 3267–3274 (2003)
177. Quinto, E.T.: Support theorems for the spherical Radon transform on manifolds. Int. Math. Res. Not. **2006**, 1–17 (2006). Article ID 67205
178. Quinto, E.T.: Helgason's support theorem and spherical Radon transforms. Contemp. Math. (2008)
179. Quinto, E.T., Zhou, Y.: Two-radius support theorems for spherical Radon transforms on manifolds. Contemp. Math. **251**, 501–508 (2000)
180. Radon, J.: Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten. Ber. Verh. Sächs. Akad. Wiss. Leipz., Math.-Nat. Wiss. Kl. **69**, 262–277 (1917)
181. Rawat, R., Sitaram, A.: Injectivity sets for spherical means on $\mathbb{R}^n$ and on symmetric spaces. J. Fourier Anal. Appl. **6**(3) (2000)

182. Riekstyn'sh, Z.Ya.: Asymptotic Expansions of Integrals, vol. I. Zinatne, Riga (1974)
183. Riordan, J.: Combinatorial Identities. Wiley, New York (1968)
184. Rudin, W.: Function Theory in the Unit Ball of $\mathbb{C}^n$. Springer, New York (1980)
185. Sajith, G., Thangavelu, S.: On the injectivity of twisted spherical means on $\mathbb{C}^n$. Isr. J. Math. **122**, 79–92 (2001)
186. Sarkar, R., Sitaram, A.: The Helgason Fourier transform for semisimple Lie groups. I. The case of $SL_2(\mathbb{R})$. Bull. Aust. Math. Soc. **73**(3), 413–432 (2006)
187. Schneider, R.: Functions on a sphere with vanishing integrals over certain subspheres. J. Math. Anal. Appl. **26**, 381–384 (1969)
188. Schwartz, L.: Théorie générale des fonctions moyenne périodiques. Ann. Math. **48**, 857–929 (1947)
189. Schwartz, L.: Theorie des distributions, 2nd ed. Hermann, Paris (1966)
190. Sedletskii, A.M.: On mean periodic functions. Izv. Akad. Nauk SSSR Ser. Mat. **34**(6), 1391–1415 (1970)
191. Sedletskii, A.M.: Mean periodic continuation and bases of exponential functions in $L^p(-\pi, \pi)$. Mat. Zametki **12**(1), 37–42 (1972). English transl.: Math. Notes **12**(1–2), 455–458 (1972)
192. Sedletskii, A.M.: Biorthogonal expansions of functions in exponential series on intervals of the real axis. Usp. Mat. Nauk **37**(5), 51–95 (1982).
193. Shahshahani, M., Sitaram, A.: The Pompeiu problem in exterior domains in symmetric spaces. Contemp. Math. **63**, 267–277 (1987)
194. Shapiro, H.S.: The expansion of mean-periodic functions in series of exponentials. Commun. Pure Appl. Math. **11**, 1–21 (1958)
195. Sherman, T.: The Helgason Fourier transform for compact Riemannian symmetric spaces of rank one. Acta Math. **164**, 73–144 (1990)
196. Shidlovskiǐ, A.B.: Transcendental Numbers. Nauka, Moscow (1987). English transl.: de Gruyter, Berlin (1989)
197. Sitaram, A.: Fourier analysis and determining sets for Radon measures on $\mathbb{R}^n$. Ill. J. Math. **28**, 339–347 (1984)
198. Sitaram, A.: Some remarks on measures on non-compact semi-simple Lie groups. Pac. J. Math. **2**, 429–434 (1984)
199. Sitaram, A.: On an analogue of the Wiener Tauberian theorem for symmetric spaces of the non-compact type. Pac. J. Math. **133**, 197–208 (1988)
200. Smith, J.D.: Harmonic analysis of scalar and vector fields in $\mathbb{R}^n$. Math. Proc. Camb. Philos. Soc. **72**, 403–416 (1972)
201. Smith, R.T.: The spherical representations of groups transitive on $\mathbb{S}^n$. Ind. Univ. Math. J. **24**(4), 307–325 (1974)
202. Springer, T., Veldkamp, F.: Elliptic and hyperbolic octave planes. Indag. Math. **25**, 413–451 (1963)
203. Stein, E., Weiss, G.: Introduction to Fourier Analysis on Euclidean Spaces. Princeton University Press, Princeton (1971)
204. Sternberg, S.: Lectures on Differential Geometry. Prentice-Hall, Englewood Cliffs (1964)
205. Struppa, D.C.: The Fundamental Principle for Systems of Convolution Equations. Mem. Am. Math. Soc. **273** (1983)

206. Suetin, P.K.: Classical Orthogonal Polynomials. Nauka, Moscow (1979)
207. Székelyhidi, L.: On mean periodic functions. Aequ. Math. **58**, 203–213 (1999)
208. Thangavelu, S.: Lectures on Hermite and Laguerre Expansions. Princeton University Press, Princeton (1993)
209. Thangavelu, S.: Spherical means and CR functions on the Heisenberg group. J. Anal. Math. **63**, 255–286 (1994)
210. Thangavelu, S.: Mean periodic functions on phase space and the Pompeiu problem with a twist. Ann. Inst. Fourier **45**, 1007–1035 (1995)
211. Thangavelu, S.: A Paley–Wiener theorem for the inverse Fourier transform on some homogeneous spaces. Hiroshima Math. J. **37**, 145–159 (2007)
212. Titchmarsh, E.C.: The Theory of Functions, 2nd ed. Oxford University Press, New York (1939)
213. Tits, J.: Le plan projectif des octaves et les groupes de Lie exeptionnels. Bull. Acad. R. Belg. **39**, 309–329 (1953)
214. Treves, J.F.: Lectures on Linear Partial Differential Equations with Constant Coefficients. Rio De Janeiro (1961)
215. Treves, J.F.: Topological Vector Spaces, Distributions and Kernels. Academic Press, New York (1967)
216. Trigub, R.M., Belinsky, E.S.: Fourier Analysis and Approximation of Functions. Kluwer Academic, Dordrecht (2004)
217. Trimèche, K.: Transformation intégrale de Weyl et théorème de Paley–Wiener associés à un opérateur différentiel singulier sur $(0, \infty)$. J. Math. Pures Appl. **60**, 51–98 (1981)
218. Trimèche, K.: Transmutation Operators and Mean-Periodic Functions Associated with Differential Operators. Harwood Academic, New York (1988)
219. Ungar, P.: Freak theorem about functions on a sphere. J. Lond. Math. Soc. (2) **29**, 100–103 (1954)
220. Vilenkin, N.Y.: Special Functions and the Theory of Group Representations. Nauka, Moscow (1991). English transl. of 1st ed.: Am. Math. Soc., Providence (1968)
221. Vladimirov, V.S.: The Equations of Mathematical Physics. Nauka, Moscow (1981)
222. Volchkov, V.V.: A definitive version of the local two-radii theorem. Mat. Sb. **186**(6), 15–34 (1995). English transl.: Sb. Math. **186**, 783–802 (1995)
223. Volchkov, V.V.: Theorems on ball means values in symmetric spaces. Mat. Sb. **192**(9), 17–38 (2001). English transl.: Sb. Math. **192**, 1275–1296 (2001)
224. Volchkov, V.V.: A definitive version of the local two-radii theorem on hyperbolic spaces. Izv. Ross. Akad. Nauk Ser. Mat. **65**(2), 3–26 (2001). English transl.: Izv. Math. **65**(2), 207–229 (2001)
225. Volchkov, V.V.: Integral Geometry and Convolution Equations. Kluwer Academic, Dordrecht (2003)
226. Volchkov, V.V.: Uniqueness theorems for solutions of a convolution equation on symmetric spaces. Izv. Ross. Acad. Nauk Ser. Mat. **70**(6), 3–18 (2006). English transl.: Izv. Math. **70**(6), 1077–1092 (2006)

227. Volchkov, V.V.: A local two-radii theorem on symmetric spaces. Mat. Sb. **198**(11), 21–46 (2007). English transl.: Sb. Math. **198**(11–12), 1553–1577 (2007)
228. Volchkov, V.V., Volchkov, Vit.V.: New results in integral geometry. Contemp. Math. **382**, 417–432 (2005)
229. Volchkov, V.V., Volchkov, Vit.V.: Uniqueness theorems and description of solutions for convolution equations on symmetric spaces and for the twisted convolution equations on $\mathbb{C}^n$. Preprint, Donetsk National University (2005)
230. Volchkov, V.V., Volchkov, Vit.V.: On mapping with generalized transmutation property. Proc. Inst. Appl. Math. Mech. NASU **13**, 31–38 (2006)
231. Volchkov, V.V., Volchkov, Vit.V.: Convolution equations and the local Pompeiu property on symmetric spaces and on phase space associated to the Heisenberg group. J. Anal. Math. **105**, 43–124 (2008)
232. Volchkov, V.V., Volchkov, Vit.V.: On a problem of Berenstein–Gay and its generalizations. Izv. Ross. Acad. Nauk Ser. Mat. (2009)
233. Volchkov, Vit.V.: Theorems on spherical means in complex hyperbolic spaces. Dopov. Nats. Akad. Nauk Ukr. No. 4, 7–10 (2000)
234. Volchkov, Vit.V.: Convolution equations on complex hyperbolic spaces. Dopov. Nats. Akad. Nauk Ukr. No. 2, 11–14 (2001)
235. Volchkov, Vit.V.: On functions with zero spherical means on a quaternion hyperbolic space. Izv. Ross. Acad. Nauk Ser. Mat. **66**(5), 3–32 (2002). English transl.: Izv. Math. **66**(5), 875–903 (2002)
236. Volchkov, Vit.V.: Uniqueness theorems for mean periodic functions on complex hyperbolic spaces. Anal. Math. **28**, 61–76 (2002)
237. Volchkov, Vit.V.: Mean-periodic functions on complex hyperbolic spaces. Sib. Mat. Zh. **43**(2), 271–280 (2002)
238. Volchkov, Vit.V.: A definitive version of the local two-radii theorem on quaternionic hyperbolic space. Dokl. Akad. Nauk **384**(4), 449–451 (2002)
239. Volchkov, Vit.V.: Convolution equation on quaternionic hyperbolic space. Dopov. Nats. Akad. Nauk Ukr., No. 12, 12–14 (2002)
240. Volchkov, Vit.V.: Injectivity theorems for the local Pompeiu transform on a sphere. Dokl. Akad. Nauk **379**(3), 309–311 (2003)
241. Volchkov, Vit.V.: Uniqueness theorems for functions periodic in the mean on a quaternionic hyperbolic space. Mat. Zametki **74**(1), 32–40 (2003). English transl.: Math. Notes **74**(1–2), 30–37 (2003)
242. Volchkov, Vit.V.: A local two-radii theorem on the sphere. Alg. Anal. **16**(3), 24–55 (2004). English transl.: St. Petersbg. Math. J. **16**(3), 453–475 (2005)
243. Volchkov, Vit.V.: Analogues of the local Pompeiu transform on a sphere. Dopov. Nats. Akad. Nauk Ukr. No. 2, 18–22 (2004)
244. Volchkov, Vit.V.: A local two radii theorem for compact Riemannian symmetric spaces of rank one and the Heisenberg group. Preprint, Donetsk National University (2004)
245. Volchkov, Vit.V.: Spherical means on the Heisenberg group and symmetric spaces. Dopov. Nats. Acad. Nauk Ukr. No. 3, 7–11 (2005)
246. Volchkov, Vit.V.: Functions with vanishing integrals over certain sets and their applications. Dopov. Nats. Acad. Nauk Ukr. No. 2, 18–20 (2005)

247. Volchkov, Vit.V.: On functions with zero spherical means on the Cayley projective plane. Dopov. Nats. Acad. Nauk Ukr. No. 12, 11–15 (2006)
248. Volchkov, Vit.V.: On functions with zero ball means on compact two-point homogeneous spaces. Mat. Sb. **198**(4), 21–46 (2007). English transl.: Sb. Math. **198**(3–4), 465–490 (2007)
249. Volchkov, Vit.V., Volchkova, N.P.: Theorems on the inversion of the local Pompeiu transform on a quaternion hyperbolic space. Alg. Anal. **15**(5), 169–197 (2003). English transl.: St. Petersbg. Math. J. **15**(5), 753–771 (2004)
250. Watson, G.N.: A Treatise on the Theory of Bessel Functions. Cambridge University Press, Cambridge (1922)
251. Wawrzrynczyk, A.: Spectral analysis and mean periodic functions on rank one symmetric spaces. Bol. Soc. Math. Mex. **30**, 15–29 (1985)
252. Wawrzrynczyk, A.: Spectral analysis and synthesis on homogeneous spaces. In: Proceedings of the XIXth National Congress of the Mexican Mathematical Society (Guadalajara), vol. 2, pp. 61–85 (1986)
253. Wawrzrynczyk, A.: Spectral analysis and synthesis on symmetric spaces. J. Math. Anal. Appl. **127**, 1–17 (1987)
254. Wawrzrynczyk, A.: Plane waves and spectral synthesis on symmetric spaces. Bull. Pol. Acad. Sci. Math. **36**(11–12), 737–739 (1988)
255. Wawrzrynczyk, A.: Solution of the spectral analysis problem on the space of horocycles. Bol. Soc. Mat. Mex. **35**(2), 49–59 (1990)
256. Weit, Y.: On Schwartz's theorem for the motion group. Ann. Inst. Fourier **30**, 91–107 (1980)
257. Weit, Y.: On spectral analysis in locally compact motion groups. J. Funct. Anal. **40**, 45–53 (1981)
258. Weit, Y.: A characterization of polynomials by convolution equations. J. Lond. Math. Soc. (2) **23**, 455–459 (1981)
259. Weit, Y.: On spectral synthesis in a non-commutative locally compact motion group. Am. J. Math. **40**, 45–53 (1982)
260. Weit, Y.: Spaces of continuous functions invariant with respect to radial action. Am. J. Math. **112**, 631–656 (1990)
261. Wiener, N.: Extrapolation, Interpolation, and Smoothing of Stationary Time Series. MIT Press, Cambridge (1949)
262. Williams, S.A.: Analyticity of the boundary for Lipschitz domains without the Pompeiu property. Indiana Univ. Math. J. **30**, 357–369 (1981)
263. Wolf, J.A.: Spaces of Constant Curvature. McGraw-Hill, New York (1967)
264. Yano, K., Bochner, S.: Curvature and Betti Numbers. Princeton University Press, Princeton (1953)
265. Yger, A.: Fonctions moyenne périodiques de deux variables; étude des systèmes d'équations de convolution envisagés par J. Delsarte. In: Sem. d'Analyse Harmonique, Orsay (1976–77)
266. Yger, A.: Propriétés de certains systèmes d'équations de convolutions dans. C. R. Acad. Sci. A **289**, 169–171 (1980)
267. Zalcman, L.: Analyticity and the Pompeiu problem. Arch. Rat. Anal. Mech. **47**, 237–254 (1972)

268. Zalcman, L.: Mean values and differential equations. Isr. J. Math. **14**, 339–352 (1973)
269. Zalcman, L.: Offbeat integral geometry. Am. Math. Mon. **87**(3), 161–175 (1980)
270. Zalcman, L.: Uniqueness and nonuniqueness for the Radon transform. Bull. Lond. Math. Soc. **14**, 241–245 (1982)
271. Zalcman, L.: Netradiční integrální geometrie. Pokroky Mat. Fyz. Astron. **27**, 9–23 (1982)
272. Zalcman, L.: Determining sets for functions and measures. Real Anal. Exch. **11**, 40–55 (1985–86)
273. Zalcman, L.: A bibliographic survey of the Pompeiu problem. In: Fuglede, B., et al. (eds.) Approximation by Solutions of Partial Differential Equations, pp. 185–194. Kluwer Academic, Dordrecht (1992)
274. Zalcman, L.: Supplementary bibliography to "A bibliographic survey of the Pompeiu problem". In: Radon Transforms and Tomography. Contemp. Math., vol. 278, pp. 69–74. Am. Math. Soc., Providence (2001)
275. Zaraisky, D.A.: An improvement of the uniqueness theorem for solutions of a convolution equation. Proc. Inst. Appl. Math. Mech. NASU **12**, 69–75 (2006)
276. Zaraisky, D.A.: Approximation of functions with zero integrals over balls by linear combinations of solutions of the Helmholtz equation. Proc. Inst. Appl. Math. Mech. NASU **16**, 80–87 (2008)
277. Zariski, O., Samuel, P.: Commutative Algebra, vol. I. Van Nostrand-Reinhold, New York (1958)
278. Zhou, Y.: Two radius support theorem for the sphere transform. J. Math. Anal. Appl. **254**, 120–137 (2001)

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